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Probing quantum spacetime with Dirac quasinormal modes

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Abstract Noncommutative (NC) geometry may open an alternative route to quantum gravity. We study the signatures that quantum structure of spacetime may leave on Dirac quasinormal mode spectrum in the setting defined by a common astrophysical background. For that purpose we examine the influence of spacetime noncommutativity on the Dirac quasinormal modes in modified Reissner–Nordström black hole spacetime. The framework for the latter study is provided by the effective model of NC gravity coupled to fermions introduced in Dimitrijević Ćirić et al. (Eur Phys J C 83:387, 2023). This model describes a classical Dirac field coupled to a modified Reissner–Nordström geometry where the corresponding metric acquires an additional nonvanishing $r - \varphi$ component. As the earlier study shows, this model appears to be equivalent to a model of semiclassical NC gauge theory in which a NC gauge field is coupled to a NC fermion field on the one side and the classical Reissner–Nordström background on the other. We compute the resulting Dirac quasinormal modes and compare them with those of the undeformed Reissner–Nordström spacetime. The results show that spacetime noncommutativity modifies both the oscillation frequencies and damping rates, and induces features in the effective potential and quasinormal mode spectrum reminiscent of a Zeeman-like splitting. Since such geometric modifications are expected to become relevant only near the Planck scale, these effects are more naturally associated with microscopic rather than astrophysical black holes.

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1 Introduction

Black hole quasinormal modes (QNMs) are the characteristic frequencies of black holes in a ringdown phase calculated at linear order in perturbation theory. They appear as an outcome of a black hole's response to an external perturbation and may be characterized by a superposition of exponentially damped oscillations with generally an infinite set of discrete frequencies and damping times. With the discovery of gravitational waves, the interest in their study increased considerably as they can be linked directly and compared with the experimentally observed complex gravitational-wave frequencies of black hole merger remnants. The QNMs and their frequencies have a long history [1–8]. They carry information about intrinsic properties of black holes. Besides the characteristic parameters of black holes like mass, charge and angular momentum, they also give information about the stability of black holes under perturbation by matter fields that evolve in their exterior region without backreacting on the metric. In general, the frequencies themselves are complex, with the real part representing the oscillation frequency and the imaginary part describing the rate at which this oscillation is attenuated. Accordingly, the stability of a black hole is guaranteed by the imaginary part of QNMs being negative.

Rising excitement in QNMs, caused by experimental discovery of gravitational waves [9], also spurred the increased interest in the black hole spectroscopy which has been given a whole new set of possibilities. Indeed, quasinormal modes are closely related to a notion of black hole spectroscopy which itself is based on the correspondence between atomic and black hole spectra. This correspondence has a root in an analogy between the theory that describes scattering of gravitational waves on a black hole on the one side and the scattering theory in quantum mechanics on the other. It provides an appropriate frame for describing puzzling physical phenomena like Hawking radiation as a process of transitioning between different states in black hole QNM spectrum.

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This point of view paves the way to an intriguing connection between Hawking radiation and black hole quasinormal modes [10–15]. Therefore, this correspondence by itself appears to be important in the route to quantizing gravity, because one can naturally interpret black hole quasinormal modes in terms of quantum energy levels.

In this paper, we study the propagation and decay of noncommutative massless fermionic fields in the Reissner–Nordström (RN) black hole background in order to document a possible appearance of the fine structure in the spectrum, including some nonstandard features like the presence of anomalous decay rate behaviour. We carry out this study by using the method of continued fractions. It is worthy to mention that the study of Dirac QNMs has started with the paper by Cho, in which the massive and massless Dirac quasinormal mode frequencies in the Schwarzschild black hole spacetime have been calculated [16]. Refinement in the calculation of Dirac quasinormal modes in Schwarzschild background has been achieved by implementing the Leaver's continued fraction method [17] in the paper by Jing [18] and the high overtones of Dirac perturbations in the same black hole setting were studied in [19]. This analysis has been extended to the Reissner–Nordström background in [20]. Later on, QNMs of the Dirac field have been studied rather extensively for massless fermionic fields [21,22] and massive fermionic fields [23,24] in Schwarzschild–de Sitter and Reissner–Nordström–de Sitter backgrounds. Similarly, the Dirac QNMs of three-dimensional, four-dimensional and D-dimensional de Sitter spacetime have been respectively studied in [25–27]. These results have shown that fields with higher masses propagating in spacetimes with larger cosmological constants tend to decay more slowly in a Schwarzschild–de Sitter black hole background [23]. Likewise, by using the convergent Frobenius method it has been shown that two chiralities of massive fermions give rise to an additional fine structure in the spectrum, for Schwarzschild and Kerr backgrounds [28]. The scalar and fermionic QNMs of a Kerr–Newman–de Sitter background were analysed in [29]. Dirac QNMs have also been studied in other configurations of substantial interest [30–38].

The study of Dirac QNMs is essential for obtaining the information about the stability of fermionic perturbations in the vicinity of a black hole and may help with providing insights into the general behaviour of quantum fields in black hole backgrounds. In the limit of large angular momentum, the QNMs start to nearly mirror/reflect the properties of photons being trapped into a closed circular orbits around black hole after they fall toward the horizon. These circular orbits are null geodesics and they are usually referred to as light rings. For large angular momentum, i.e. in the eikonal limit, the orbital frequency of photons along light rings is closely given by the real part of the fundamental QNM frequency, while its imaginary part closely corresponds to the Lyapunov exponent [39–42]. The latter characterizes the stability of the photon circular orbits along light rings. Quasinormal modes of fermion perturbations in the vicinity of a black hole, as well as QNMs of other types of perturbations may be used to deduce these stability parameters related to null geodesics.

Moreover, Dirac QNMs exhibit distinctive features compared to scalar or gravitational perturbations, including spin–orbit coupling and its associated effects, as well as characteristic behaviour with respect to superradiance. Concerning the latter in particular, it is known that the Dirac field behaves in a different way than the integer spin fields when they propagate in a curved background. For example, the fermion fields do not suffer from superradiant instabilities when they are scattered by rotating black holes [43–48]. Indeed, while the bosonic fields on Kerr spacetime become unstable in a certain regime of system parameters [49–55], the fermionic fields on Kerr spacetime in that same regime remain stable under the condition of extra slow rotation.

It is thus of interest to investigate the QNM frequencies of the Dirac field in spacetimes of various types. For the same reason, it is of interest to go one step further and consider Dirac field perturbations in spacetimes that incorporate essential features of quantum gravity. This may contribute to resolving some of the open puzzles in the development of a consistent theory of quantum gravity. Noncommutative spacetimes and the corresponding gauge and gravity models [56–62] on such manifolds offer a framework in which several features anticipated in quantum gravity can be consistently implemented within effective models.

The premise of quantum nature of spacetime in the context of spacetime propagation of fermions and the associated equation of motion in the form of Dirac equation has been taken into account and considered in a series of different studies [63–75]. In particular, quasinormal modes of the Dirac field have been studied within different approaches to effective quantum gravity, including models on noncommutative spacetime [76,77], as well as certain models of effective quantum gravity which arise when canonical quantum gravity leads to a semiclassical model described by an effective Hamiltonian constraint [78,79]. Besides, it has been found that the issue of possible instability of the fermion field on the RN background due to superradiant growth remains unaffected by the presence of spacetime noncommutativity. This appears to be a direct consequence of the weak energy condition not being violated by the NC deformation and the resulting perseverance of the second law of black hole thermodynamics [80].

In this work, we study noncommutative massless Dirac field perturbations in a background of the RN black hole. They were shown to be equivalent to considering the ordinary massless Dirac field perturbations in a background of modified RN geometry, where the associated metric acquires an additional nonvanishing $r - \varphi$ component [80]. Furthermore, we calculate the corresponding quasinormal mode frequencies using the Leaver's method of continued fractions, but only after a systematic application of the Gaussian elimination procedure. The latter was devised with a purpose of bringing down the resultant 6-term recurrence relations to the solvable 3-term relations. The quasinormal modes of the Dirac field exhibit a discernable Zeeman-like splitting when the quantum parameter is introduced. As the fundamental mode is the dominating one in the signal, only that mode will be the subject of our concern. However, a note of caution is needed here. Given that the quantum nature of spacetime gives rise to modifications in geometry only at extremely small scales, the results implied in the paper may be relevant for microscopic rather than astrophysical black holes.

In Sect. 2 we introduce the modified Reissner–Nordström metric with the $\star$-product of noncommutative spacetime and the associated tetrad field. After selecting the appropriate ansatz for the Dirac equation, it reduces to two radial equations governing the two Weyl spinors. Using appropriate field redefinitions, the two equations can be condensed into a single second order ODE in the radial coordinate. In Sect. 3 we recast the equation of motion into the Schrödinger form by introducing the tortoise coordinate and we identify the boundary conditions for QNMs. We then set up the continued fraction method and solve the recurrence relations. Solutions of the recurrence relations are analysed in Sect. 4, where we focus on the modifications of the QNM spectrum due to spacetime noncommutativity. Closing remarks are given in Sect. 5.

2 Dirac equation in modified Reissner–Nordström spacetime

In Refs. [81,82], a semiclassical model describing a charged NC scalar field $\hat{\Phi}$ and an NC $U(1)$ gauge field $\hat{A}$ has been introduced. Both scalar and gauge NC fields interact with each other and they also interact with a classical gravitational background of the RN type. The model is semiclassical in a sense that only gauge and scalar fields are considered to be affected by the NC deformation, while on the other side, the gravitational field is not. Instead, it is considered to be described by classical degrees of freedom, completely unaffected by the NC nature of spacetime. The model introduced in Refs. [81,82] therefore deals with a situation where the gauge and scalar fields are quantized and the gravitational field is not. It is important to stress that the term "quantized", when applied to the gauge and scalar fields, refers here exclusively to the situation when these fields are considered as noncommutative variables and does not refer to the usual notion of second quantization in quantum field theory.

The model was built using deformation quantization techniques based on Drinfeld twist operator and the explicit twist operator that was used in the construction was the so-called angular twist operator [81,82]

$$\begin{aligned}\mathcal{F} &= e^{-\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta}\\ &= e^{-\frac{ia}{2}(\partial_t\otimes\partial_\varphi-\partial_\varphi\otimes\partial_t)},\end{aligned}\tag{1}$$

with $\alpha,\beta\in\{t,r,\theta,\varphi\}$ and $\theta^{t\varphi}=-\theta^{\varphi t}=a$ as the only nonzero components of the deformation tensor $\theta^{\alpha\beta}$. Here, $a=1/\kappa$ is the deformation parameter that sets up the NC scale, commonly related to the Planck length. This twist operator is constructed from Killing fields of the geometry that we consider – it is a Killing twist.

The $\star$-product, the wedge $\star$-product between forms, the coproduct and other structural maps of the related symmetry algebra can all be obtained from the twist operator (1). In particular, the $\star$-product between functions is given by

$$\begin{aligned}f\star g &= \mu\big(e^{\frac{ia}{2}(\partial_t\otimes\partial_\varphi-\partial_\varphi\otimes\partial_t)}f\otimes g\big)\\ &= fg+\frac{ia}{2}(\partial_t f(\partial_\varphi g)-\partial_t g(\partial_\varphi f))+\mathcal{O}(a^2),\end{aligned}\tag{2}$$

where μ is the usual commutative pointwise multiplication of functions. The remaining ingredients of the differential calculus are described in [81].

It is noteworthy that some spacetime metrics may be deduced from certain duality arguments, as demonstrated in [80]. These arguments are based on a recognition that in some cases twist deformed $U(1)$ gauge theories on curved space behave in as much the same way as their commutative counterparts, albeit in a modified spacetime geometry. In particular, in [80] it has been shown that the equation of motion for a charged NC scalar field in a classical RN background coupled to NC $U(1)$ gauge field may be rewritten in terms of the equation of motion governing the behaviour of a charged commutative scalar field, having the same charge q as its NC counterpart, and propagating in a modified RN geometry

$$\begin{aligned}\mathrm{d}s^2 &= \Big(1-\frac{2MG}{r}+\frac{Q^2G}{r^2}\Big)\mathrm{d}t^2-\frac{\mathrm{d}r^2}{1-\frac{2MG}{r}+\frac{Q^2G}{r^2}}\\ &\quad -aqQ\sin^2\theta\,\mathrm{d}r\,\mathrm{d}\varphi-r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2).\end{aligned}\tag{3}$$

It appears that this novel, first order effective dual metric (3) acquires an additional off-diagonal term which is induced purely by noncommutative nature of spacetime. This feature comes into play only in the presence of charged matter.

The geometry (3) can be considered as a noncommutative deformation of the Reissner–Nordström (RN) metric

$$\begin{aligned}\mathrm{d}s^2 &= \Big(1-\frac{2MG}{r}+\frac{Q^2G}{r^2}\Big)\mathrm{d}t^2-\frac{\mathrm{d}r^2}{1-\frac{2MG}{r}+\frac{Q^2G}{r^2}}\\ &\quad -r^2(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\varphi^2),\end{aligned}\tag{4}$$

that represents a charged non-rotating black hole with mass M and charge Q. For later purposes we shall introduce the usual abbreviation $f = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$. Being static and spherically symmetric, the spacetime of RN black hole has four Killing vectors, among which ∂_t and ∂_φ are included, and t and φ are the time and polar variables of the spherical coordinate system $\mu \in \{t, r, \theta, \varphi\}$. It is now plainly seen that the twist (1) is a Killing twist, as it is formed from the operators that are actually the Killing vectors for the metric (4). Note that the implementation of the twist (1) is compatible with the semiclassical nature of the model that we consider because it ensures that the geometry (4) remains unaffected by the deformation via this twist. This is because the twist (1) does not act on the RN metric.

In this paper we consider a Dirac equation on the background geometry given by the modified RN metric (3), where in addition we introduce a gauge potential A_μ minimally coupled to the Dirac operator $\gamma^a \nabla_a$,

$$\big(i\gamma^a(\nabla_a + A_a) - m\big)\Psi = \big(i\gamma^a e_a{}^\mu(\nabla_\mu + A_\mu) - m\big)\Psi = 0. \tag{5}$$

Here the Latin index $a \in \{0, 1, 2, 3\}$ refers to the intrinsic coordinates and γ^a are the standard flat space Dirac gamma matrices satisfying $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, where

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{6}$$

The Dirac operator $\gamma^a \nabla_a$ on a curved space is introduced in terms of tetrads (vierbeins) $e^a{}_\mu$ and their inverse $e_a{}^\mu$, satisfying $e^a{}_\mu e_a{}^\nu = \delta_\mu^\nu$ and $e^a{}_\mu e_b{}^\mu = \delta^a{}_b$. Tetrads written in components are $e^a{}_\mu = (e^a{}_t, e^a{}_r, e^a{}_\theta, e^a{}_\varphi)$ and $e_a{}^\mu = (e_0{}^\mu, e_1{}^\mu, e_2{}^\mu, e_3{}^\mu)$. They also satisfy $g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}$ and $g^{\mu\nu} = e_a{}^\mu e_b{}^\nu \eta^{ab}$. In what follows we use the setting defined in [48]. This setting consists of the vierbein frame chosen to be

$$e^a{}_\mu = \begin{pmatrix} \sqrt{f} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{f}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & \frac{aqQ}{2r}\sin\theta & 0 & r\sin\theta \end{pmatrix}$$

with the corresponding inverse matrix

$$e_a{}^\mu = \begin{pmatrix} \frac{1}{\sqrt{f}} & 0 & 0 & 0 \\ 0 & \sqrt{f} & 0 & -\frac{aqQ}{2r^2}\sqrt{f} \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r\sin\theta} \end{pmatrix} \tag{7}$$

and the following representation of gamma matrices

$$\gamma^0 = i\tilde{\gamma}^0 = i\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \gamma^1 = i\tilde{\gamma}^3 = i\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix},$$
$$\gamma^2 = i\tilde{\gamma}^1 = i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, \quad \gamma^3 = i\tilde{\gamma}^2 = i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \tag{8}$$

where $\tilde{\gamma}^0$, $\tilde{\gamma}^1$, $\tilde{\gamma}^2$ and $\tilde{\gamma}^3$ are gamma matrices in chiral/Weyl representation, while σ_i, $(i = 1, 2, 3)$ are the usual Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

With the spinor field Ψ written in terms of two two-component spinors Ψ_1 and Ψ_2, namely $\Psi = (\Psi_1, \Psi_2)^T$ and the gauge potential $A_\mu = (A_t, \mathbf{A}) = (-\frac{qQ}{r}, \mathbf{0})$, the Eq. (5) splits into two two-component equations

$$\begin{aligned}
&\Big[-\frac{1}{\sqrt{f}}\mathbb{1}\partial_t - \sqrt{f}\sigma_3\partial_r - \frac{1}{2}\frac{Mr - Q^2}{r^3}\frac{1}{\sqrt{f}}\sigma_3 - \frac{\sqrt{f}}{r}\sigma_3 - \frac{1}{r}\sigma_1\partial_\theta \\
&+\frac{aqQ}{2r^2}\sqrt{f}\sigma_3\partial_\varphi - \frac{1}{r\sin\theta}\sigma_2\partial_\varphi - \frac{1}{2r}\cot\theta\sigma_1 - \frac{iqQ}{r\sqrt{f}}\mathbb{1}\Big]\Psi_2 \\
&-m\mathbb{1}\Psi_1 = 0, \\
&\Big[-\frac{1}{\sqrt{f}}\mathbb{1}\partial_t + \frac{1}{2}\frac{Mr - Q^2}{r^3}\frac{1}{\sqrt{f}}\sigma_3 + \sqrt{f}\sigma_3\partial_r + \frac{\sqrt{f}}{r}\sigma_3 + \frac{1}{r}\sigma_1\partial_\theta \\
&-\frac{aqQ}{2r^2}\sqrt{f}\sigma_3\partial_\varphi + \frac{1}{r\sin\theta}\sigma_2\partial_\varphi + \frac{1}{2r}\cot\theta\sigma_1 - \frac{iqQ}{r\sqrt{f}}\mathbb{1}\Big]\Psi_1 \\
&-m\mathbb{1}\Psi_2 = 0.
\end{aligned} \tag{10}$$

For details, we refer the reader to the reference [80]. In order to separate this system of equations, we impose the following ansatz

$$\Psi = e^{i(\nu\varphi - \omega t)}\begin{pmatrix} \psi_1(r,\theta) \\ \psi_2(r,\theta) \end{pmatrix} = e^{i(\nu\varphi - \omega t)}\begin{pmatrix} pR_2(r)S_1(\theta) \\ -\frac{1}{r}R_1(r)S_2(\theta) \\ \frac{1}{r}R_1(r)S_1(\theta) \\ R_2(r)S_2(\theta) \end{pmatrix}. \tag{11}$$

In the above ansatz p is a free parameter which can acquire two possible values, $p \in \{-1, +1\}$. We keep it in order to be able to cover and analyse a somewhat more general scope of possibilities.

After inserting the ansatz (11) into (10) and dividing the resulting equations respectively with R_2S_1, R_1S_2, R_1S_1, R_2S_2, one finds that this system of equations is completely separable,

$$\begin{aligned}
&\frac{i\omega}{\sqrt{f}}\frac{R_1}{R_2} - r\sqrt{f}\frac{1}{R_2}\partial_r\Big(\frac{R_1}{r}\Big) - \frac{1}{2}\frac{Mr - Q^2}{r^3}\frac{1}{\sqrt{f}}\frac{R_1}{R_2} - \frac{\sqrt{f}}{r}\frac{R_1}{R_2} \\
&\quad + i\nu\frac{aqQ}{2r^2}\sqrt{f}\frac{R_1}{R_2} - \frac{iqQ}{r\sqrt{f}}\frac{R_1}{R_2} - pmr \\
&\quad = \frac{\partial_\theta S_2}{S_1} + \frac{\nu}{\sin\theta}\frac{S_2}{S_1} + \frac{1}{2}\cot\theta\frac{S_2}{S_1} \equiv \lambda, \\
&\frac{i\omega r^2}{\sqrt{f}}\frac{R_2}{R_1} + r^2\sqrt{f}\frac{1}{R_1}\partial_r R_2 + \frac{1}{2}\frac{Mr - Q^2}{r}\frac{1}{\sqrt{f}}\frac{R_2}{R_1} + r\sqrt{f}\frac{R_2}{R_1} \\
&\quad - i\nu\frac{aqQ}{2}\sqrt{f}\frac{R_2}{R_1} - \frac{iqQr}{\sqrt{f}}\frac{R_2}{R_1} + mr \\
&\quad = \frac{\partial_\theta S_1}{S_2} - \frac{\nu}{\sin\theta}\frac{S_1}{S_2} + \frac{1}{2}\cot\theta\frac{S_1}{S_2} \equiv \lambda_1, \\
&p\frac{i\omega r^2}{\sqrt{f}}\frac{R_2}{R_1} + r^2\sqrt{f}\frac{p}{R_1}\partial_r R_2
\end{aligned}$$

$$+\frac{1}{2}\frac{Mr-Q^2}{r}\frac{p}{\sqrt{f}}\frac{R_2}{R_1}+pr\sqrt{f}\frac{R_2}{R_1}$$
$$-piv\frac{aqQ}{2}\sqrt{f}\frac{R_2}{R_1}-p\frac{iqQr}{\sqrt{f}}\frac{R_2}{R_1}-mr$$
$$=\frac{\partial_\theta S_2}{S_1}+\frac{\nu}{\sin\theta}\frac{S_2}{S_1}+\frac{1}{2}\cot\theta\frac{S_2}{S_1}=\lambda,$$
$$-\frac{i\omega}{\sqrt{f}}\frac{R_1}{R_2}+r\sqrt{f}\frac{1}{R_2}\partial_r\Big(\frac{R_1}{r}\Big)$$
$$+\frac{1}{2}\frac{Mr-Q^2}{r^3}\frac{1}{\sqrt{f}}\frac{R_1}{R_2}$$
$$+\frac{\sqrt{f}}{r}\frac{R_1}{R_2}-iv\frac{aqQ}{2r^2}\sqrt{f}\frac{R_1}{R_2}+\frac{iqQ}{r\sqrt{f}}\frac{R_1}{R_2}-mr$$
$$=-p\frac{\partial_\theta S_1}{S_2}+p\frac{\nu}{\sin\theta}\frac{S_1}{S_2}-p\frac{1}{2}\cot\theta\frac{S_1}{S_2}$$
$$=-p\Big(\frac{\partial_\theta S_1}{S_2}-\frac{\nu}{\sin\theta}\frac{S_1}{S_2}+\frac{1}{2}\cot\theta\frac{S_1}{S_2}\Big)=-p\lambda_1. \qquad (12)$$

It is easily seen that the two separation constants λ and λ_1, which have appeared in the process of separation, are not mutually independent, but subject to a certain constraint, as we shall see shortly. Moreover, it appears that a separation of the system (10) can be achieved for both values of p. Let us now analyse different possibilities as related to the choice of p and the implications this choice has for the relation between the separation constants λ and λ_1.

For that purpose, we note that while the match between the first and the fourth equation in (12) leads to the condition $\lambda + pmr = p\lambda_1 - mr$, the match between the second and the third yields $\lambda_1 - mr = p\lambda + pmr$. These two conditions are in fact equivalent to each other, as can readily be seen by multiplying any of them by p. For $p = -1$, two separation constants λ and λ_1 can be related and the relation among them is given by $\lambda_1 = -\lambda$. On the opposite, for $p = +1$, the only viable possibility is attained when $m = 0$.[1] Although the system (10) is still separable for $p = +1$, the separation constants λ and λ_1 can be related only if the spinor field is massless, in which case the corresponding relation is $\lambda_1 = \lambda$.

We can conclude that for the spinor field with nonzero mass, the parameter p must necessarily be equal to $p = -1$. On the contrary, for massless field, the parameter p may take any of the two possible values, $p = +1$ or $p = -1$. Generally, we can write $\lambda_1 = p\lambda$ with a remark that for $p = +1$, the mass of the field must be zero. For $p = -1$ the spinor field may be massive, as well as massless.

In effect, the system of equations (10) gives rise to two independent angular equations

$$\partial_\theta S_2+\frac{\nu}{\sin\theta}S_2+\frac{1}{2}\cot\theta S_2=\lambda S_1,$$
$$\partial_\theta S_1-\frac{\nu}{\sin\theta}S_1+\frac{1}{2}\cot\theta S_1=\lambda_1 S_2, \qquad (13)$$

and two independent radial equations

$$\frac{i\omega}{\sqrt{f}}R_1-\sqrt{f}\partial_r R_1-\frac{1}{2}\frac{Mr-Q^2}{r^3}\frac{1}{\sqrt{f}}R_1$$
$$+iv\frac{aqQ}{2r^2}\sqrt{f}R_1-\frac{iqQ}{r\sqrt{f}}R_1=(\lambda+pmr)R_2,$$
$$\frac{i\omega r^2}{\sqrt{f}}R_2+r^2\sqrt{f}\partial_r R_2+\frac{1}{2}\frac{Mr-Q^2}{r}\frac{1}{\sqrt{f}}R_2+r\sqrt{f}R_2$$
$$-iv\frac{aqQ}{2}\sqrt{f}R_2-\frac{iqQr}{\sqrt{f}}R_2=(p\lambda-mr)R_1. \qquad (14)$$

This system of radial equations can be used to study the behaviour of fermion quasinormal modes in the modified RN background (3).

Upon decoupling, these two equations give rise to the following two 2nd order differential equations,

$$-r^2 f\partial_r^2 R_1+\Big(M-r-\frac{Mr-Q^2}{r}+ivaqQf+\frac{pmr^2}{\lambda+pmr}f\Big)\partial_r R_1$$
$$+\Big[\frac{Mr-Q^2}{4r}\frac{\partial_r f}{f}-\frac{1}{2}i\omega r^2\frac{\partial_r f}{f}-\frac{i\omega pmr^2}{\lambda+pmr}$$
$$-\frac{1}{2}\frac{-2\lambda Mr^3-3pmMr^4+3\lambda Q^2r^2+4pmQ^2r^3}{r^4(\lambda+pmr)}$$
$$+\frac{ivaqQ}{4}\partial_r f-\frac{ivaqQ}{2}f\frac{2\lambda r+3pmr^2}{r^2(\lambda+pmr)}$$
$$+\frac{1}{2}iqQr\frac{\partial_r f}{f}+iqQ\frac{\lambda+2pmr}{\lambda+pmr}+\Big(i\omega r^2+\frac{1}{2}\frac{Mr-Q^2}{r}$$
$$+rf-\frac{ivaqQ}{2}f-iqQr\Big)\Big(\frac{i\omega}{f}$$
$$-\frac{1}{2f}\frac{Mr-Q^2}{r^3}+\frac{ivaqQ}{2r^2}-\frac{iqQ}{r}\frac{1}{f}\Big)\Big]R_1$$
$$=(\lambda+pmr)(p\lambda-mr)R_1, \qquad (15)$$

for the first component $R_1 \equiv R_{s=-\frac{1}{2}}$ and

$$-r^2 f\partial_r^2 R_2+\Big(5M-3r-\frac{2Q^2}{r}$$
$$-\frac{Mr-Q^2}{r}+ivaqQf-\frac{mr^2}{p\lambda-mr}f\Big)\partial_r R_2$$
$$+\Big[\Big(i\omega r^2+\frac{1}{2}\frac{Mr-Q^2}{r}+rf-\frac{ivaqQ}{2}f-iqQr\Big)$$
$$\times\Big(\frac{i\omega}{f}-\frac{1}{2f}\frac{Mr-Q^2}{r^3}+\frac{ivaqQ}{2r^2}-\frac{iqQ}{r}\frac{1}{f}+\frac{1}{2}\frac{\partial_r f}{f}-\frac{m}{p\lambda-mr}\Big)$$
$$-2i\omega r-\frac{1}{2}\frac{Q^2}{r^2}-f-r\partial_r f+\frac{ivaqQ}{2}\partial_r f+iqQ\Big]R_2$$
$$=(p\lambda-mr)(\lambda+pmr)R_2, \qquad (16)$$

for the second component $R_2 \equiv R_{s=+\frac{1}{2}}$, where s refers to chirality.[2] Separating the right-handed component of the Dirac spinor from the left-handed one implies that λ satisfies $\lambda^2 = (j-s)(j+s+1)$ [48]. Interestingly, if we take

[1] For $p = +1$ the above obtained condition leads to $\lambda_1 = \lambda + 2mr$, which is inconsistent with λ and λ_1 both being the separation constants (independent of r and θ).

[2] For massless particles chirality turns out to be the same as helicity. However, for massive particles these two notions need to be distinguished.

the notation $\lambda \equiv \lambda_s$, where $s \in \{-1/2, +1/2\}$, then it is straightforward to see that $\lambda^2_{-\frac{1}{2}} = \lambda^2_{+\frac{1}{2}} + 1$. We shall make use of this relation in the subsequent analysis when trying to relate the separation constants appearing in the two component equations (17) and (18) down below. At this point, it is also convenient to introduce the label $\Delta = r^2 f$.

As a next step, note that if the transformation $R_1 \equiv R_{s=-\frac{1}{2}} = r^{\beta}\Delta^{\alpha/2}\xi_{s=-\frac{1}{2}}$ is carried out on the Eq. (15), then for $\beta = \frac{1}{2}$ and $\alpha \equiv s = -\frac{1}{2}$, this equation transforms into

$$\begin{aligned}
&\Delta\partial_r^2\xi_{-\frac{1}{2}} + \left(2(\alpha+1)(r-M) - ivaqQf - \frac{pmr^2}{\lambda+pmr}f\right)\partial_r\xi_{-\frac{1}{2}}\\
&+\Bigg[\frac{(\omega r^2 - qQr)^2 + i(r-M)(\omega r^2 - qQr)}{\Delta}\\
&+iqQ - 2i\omega r + p\lambda^2\Bigg]\xi_{-\frac{1}{2}}\\
&-\Bigg[\frac{ivaqQ}{r^3}\left(\alpha r^2 + (1-\alpha)Mr - Q^2\right)\\
&+\frac{pm}{\lambda+pmr}\left((\alpha+\frac{1}{2})(r-M) - i\omega r^2 + iqQr - \frac{ivaqQ}{2}f\right)\\
&+pm^2r^2\Bigg]\xi_{-\frac{1}{2}} = 0. \qquad (17)
\end{aligned}$$

Likewise, if the same transformation $R_2 \equiv R_{s=\frac{1}{2}} = r^{\beta}\Delta^{\alpha/2}\xi_{s=\frac{1}{2}}$ is applied to the Eq. (16) for the component R_2, then the choice $\beta = -\frac{1}{2}$ and $\alpha \equiv s = \frac{1}{2}$ yields[3]

$$\begin{aligned}
&\Delta\partial_r^2\xi_{+\frac{1}{2}} + \left(2(\alpha+1)(r-M) - ivaqQf + \frac{mr^2}{p\lambda-mr}f\right)\partial_r\xi_{+\frac{1}{2}}\\
&+\Bigg[\frac{(\omega r^2 - qQr)^2 - i(r-M)(\omega r^2 - qQr)}{\Delta}\\
&-iqQ + 2i\omega r + p\lambda^2 + 1\Bigg]\xi_{+\frac{1}{2}}\\
&-\Bigg[\frac{ivaqQ}{r^3}\left(\alpha r^2 + (1-\alpha)Mr - Q^2\right)\\
&-\frac{m}{p\lambda-mr}\left((\alpha+\frac{1}{2})(r-M) + i\omega r^2 - iqQr - \frac{ivaqQ}{2}f\right)\\
&+pm^2r^2\Bigg]\xi_{+\frac{1}{2}} = 0. \qquad (18)
\end{aligned}$$

[3] At this point, $\alpha \equiv s$ is a generic parameter which may take one of two values, $\alpha \equiv s = -\frac{1}{2}$ or $\alpha \equiv s = +\frac{1}{2}$. Which one of these two values will it take depends on the component it refers to, as well as the equation in which it appears. Therefore, if the parameter s appears in the Eq. (17) and refers to the first component $R_1 \equiv R_s$, it will acquire the value $-\frac{1}{2}$, and when it appears in the Eq. (18) that governs behaviour of the second component $R_2 \equiv R_s$, it will take the value $+\frac{1}{2}$. Note that in all terms except the middle one, we have retained a generic label α, avoiding to fix it to a particular value, so that later on we can more easily deduce a generic form of the 2nd order differential equation that would encompass both components within a single equation.

We point out that in the limit $m, a \to 0$ the above equations both reduce to the equations studied in [83].

From now on, we put the parameter $p = -1$ in the above 2nd order equations. This is because it is the only viable choice in the case of the spinor field with mass and it is also valid in the case of massless spinor field. Setting $p = -1$ leads to

$$\begin{aligned}
&\Delta\partial_r^2\xi_{-\frac{1}{2}} + \left(2(\alpha+1)(r-M) - ivaqQf + \frac{m\Delta}{\lambda-mr}\right)\partial_r\xi_{-\frac{1}{2}}\\
&+\Bigg[\frac{(\omega r^2 - qQr)^2 + i(r-M)(\omega r^2 - qQr)}{\Delta}\\
&+iqQ - 2i\omega r - \lambda^2\Bigg]\xi_{-\frac{1}{2}}\\
&-\Bigg[\frac{ivaqQ}{r^3}\left(\alpha r^2 + (1-\alpha)Mr - Q^2\right)\\
&-\frac{m}{\lambda-mr}\Big((\alpha+\frac{1}{2})(r-M) - i\omega r^2\\
&+iqQr - \frac{ivaqQ}{2}f\Big) - m^2r^2\Bigg]\xi_{-\frac{1}{2}} = 0 \qquad (19)
\end{aligned}$$

$$\begin{aligned}
&\Delta\partial_r^2\xi_{+\frac{1}{2}} + \left(2(\alpha+1)(r-M) - ivaqQf - \frac{m\Delta}{\lambda+mr}\right)\partial_r\xi_{+\frac{1}{2}}\\
&+\Bigg[\frac{(\omega r^2 - qQr)^2 - i(r-M)(\omega r^2 - qQr)}{\Delta}\\
&-iqQ + 2i\omega r - \lambda^2 + 1\Bigg]\xi_{+\frac{1}{2}}\\
&-\Bigg[\frac{ivaqQ}{r^3}\left(\alpha r^2 + (1-\alpha)Mr - Q^2\right)\\
&+\frac{m}{\lambda+mr}\Big((\alpha+\frac{1}{2})(r-M)\\
&+i\omega r^2 - iqQr - \frac{ivaqQ}{2}f\Big) - m^2r^2\Bigg]\xi_{+\frac{1}{2}} = 0. \qquad (20)
\end{aligned}$$

The transformation $R_s = r^{-s}\Delta^{s/2}\xi_s$ has a generic form that enables both components R_1 and R_2 to be factorised at once. We want to achieve a similar thing with the system of two 2nd order differential equations (19) and (20). More precisely, we want to write a generic form of the equation that would unify both Eqs. (19) and (20) into a single equation valid for both components $\xi_{-\frac{1}{2}}$ and $\xi_{+\frac{1}{2}}$. This generalization is straightforward and is achieved by unification in the form

$$\begin{aligned}
&\Delta\partial_r^2\xi_s + \left(2(s+1)(r-M) - ivaqQf - 2s\frac{m\Delta}{\lambda_s + 2smr}\right)\partial_r\xi_s\\
&+\Bigg[\frac{(\omega r^2 - qQr)^2 - 2is(r-M)(\omega r^2 - qQr)}{\Delta}\\
&+4is\omega r - 2isqQ - \lambda_s^2\Bigg]\xi_s\\
&-\Bigg[\frac{ivaqQ}{r^3}\left(sr^2 + (1-s)Mr - Q^2\right)
\end{aligned}$$

$$+\frac{m}{\lambda_s+2smr}\Big(2s(s+\frac{1}{2})(r-M)+i\omega r^2-iqQr$$
$$-2s\frac{ivaqQ}{2}f\Big)-m^2r^2\Big]\xi_s=0. \tag{21}$$

Here we took the notation $\lambda \equiv \lambda_s$, where $s \in \{-1/2, +1/2\}$, as explained earlier, and used the identity $\lambda^2_{-\frac{1}{2}} = \lambda^2_{+\frac{1}{2}} + 1$.

This is the required equation that has the right commutative limit coinciding with the known results in the literature [83]. Moreover, from this equation the contribution of noncommutativity, as well as the field mass contribution can be precisely isolated. In order to proceed further in complete generality, one needs to find the appropriate tortoise coordinate, which again in the limit $a, m \to 0$ reduces to the known tortoise coordinate for RN. However, we shall not pursue the analysis further in its whole generality and in the rest of the paper will rather focus on massless, but noncommutative fermion perturbations. We will address the massive case in a separate study.

For the end of this section, note that we could arrive at the Eq. (21) directly from (10) by assuming the following ansatz from the very beginning,

$$\Psi = e^{i(\nu\varphi-\omega t)}\begin{pmatrix}\psi_1(r,\theta)\\ \psi_2(r,\theta)\end{pmatrix}$$
$$= e^{i(\nu\varphi-\omega t)}\begin{pmatrix} pr^{-1/2}\Delta^{1/4}\xi_{+\frac{1}{2}}(r)S_1(\theta)\\ -r^{-1/2}\Delta^{-1/4}\xi_{-\frac{1}{2}}(r)S_2(\theta)\\ r^{-1/2}\Delta^{-1/4}\xi_{-\frac{1}{2}}(r)S_1(\theta)\\ r^{-1/2}\Delta^{1/4}\xi_{+\frac{1}{2}}(r)S_2(\theta)\end{pmatrix}. \tag{22}$$

3 Tortoise coordinate and the boundary conditions

3.1 The tortoise coordinate and the effective potential

Let us focus on the case of vanishing field mass, $m = 0$. Then the tortoise coordinate y can be introduced in which the radial equation (21) takes the Schrödinger form

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}y^2} + V\chi = 0, \tag{23}$$

and which is defined by the following change of coordinates

$$\frac{\mathrm{d}y}{\mathrm{d}r} = \frac{1}{f\left(1+iav\frac{qQ}{r}\right)}. \tag{24}$$

Integrating (24) we find

$$y = y^{(0)} - iavqQ\left\{\frac{r_+}{r_+-r_-}\ln(r-r_+) - \frac{r_-}{r_+-r_-}\ln(r-r_-)\right\}$$
$$= r + \frac{r_+}{r_+-r_-}\Big(r_+ - iavqQ\Big)\ln(r-r_+)$$
$$-\frac{r_-}{r_+-r_-}\Big(r_- - iavqQ\Big)\ln(r-r_-), \tag{25}$$

where $y^{(0)}$ is the standard tortoise coordinate for the Reissner–Nordström metric given by

$$y^{(0)} \equiv r_*^{RN} = r + \frac{r_+^2}{r_+-r_-}\ln(r-r_+) - \frac{r_-^2}{r_+-r_-}\ln(r-r_-). \tag{26}$$

More precisely, for $m = 0$ the transformation $\chi_s(r) = \Delta^{s/2}r\xi_s(r)$, followed by the change of coordinate (24), brings the Eq. (21) into the form

$$\frac{\partial^2\chi_s}{\partial y^2} + \frac{\Delta}{r^4}\Big[\frac{2Q^2}{r^2} - \frac{2M}{r} - j(j+1)$$
$$+s^2 + \frac{\big(\omega r^2 - qQr - is(r-M)\big)^2}{\Delta}$$
$$+4is\omega r - 2isqQ + \frac{iavqQ\Delta}{r^3}$$
$$+isavqQ\frac{r-M}{r^2} - \frac{iavqQ}{r^3}\Big(sr^2 + (1-s)Mr - Q^2\Big)$$
$$+2iav\frac{qQ}{r}\Big(\frac{2Q^2}{r^2} - \frac{2M}{r} - j(j+1) + s^2\Big)$$
$$+2iav\frac{qQ}{r}\frac{\big(\omega r^2 - qQr - is(r-M)\big)^2}{\Delta}$$
$$-8sav\omega qQ + 4sav\frac{q^2Q^2}{r}\Big]\chi_s = 0. \tag{27}$$

It has the form of the Eq. (23), with the potential being given by

$$V = \frac{\Delta}{r^4}\Big[\frac{2Q^2}{r^2} - \frac{2M}{r} - j(j+1)$$
$$+s^2 + \frac{\big(\omega r^2 - qQr - is(r-M)\big)^2}{\Delta}$$
$$+4is\omega r - 2isqQ + \frac{iavqQ\Delta}{r^3}$$
$$+isavqQ\frac{r-M}{r^2} - \frac{iavqQ}{r^3}\Big(sr^2 + (1-s)Mr - Q^2\Big)$$
$$+2iav\frac{qQ}{r}\Big(\frac{2Q^2}{r^2} - \frac{2M}{r} - j(j+1) + s^2\Big)$$
$$+2iav\frac{qQ}{r}\frac{\big(\omega r^2 - qQr - is(r-M)\big)^2}{\Delta}$$
$$-8sav\omega qQ + 4sav\frac{q^2Q^2}{r}\Big]. \tag{28}$$

We point out that this equation is valid up to the first order in the deformation parameter a. Note also that (24) is the same change of coordinate that was used in [82] in order to accomplish the same task of retrieving the effective potential in the context of NC scalar perturbations.

3.2 Boundary conditions

In general, the boundary conditions impose the constraints on the shape of the solutions in the asymptotic regions of space. In order to deduce the appropriate boundary conditions for the QNM spectrum of the fermionic perturbations, we need to consider the limiting forms of the Eq. (27) in two separate asymptotic regions of space – the spatial infinity $r \to \infty$ and the region near the event horizon $r \to r_+$. The required limiting forms of the solution are obtained by solving Eq. (27) analytically in two stated regions and then imposing the QNM boundary condition of purely incoming waves on the horizon and purely outgoing waves at the infinity.

We first consider the $r \to \infty$ limit. The Eq. (27) in this limit reduces to

$$\frac{\mathrm{d}^2\chi_s}{\mathrm{d}y^2} + \left[\omega^2 - 2\frac{\omega q Q}{y} + 2is\frac{\omega}{y} + 2iav\frac{qQ}{y}\omega^2\right]\chi_s = 0. \tag{29}$$

Note that in this limit the tortoise coordinate does not actually differ from the radial coordinate. The solution to the Eq. (29) is given by

$$\xi_s = \frac{\chi_s}{r\Delta^{s/2}} \sim e^{\pm i\omega y} y^{-1-iqQ-2s-avqQ\omega}. \tag{30}$$

Likewise, in the near horizon limit $r \to r_+$, the Eq. (27) boils down to

$$\frac{\mathrm{d}^2\chi_s}{\mathrm{d}y^2} + \frac{1}{r_+^4}\left(\omega r_+^2 - qQr_+ - is(r_+ - M)\right)^2 \times\left(1 + 2iav\frac{qQ}{r_+}\right)\chi_s = 0, \tag{31}$$

with the solution

$$\xi_s = \frac{\chi_s}{r\Delta^{s/2}} \sim \frac{1}{(r-r_+)^{s/2}} e^{\pm i\left(\omega - \frac{qQ}{r_+} - is\frac{r_+-r_-}{2r_+^2}\right)\left(1+iav\frac{qQ}{r_+}\right)y}. \tag{32}$$

It should be stressed that the solutions (30) and (32) are perturbative in the NC parameter a and are valid only up to first order in a. The QNM boundary conditions, purely outgoing in the infinity and purely incoming at the horizon, select signs in (30) and (32). Consequently, the asymptotic form of the quasinormal mode solutions can be summarized as

$$\xi_s(r) \to \begin{cases} Z_{\text{out}} e^{i\omega y} y^{-1-iqQ-2s-avqQ\omega}, & \text{for } r \to \infty, \ (y \to \infty) \\ Z_{\text{in}} \frac{1}{(r-r_+)^{s/2}} e^{-i\left(\omega - \frac{qQ}{r_+} - is\frac{r_+-r_-}{2r_+^2}\right)\left(1+iav\frac{qQ}{r_+}\right)y}, & \text{for } r \to r_+, \ (y \to -\infty) \end{cases}. \tag{33}$$

Here Z_{out} and Z_{in} are the amplitudes of the outgoing and ingoing waves, respectively, and they do not depend on r (or y). In the special case of vanishing spacetime deformation $a = 0$, these asymptotic solutions reduce to the asymptotic solutions of [83].

Equation (21) has an irregular singularity at $r = +\infty$ and three regular singularities at $r = 0$, $r = r_-$ and $r = r_+$. In order to apply the Leaver's method of continued fractions, we expand the general solution of (21) in terms of powers series around $r = r_+$. Then the radial part of the spin 1/2 field takes the form

$$\xi_s(r) = e^{i\omega r}(r - r_-)^{\epsilon} \sum_{n=0}^{\infty} a_n \left(\frac{r - r_+}{r - r_-}\right)^{n+\delta}. \tag{34}$$

At this stage, the parameters δ and ϵ are still unknown. However, they can be fixed by demanding that the solution (34) satisfies the boundary conditions (33) at the horizon and at the infinity. From the general solution (34) to the Eq. (21), it is possible to infer its most prevailing behaviour in both critical regimes, that close to the event horizon, as well as that at far infinity. While the dominant behaviour of (34) in the regime $r \to \infty$ is governed by the term $r^{\epsilon}e^{i\omega r}$, its dominant behaviour in the regime $r \to r_+$ is determined by the term $(r - r_+)^{\delta}$. Moreover, an identification of the leading contributions to the general solution (34) may also be made by inserting the tortoise coordinate (25) into (30) and (32). As the limiting patterns of the solution (34), obtained in two different ways described above must obviously match each other, bringing them together directly fixes the parameters ϵ and δ:

$$\delta = -i\frac{r_+^2}{r_+ - r_-}\left(\omega - \frac{qQ}{r_+}\right) - s,$$
$$\epsilon = i\omega(r_+ + r_-) - 1 - 2s - iqQ. \tag{35}$$

It is worthy to note that these parameters are not affected by the spacetime noncommutativity. Furthermore, they coincide with the corresponding parameters obtained in the Ref. [84].

3.3 Recurrence relations and Nollert method

After inserting the power series solution (34) with the exact values of the parameters (35) into the Eq. (21), we obtain the recurrence relations for the coefficients a_n. While in commutative case we get 3-term recurrence relations, in the non-commutative case we get the 6-term recurrence relation

$$A_n a_{n+1} + B_n a_n + C_n a_{n-1} + D_n a_{n-2} + E_n a_{n-3} + F_n a_{n-4} = 0, \quad n \geqslant 4$$
$$A_3 a_4 + B_3 a_3 + C_3 a_2 + D_3 a_1 + E_3 a_0 = 0, \quad n = 3$$
$$A_2 a_3 + B_2 a_2 + C_2 a_1 + D_2 a_0 = 0, \quad n = 2$$

$$A_1 a_2 + B_1 a_1 + C_1 a_0 = 0, \quad n = 1$$
$$A_0 a_1 + B_0 a_0 = 0, \quad n = 0. \tag{36}$$

The coefficients A_n, B_n, C_n, D_n, E_n and F_n are given by

$$\begin{aligned}
A_n &= r_+^3 \alpha_n,\\
B_n &= r_+^3 \beta_n - 3r_+^2 r_- \alpha_{n-1}\\
&\quad - iavqQr_+\Big(\frac{r_+ - r_-}{2} + (n-s)(r_+ - r_-)\\
&\quad - ir_+(\omega r_+ - qQ) + (r_+ - r_-)\frac{s}{2}\Big),\\
C_n &= r_+^3 \gamma_n + 3r_+ r_-^2 \alpha_{n-2} - 3r_+^2 r_- \beta_{n-1}\\
&\quad + avqQ\omega r_+(r_+ - r_-)^3 + iavqQ(r_+ - r_-)\big((r_+ - r_-)^2\\
&\quad + \frac{r_-}{2}(r_+ - r_-)\big)\\
&\quad - iavqQ(r_+ - r_-)^2(-1 - 2s - iqQ + i\omega(r_+ + r_-))r_+\\
&\quad + iavqQ(r_+ - r_-)(2r_+ + r_-)\big((n-1-s)(r_+ - r_-)\\
&\quad - ir_+(\omega r_+ - qQ)\big) + iavqQsr_+(r_+ - r_-)(r_+ + 2r_-)\\
&\quad - iavqQs(r_+ - r_-)(2r_+ + r_-)\frac{r_+ + r_-}{2},\\
D_n &= -r_-^3 \alpha_{n-3} + 3r_+ r_-^2 \beta_{n-2} - 3r_+^2 r_- \gamma_{n-1}\\
&\quad + \frac{i}{2} avqQr_+(r_+ - r_-)^2 - iavqQ(r_+ - r_-)(2r_+ + r_-)\\
&\quad \times \big((n-2-s)(r_+ - r_-) - ir_+(\omega r_+ - qQ)\big)\\
&\quad + iavqQ(r_+ - r_-)^2(r_+ + r_-)\\
&\quad \times \big(-1 - 2s - iqQ + i\omega(r_+ + r_-)\big)\\
&\quad - iavqQ(r_+ - r_-)^3\\
&\quad - avqQr_-\omega(r_+ - r_-)^3\\
&\quad + \frac{i}{2} avqQs(r_+ - r_-)(r_+ + 2r_-)(r_+ + r_-)\\
&\quad - iavqQsr_-(r_+ - r_-)(2r_+ + r_-),\\
E_n &= 3r_+ r_-^2 \gamma_{n-2} - r_-^3 \beta_{n-3}\\
&\quad - iavqQ(r_+ - r_-)^2 \frac{r_-}{2}\\
&\quad - iavqQ(r_+ - r_-)^2\big(-1 - 2s - iqQ + i\omega(r_+ + r_-)\big)r_-\\
&\quad + iavqQ(r_+ - r_-)r_-\big((n-3-s)(r_+ - r_-)\\
&\quad - ir_+(\omega r_+ - qQ)\big)\\
&\quad - iavqQsr_-(r_+ - r_-)(2r_+ + r_-)\\
&\quad + \frac{i}{2} avqQs(r_+ - r_-)(r_+ + 2r_-)(r_+ + r_-),\\
F_n &= -r_-^3 \gamma_{n-3}.
\end{aligned} \tag{37}$$

The coefficients α_n, β_n, γ_n are

$$\begin{aligned}
\alpha_n &= -(n+1)\Big(r_-(n-s+1) + r_+\\
&\quad \times (-n + s - 1 - 2iqQ + 2ir_+\omega)\Big),
\end{aligned} \tag{38}$$

$$\begin{aligned}
\beta_n &= -r_+\Big(\lambda_s + 2n^2 - 4ir_+\omega(2n + 1 + 3iqQ)\\
&\quad + 6inqQ + 2n - 4(qQ)^2 + 3iqQ - 8r_+^2\omega^2 + s + 1\Big) +\\
&\quad + r_-\big(\lambda_s + 2n(n + 1 + iqQ) + iqQ + s + 1\big)\\
&\quad - 2i(2n+1)r_+ r_- \omega,
\end{aligned} \tag{39}$$

$$\begin{aligned}
\gamma_n &= -\Big(n + 2i\big(qQ - \omega(r_+ + r_-)\big)\Big)\\
&\quad \times \Big(n(r_- - r_+) + ir_+(-2qQ + 2r_+\omega + is) + r_- s\Big).
\end{aligned} \tag{40}$$

The first relation in (36), for $n \geqslant 4$, is a general 6-term recurrence relation. The remaining four relations are the indicial equations relating the lowest order coefficients a_n in the general expansion (34). They may be thought as boundary conditions for the first relation in (36). In contrast to the commutative case [83], where we have the 3-term recurrence relations, in the NC case we have these 6-term recurrence relations. Due to differing orders of recursion, comparison with the commutative case in [83] is nontrivial, the main point being that the $a \to 0$ limit should be taken not at the very end, but rather at some earlier step in the analysis. For more details we refer the reader to [82].

Having the recurrence relations that involve more than 3 expansion coefficients a_n, as we do have here, we cannot straightforwardly apply the usual method for solving the recurrence relations [85]. Instead, we should first use the Gaussian elimination method to gradually reduce the initial recurrence relation from the 6-term recurrence relation to a 3-term recurrence relation. The situation here resembles that described in reference [86], where finding the QNM spectrum related to massive scalar perturbations of higher-dimensional black holes (with dimensions higher than four) requires a repeated application of the Gaussian method of elimination prior to applying the Nollert's method [87]. In our case the Gaussian elimination method needs to be applied 3 times in a row. The details of implementing the Gaussian elimination procedure are presented in the Appendix, first in the most general case and then specialized to the situation considered here where a reduction from the 6-term recurrence relation to the 3-term recurrence relation is needed. The whole procedure was done in Wolfram Mathematica straightforwardly. The initial recurrence relation can ultimately be reduced to a three-term recurrence relation, expressed as

$$A_0^{(3)} a_1 + B_0^{(3)} a_0 = 0, \tag{41}$$
$$A_n^{(3)} a_{n+1} + B_n^{(3)} a_n + C_n^{(3)} a_{n-1} = 0 \quad \text{for } n \geq 1. \tag{42}$$

As explained in the appendix, the number in the superscript refers to a number of times that the Gaussian elimination procedure has been performed in order to yield the recurrence relation in question and the corresponding coefficients. Owing to the recursive nature of this procedure, a general analytic form for the coefficients $A_n^{(3)}$, $B_n^{(3)}$ and $C_n^{(3)}$ is unavailable. These coefficients are complex and must be computed numerically.

The convergence of the series $\sum_n a_n$ is essential for ensuring that the ansatz (34) exhibits the appropriate asymptotic behaviour. For convergence to hold, the coefficients $A_n^{(3)}$, $B_n^{(3)}$ and $C_n^{(3)}$ must satisfy the following continued fraction condition [17]

$$B_0^{(3)} = \frac{A_0^{(3)} C_1^{(3)}}{B_1^{(3)} - \frac{A_1^{(3)} C_2^{(3)}}{B_2^{(3)} - \dots}} = \frac{A_0^{(3)} C_1^{(3)}}{B_1^{(3)} -} \frac{A_1^{(3)} C_2^{(3)}}{B_2^{(3)} -} \frac{A_2^{(3)} C_3^{(3)}}{B_3^{(3)} -} \dots \tag{43}$$

The QNM frequencies are determined by finding the roots of the continued fraction (43). However, as the continued fraction represents an infinite series, it must be truncated to a finite number of terms to facilitate the numerical evaluation. In our numerical computations, we employed approximately $N \sim 200$ terms, depending on the required precision (up to six decimal places or more). As a validation measure, we confirmed that increasing the number of terms consistently improves the accuracy of the computed QNMs.

To enhance accuracy, we apply Nollert's improvement method to estimate the contribution from the truncated tail of the series [87]. The Nollert method involves recognizing that the quantity $R_N = -a_{N+1}/a_N$ satisfies the recursion relation

$$R_N = \frac{C_{N+1}^{(3)}}{B_{N+1}^{(3)} - A_{N+1}^{(3)} R_{N+1}}, \tag{44}$$

and accurately represents the remainder of the continued fraction (43) truncated at order N. This remainder R_N is subsequently expanded in terms of inverse powers of $N^{1/2}$ as follows[4]

$$R_N = \sum_{k=0}^{\infty} C_k N^{-k/2}. \tag{45}$$

Substituting this series expansion into Eq. (44) allows one to determine the coefficients C_k. Since analytic expressions for $A_{N+1}^{(3)}$, $B_{N+1}^{(3)}$ and $C_{N+1}^{(3)}$ are not feasible (they are evaluated numerically using the Gaussian elimination procedure), we rely on the commutative expressions for C_k provided in [83].[5]

Using described methods and corresponding inputs, we employ a numerical root-finding algorithm to calculate the noncommutative QNM frequencies, whose results are presented in the subsequent section. However, before we proceed with applying these robust numerical algorithms in a most general situation, let us consider a possibility of getting an analytical solution for the spectrum, at least within some restricted range of the system parameters.

[4] The constants C_k appearing in the expansion (45) should not be confused with the coefficients C_n defined in (37).

[5] The Nollert method enhances the convergence of the continued fraction equation. Since the commutative tail approximation suffices for our analysis, we adopt this approach in our study.

3.4 Analytic solution in the large qQ limit

As it is widely known, the analytic solutions across different types of black hole QNM spectra are very rare and the problem of Dirac type of black hole perturbations in the noncommutative setting considered here is not the exception to this general rule. Despite that, in some cases of black hole perturbations, certain asymptotic limits may allow for a solution in a closed analytic form. In the following we show that this may also happen for the situation considered in this paper, where the fermionic matter perturbs the RN black hole under conditions of prevailing quantum spacetime structure.

That this might indeed be the case, we observe that in the limit of large qQ the 3−term recurrence relation (42) allows for the analytic solution. This conclusion may be reached by following the same line of reasoning outlined in Ref. [88], where a massive charged scalar field in the Kerr-Newman background and the corresponding QNM spectrum in the large qQ limit were considered.

As in [88], in our case the coefficients $\alpha_n, \beta_n, \gamma_n$ pertaining to the 3−term recurrence relations

$$\alpha_0 a_1 + \beta_0 a_0 = 0, \tag{46}$$

$$\alpha_n a_{n+1} + \beta_n a_n + \gamma_n a_{n-1} = 0 \quad \text{for } n \geq 1, \tag{47}$$

that hold in the absence of noncommutativity ($a = 0$) have the same asymptotic form for $qQ \gg 1$: $\alpha_n = \mathcal{O}(qQ)$, $\beta_n = \mathcal{O}(q^2Q^2)$ and $\gamma_n = \mathcal{O}(q^2Q^2)$. The frequency ω must be of the order $\omega = \mathcal{O}(qQ)$. In that case we have that

$$\frac{\beta_n}{(qQ)^2} + \mathcal{O}\left(\frac{1}{qQ}\right) = 0. \tag{48}$$

Furthermore, it should be noted that after three consecutive Gaussian eliminations the coefficient $B_n^{(3)}$ will have the following form in the limit of $qQ \gg 1$:

$$\begin{aligned} B_n^{(3)} &= iqQ(2n+1)(r_- - 3r_+) + 4r_+(qQ)^2 - 8r_+^3\omega^2 \\ &\quad - 12r_+^2\omega qQ + 2ir_+\omega(2r_+ - r_-)(2n+1) \\ &\quad - \frac{ia\nu}{r_+^2}((r_+ - r_-)(2n+1-s) - ir_+(\omega r_+ - qQ)). \end{aligned} \tag{49}$$

If we further assume that $\omega = \omega^{(0)} + a\omega^{(1)}$ and $\omega^{(i)} = W^{(i)}qQ + Y^{(i)}$, for some $W^{(i)}, Y^{(i)}$ independent of qQ and insert this ansatz into the Eq. (48), we obtain

$$\omega = \frac{qQ}{r_+} - i\frac{2n+1}{4r_+^2}(r_+ - r_-) + \frac{ia\nu(2n+1-2s)}{16r_+^4}(r_+ - r_-)^2. \tag{50}$$

We see that the real part of the frequency does not have any noncommutative correction in the limit of $qQ \gg 1$, while

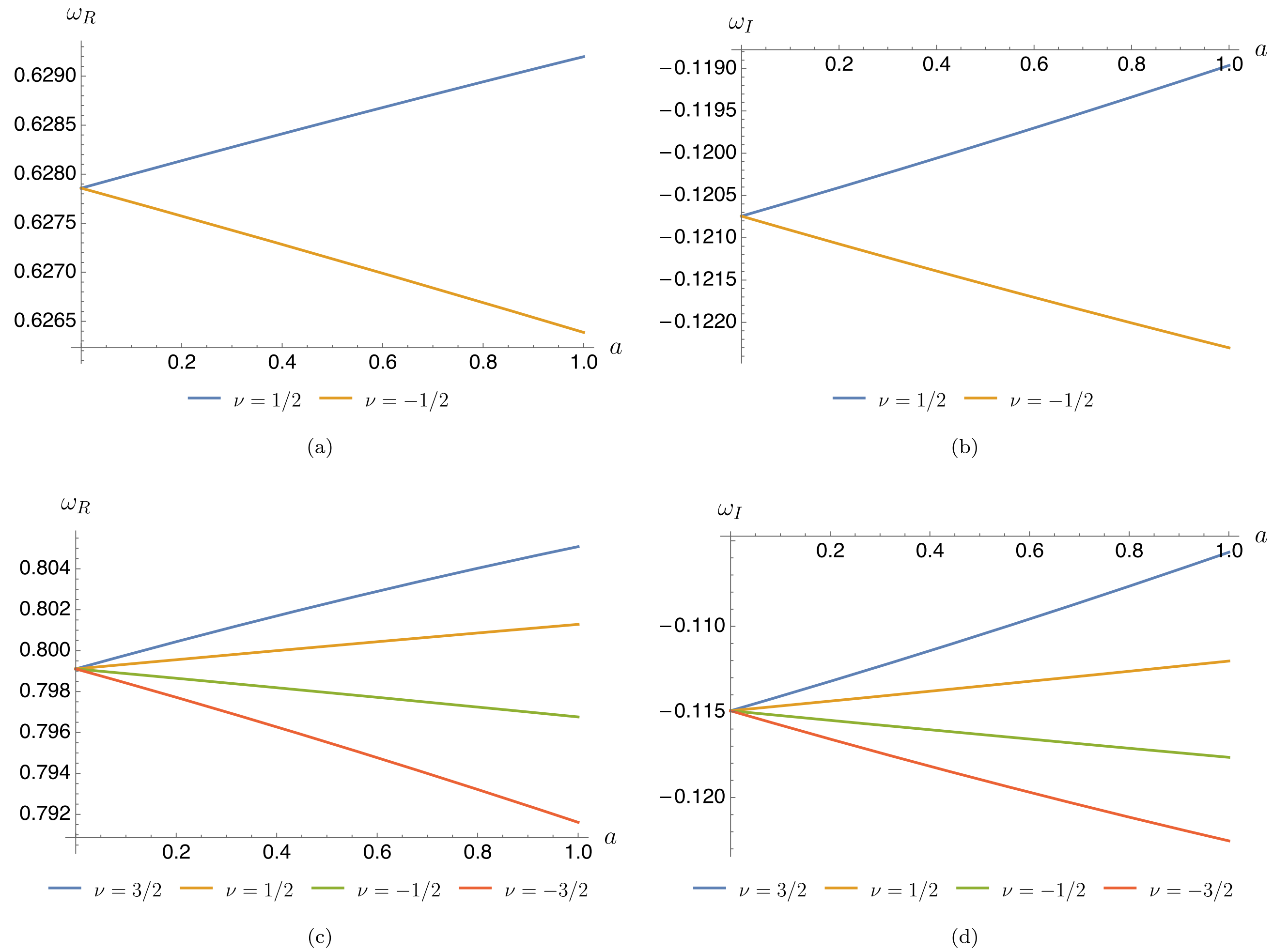


Fig. 1 Dependence of the fermionic QNM frequencies on the NC parameter a for $j = 1/2$ and $s = 1/2$. Subfigure **a** displays the splitting in the real part of QNM frequencies, while subfigure **b** illustrates the corresponding splitting in the imaginary part for various values of the magnetic quantum number ν. Subfigures **c** and **d** display the spectrum for $j = 3/2$ and $s = 1/2$. The parameters are set to $Q = 0.5$, $qQ = 1$, and $M = 1$

the imaginary part has a constant correction, which depends on the magnetic quantum number ν, implying the Zeeman-like splitting of QNM spectrum in the large qQ domain. This result in the commutative limit ($a = 0$) is also in accordance with the result obtained in [88], after the limit of vanishing black hole angular momentum is taken (i.e. for non-rotating black hole).

4 Quasinormal mode spectrum

The resulting spectrum displays interesting deviations from the commutative one. Figure 1 displays the dependence of the fundamental mode QNM frequency on the NC parameter a. Linear deviation from the commutative frequency supports the validity of continuous fraction method for obtaining the spectrum in this perturbative approach. This characteristic Zeeman-like splitting is similar to the scalar field case considered in [82] and spin-2 field analysed in [89–91].[6] It should be noted that the expected value of a is at the order of Planck Length, which is much lower than 1 in units where the black hole mass is equal to 1. Therefore the range of values on the horizontal axis of Fig. 1 is selected for illustration purpose.

In Fig. 2 the QMM spectrum is plotted as a function of qQ, while keeping Q fixed. The impact of noncommutativity on the imaginary part of the frequency (damping part) appears to be greater than the impact on the real part, especially for higher values of qQ. However, when one takes into account the scales on the y-axis, the differences are comparable in absolute value. A similar graph for the scalar field can be found in [82]. The trend is very similar, although the spectrum

[6] There it should be compared to $r - \varphi$ noncommutativity as the authors do not consider the $t - \varphi$ noncommutativity studied here.

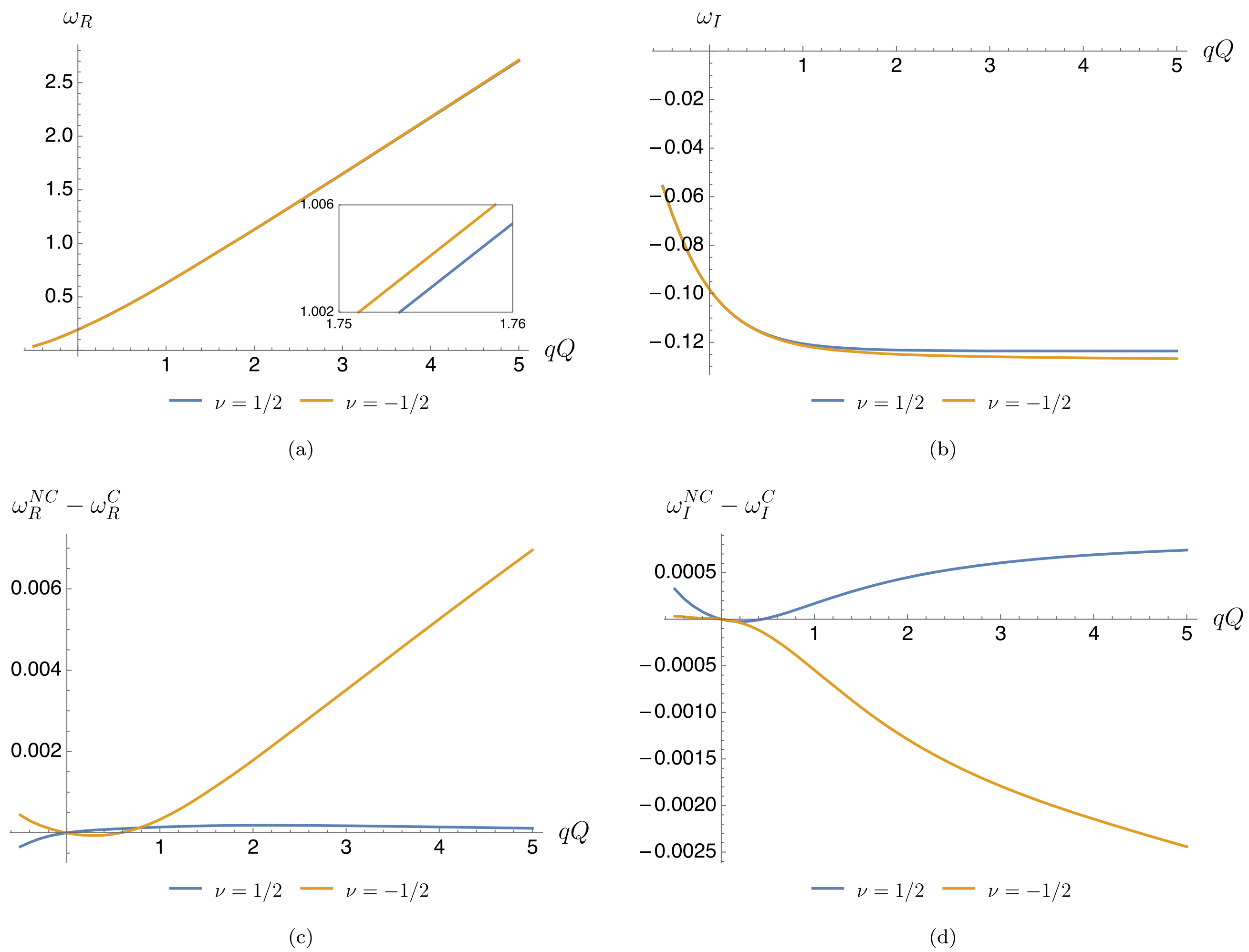


Fig. 2 Dependence of the fermionic QNMs ($j = 1/2$, $s = 1/2$) on the parameter qQ. Subfigures **a** and **b** respectively illustrate the splitting in the real (highlighted in the inset) and imaginary parts of the QNM frequencies for different magnetic quantum numbers ν. Subfigures **c** and **d** display the difference between NC and commutative QNM frequencies, $\omega^{NC} - \omega^{C}$, as functions of qQ for the real and imaginary parts, respectively. Parameters used are $Q = 0.5$, $a = 0.1$, and $M = 1$

of the scalar field seems to be affected more by spacetime noncommutativity.

In Subfigures (c) and (d) of Fig. 2, it is visible that the deviation of $\nu = -1/2$ mode from the commutative value is much larger than that of $\nu = 1/2$ mode – especially in the real part. While this behaviour looks interesting, the qualitative difference between the QNM frequencies for $\nu = \pm 1/2$ (which does not exist in commutative case [83]) could be explained by preferred direction of φ in the twist (1). Another possible source of asymmetry could be that the two components of the chirality display different behaviour and our analysis of the spectrum was concerned only with the $s = +1/2$ component of the spinor.

In Fig. 3 the fermionic QNM spectrum of the fundamental mode is plotted for $j = 3/2$ and accordingly four values of ν. In Fig. 3b one can observe an almost uniform splitting of imaginary parts of the frequencies. Even though the additional component in the modified Reissner–Nordström metric (3) cannot be related to rotation, this kind of splitting of imaginary part of frequencies temptingly points to such interpretation. Indeed, as imaginary parts represent damping, we may infer from the figures that the waves with negative value of ν are damped more strongly than the ones with positive ν. This is reminiscent of a way how frame dragging advances the waves orbiting in the same direction as the black hole rotates and damps the ones rotating in the opposite direction [17,92]. Figure 3c and d show the noncommutative deviations in the real and imaginary parts of the frequencies, respectively.

Figure 4 similarly shows the qQ dependence of the QNM frequencies for several values of Q. The most interesting subfigure is Fig. 4d, where one can see how much noncommutativity affects the spectrum depending on the value of qQ for fixed Q-values. As mentioned previously, the com-

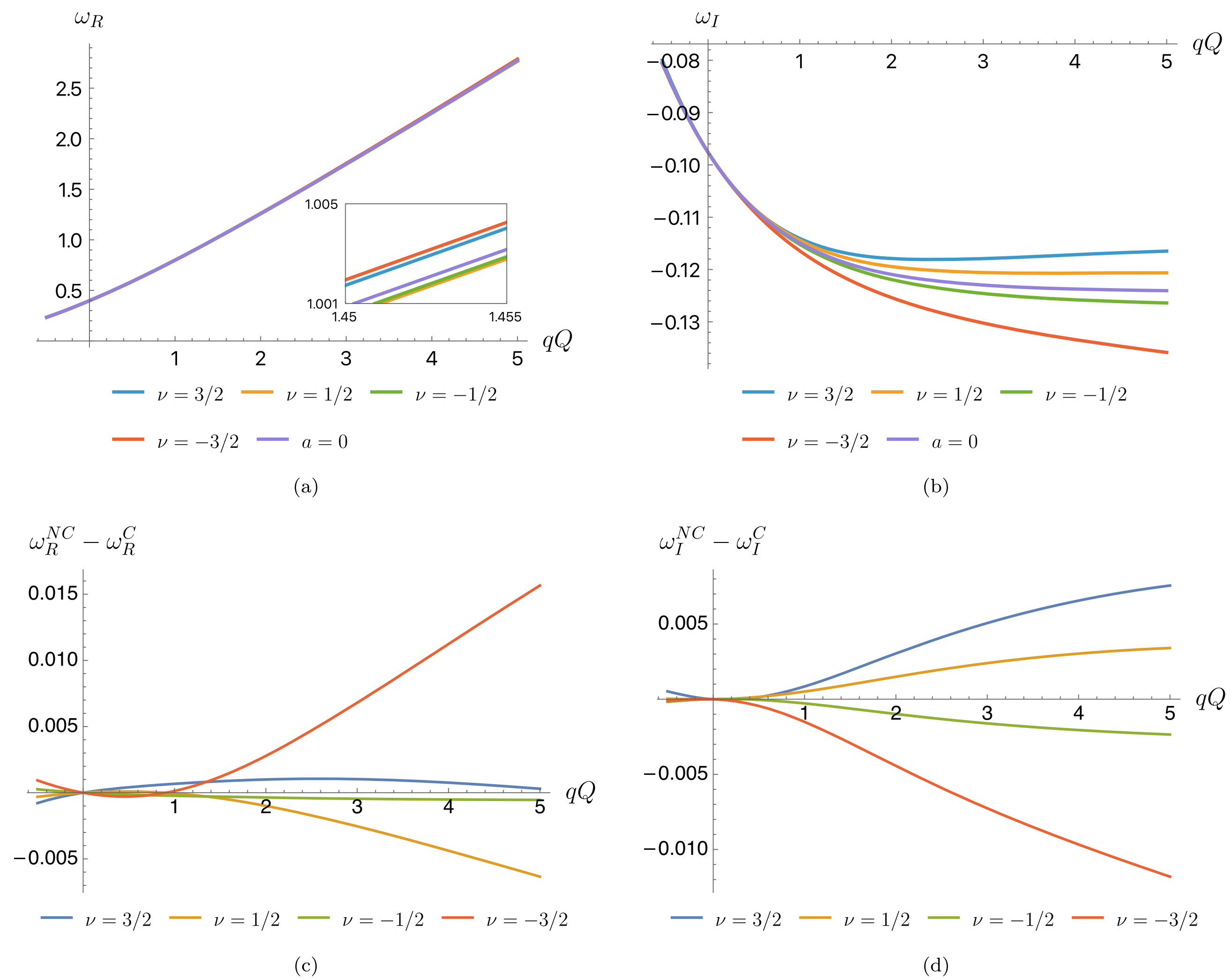


Fig. 3 Dependence of fermionic QNMs ($j = 3/2$, $s = 1/2$) on qQ. Subfigures **a** and **b** respectively illustrate the splitting in the real (highlighted in the inset) and imaginary parts of QNM frequencies for various magnetic quantum numbers ν, along with the commutative value $a = 0$. Subfigures **c** and **d** depict the differences between NC and commutative QNM frequencies, $(\omega^{NC} - \omega^C)$, as functions of qQ for the real and imaginary parts, respectively. Parameters chosen are $Q = 0.5$, $a = 0.1$, and $M = 1$

mutative spectrum does not depend on ν, so plotting for ν from $-3/2$ to $3/2$ provides an insight into the magnitude of the NC correction to the spectrum.

From Fig. 4b one may observe a phase transition-like appearance in the behaviour of the ω_I plot occurring at around $Q/M \sim 0.9$. This threshold is not significantly affected by noncommutativity [83]. For the scalar NC case, the phase transition occurs at $Q/M \sim 0.7$ [82].

Figure 5 shows the profile of the fundamental QNM in the complex plane, where the frequency plot is parametrized by Q/M. The impact of noncommutativity grows with increasing Q/M. The characteristic shape of the graph is not significantly affected by noncommutativity. Note that at values of Q/M at around 0.96 the frequencies start to diverge more rapidly.

The QNM frequencies in the $a = 0$ case (commutative limit) presented in this section agree with the values obtained in [83]. Our analysis breaks down at higher values of the black hole charge Q. The exact threshold of Q where this occurs depends additionally on parameters such as the field charge q and the noncommutative parameter a. Nevertheless, for very small values of a, our method remains robust for arbitrary choices of other parameters, except in the near-extremal limit $Q/M \to 1$. Since the magnitude of a is expected to be on the order of the Planck length, the regime of validity for our analysis essentially mirrors that of the commutative case. As in the commutative scenario, the extremal limit necessitates a separate, dedicated treatment.

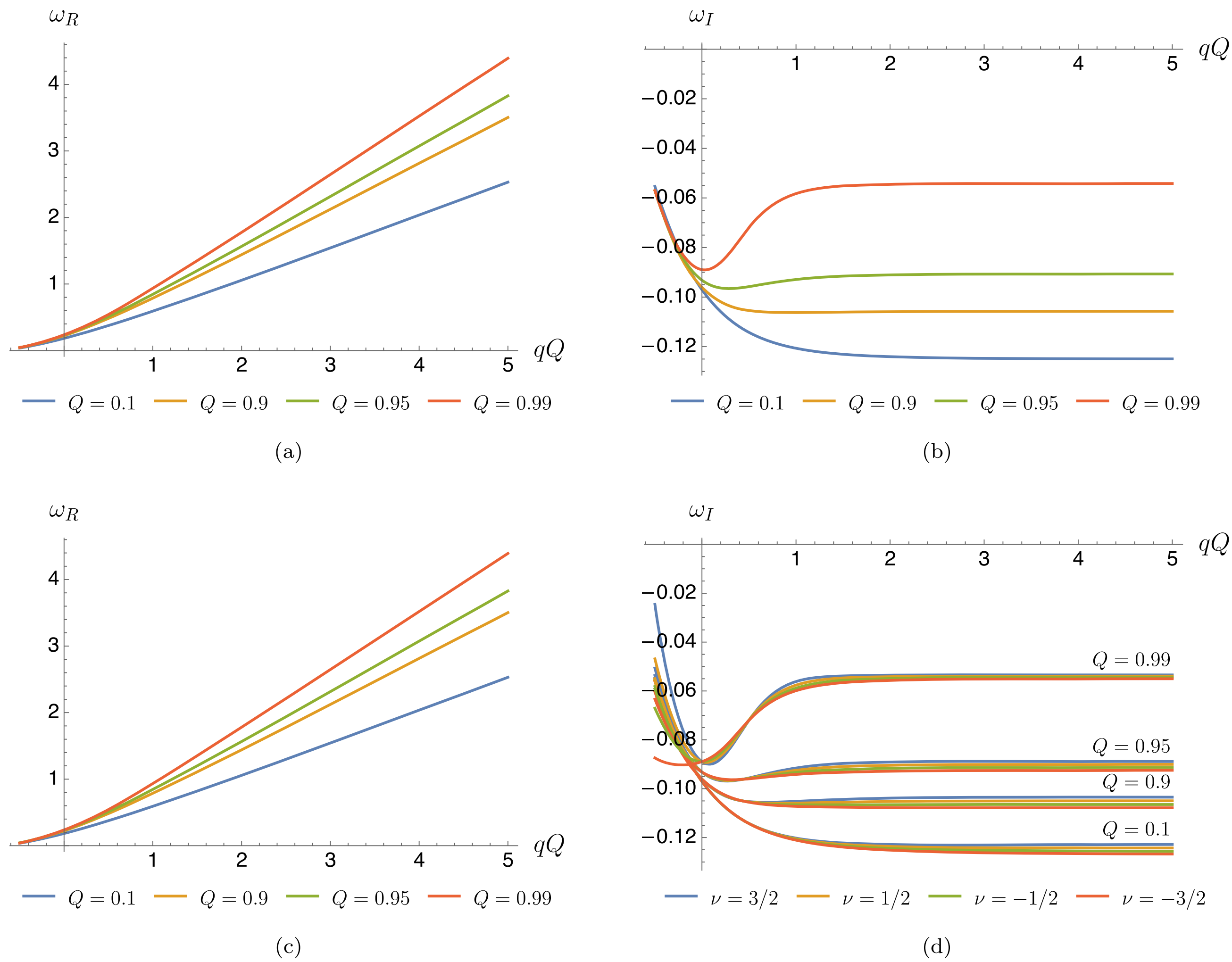


Fig. 4 Variation of fermionic QNMs ($j = 3/2$, $s = 1/2$) against the parameter qQ for different values of Q. Subfigures **a** and **b** respectively illustrate the real and imaginary parts of the QNM frequencies in the commutative case. Subfigures **c** and **d** represent the NC QNM frequencies (with $a = 0.1$) plotted for $j = 3/2, s = 1/2$ and $\nu \in \{-3/2, -1/2, 1/2, 3/2\}$. The splitting in ν is not visible in the **c** subfigure, but is clearly present in **d**. For both commutative and NC cases, we set $M = 1$

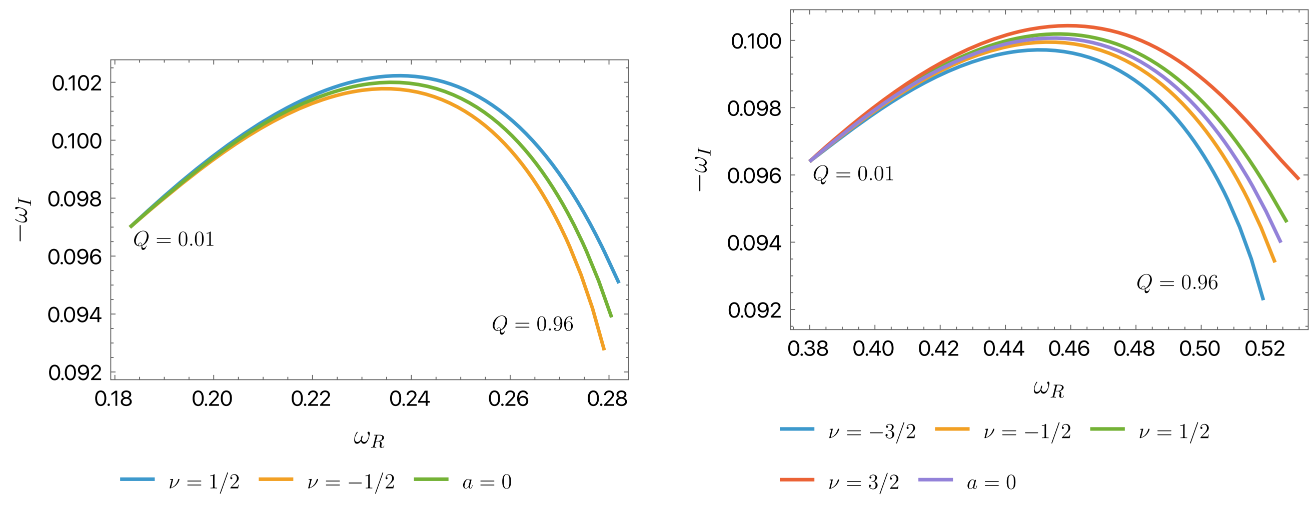
5 Final remarks

By the progress in experimental astrophysics and discovery of gravitational waves, black hole QNMs became one of the most important tools for testing predictions of general relativity in hitherto unexamined conditions, as well as for gaining insights into the characteristic parameters of massive compact objects [93]. The latter may be mainly achieved through the study of quasinormal mode spectra of black holes and other exotic compact objects.

Studying the black hole perturbations by different types of fields gives rise to the details in the spectrum that are able to uncover features like instabilities, bound states, zero modes or characteristic resonances for which perturbations of these field configurations tend to be radiated away and decay. The spectrum may also tell us a lot about the properties of the spacetime itself. In particular, if the structure of the spacetime is not in line with the usual concept of smooth continuum, then presumably this should be also reflected in the spectrum.

In order to emulate conditions like this, in this paper we have used a specific model in which a massless charged Dirac field probes a modified Reissner–Nordström geometry. This physical model was previously shown to be equivalent to the noncommutative gauge field theory where the NC Dirac field probes classical Reissner–Nordström background [80].

Features like these show that the total impact of the spacetime deformation may, at the level of effective description, be squeezed into a modification of geometry, which manifests in the appearance of an additional nonvanishing component in the original metric. This results with modified geometry of Reissner–Nordström type with the emergence of an additional $r - \varphi$ component in the metric. We then took advantage



(a) (b)

Fig. 5 Real and imaginary parts of the fermionic QNMs for different magnetic quantum numbers ν and $a = 0$ with Q varying from 0.01 to 0.96. Subfigure **a** corresponds to the case $(j = 1/2, s = 1/2)$, and Subfigure **b** corresponds to $(j = 3/2, s = 1/2)$. For both cases, we set parameters as $q = 0.1$, $a = 1$, and $M = 1$

of the stated correspondence in order to analyse the modifications in the fermion perturbation spectrum that appear due to spacetime deformation.

In brief, the present work discusses the perturbations of massless charged Dirac particles and the fermion QNMs generated by these perturbations in the vicinity of a quantum-deformed Reissner–Nordström black hole. In order to accomplish this task, we have adopted the continued fraction method extended to include the Gaussian elimination procedure. The characteristic pattern of the fundamental mode has been analysed in dependence of different black hole parameters, such as charge, mass and the NC deformation parameter. Moreover, changes in the spectrum were analysed for different extremal black hole characteristic ratios, i.e. for different mass over charge ratios and for different charges of the probing fermion field.

The feature that is the most striking one and that is at the same time the only genuinely intrinsic to noncommutative nature of space is the Zeeman-like splitting in the spectrum. The latter occurs with varying a projection of the total angular momentum. Another noteworthy feature is the dependence of the imaginary (damping) part of the frequency on ν. Even though the new nonzero component in the modified RN metric cannot be directly linked to a rotation, the pattern of how the imaginary part of the frequency depends on ν gives a spectral analogue of the rotation-induced dynamics, thus providing some arguments in support of the rotational correspondence.

Another interesting point regarding the model of Dirac perturbations of black holes in the NC setting considered here is that it allows for the analytic solution in the large qQ limit, so that the QNM frequencies may be expressed in a closed form. The result shows that in the stated limit the real part of the frequencies is unaffected by the NC deformation, while the imaginary part acquires quantum correction which scales linearly with the magnetic quantum number (the projection of the total angular momentum). The latter also reveals the Zeeman-like splitting in the spectrum, in accordance with the general and more robust analysis.

The study carried out in this work leaves ample room for several additional directions of research. For instance, the question of how spacetime deformation influences the QNM spectrum for massive fermion perturbations remains unanswered. In general, it is known that the perturbations of the massive Dirac field give rise to the real part of the QNM frequency that increases with the mass. Likewise, the imaginary part which corresponds to damping decreases, implying that the QNM frequencies due to perturbations of black hole background with more massive Dirac fields are more likely to be detected [94]. This is due to the fields with higher masses decaying more slowly. How the deformed structure of spacetime would influence these certainties still remains to be seen. Finally, it is noteworthy that Dirac fields have been extensively studied in a number of black hole models and gravitational theories and in a variety of physical processes that occur near the black hole horizon, such as the scattering and absorption processes for Dirac particles [95], the spectral power emission of Dirac fermions [96,97], including the study of superradiance [49] and gray-body factors [98–101].

A clear picture of the influence of the quantum structure of spacetime on majority of these features and processes is still lacking as the current investigation has only just began to scratch the surface of a fundamental mechanisms that lie behind them. Some of these issues, as seen from the perspective of quantum gravity, particularly massive Dirac perturbations and gray-body factors corresponding to fermion particles, will be addressed in the upcoming work.

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Data Availability Statement This manuscript has no associated data. [Author's comment: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.]

Code Availability Statement Code/software will be made available on reasonable request. [Author's comment: The code/software generated during and/or analysed during the current study is available from the corresponding author on reasonable request.]

Appendix A: Gaussian elimination procedure

In this Appendix we describe the general Gaussian elimination procedure and then specifically its variant that has been applied in Sect. 3.3 in order to reduce the initial 6-term recurrence relation into a required 3-term form.

Therefore, once the appropriate higher-term recurrence relation is identified, it has to be reduced to an appropriate 3-term form by successive and systematic application of the Gaussian method of elimination. This 3-term recurrent relation can then be solved by applying the continued fraction method.

If we start with the general p-term recurrence relation,

$$A^{(0)}_{1,n}a_{n+1}+A^{(0)}_{2,n}a_n+A^{(0)}_{3,n}a_{n-1}+\cdots+A^{(0)}_{p-1,n}a_{n-(p-2)+1} +A^{(0)}_{p,n}a_{n-(p-2)}=0, \tag{A1}$$

which involves the coefficients $A^{(0)}_{i,n}$ at the level 0 and perform k Gaussian eliminations, $k \le p-3$, in k steps, one after another, the result will be a recurrence relation with $p-k$ terms, involving coefficients $A^{(k)}_{i,n}$ at the level k,

$$A^{(k)}_{1,n}a_{n+1}+A^{(k)}_{2,n}a_n+\cdots+A^{(k)}_{p-k-1,n}a_{n-(p-k-2)+1} +A^{(k)}_{p-k,n}a_{n-(p-k-2)}=0. \tag{A2}$$

The last two expressions may be written in a more compact way,

$$\sum_{i=1}^{p}A^{(0)}_{i,n}a_{n+2-i}=0,\quad \sum_{i=1}^{p-k}A^{(k)}_{i,n}a_{n+2-i}=0. \tag{A3}$$

After the first Gaussian elimination, we obtain the recurrence relation with $p-1$ terms, involving $p-1$ coefficients $A^{(1)}_{i,n}$ at the level 1. These two sets of coefficients $\{A^{(0)}_{i,n},\ i=1,\ldots,p\}$, and $\{A^{(1)}_{i,n},\ i=1,\ldots,p-1\}$, are related by the following connecting relations:

$$\begin{aligned}
A^{(1)}_{1,n}&=A^{(0)}_{1,n},\quad n\ge p-2\\
A^{(1)}_{2,n}&=A^{(0)}_{2,n}-\frac{A^{(0)}_{p,n}}{A^{(1)}_{p-1,n-1}}A^{(1)}_{1,n-1},\quad n\ge p-2\\
A^{(1)}_{i,n}&=A^{(0)}_{i,n}-\frac{A^{(0)}_{p,n}}{A^{(1)}_{p-1,n-1}}A^{(1)}_{i-1,n-1},\quad n\ge p-2\\
A^{(1)}_{p-1,n}&=A^{(0)}_{p-1,n}-\frac{A^{(0)}_{p,n}}{A^{(1)}_{p-1,n-1}}A^{(1)}_{p-2,n-1},\quad n\ge p-2\\
A^{(1)}_{i,n}&=A^{(0)}_{i,n},\quad n\le p-3.
\end{aligned}$$

In a similar way, after k Gaussian eliminations, the recurrence relation (A2) with $p-k$ coefficients $A^{(k)}_{i,n}$ at the level k is obtained. The coefficients are given by:

$$\begin{aligned}
A^{(k)}_{1,n}&=A^{(k-1)}_{1,n}=A^{(0)}_{1,n},\quad n\ge p-2-(k-1)\\
A^{(k)}_{2,n}&=A^{(k-1)}_{2,n}-\frac{A^{(k-1)}_{p-(k-1),n}}{A^{(k)}_{p-k,n-1}}A^{(k)}_{1,n-1},\quad n\ge p-2-(k-1)\\
A^{(k)}_{p-i,n}&=A^{(k-1)}_{p-i,n}-\frac{A^{(k-1)}_{p-(k-1),n}}{A^{(k)}_{p-k,n-1}}A^{(k)}_{p-i-1,n-1},\quad n\ge p-2-(k-1)\\
A^{(k)}_{p-k,n}&=A^{(k-1)}_{p-k,n}-\frac{A^{(k-1)}_{p-(k-1),n}}{A^{(k)}_{p-k,n-1}}A^{(k)}_{p-k-1,n-1},\quad n\ge p-2-(k-1)\\
A^{(k)}_{i,n}&=A^{(k-1)}_{i,n}=A^{(0)}_{i,n}\quad n\le p-2-k.
\end{aligned} \tag{A4}$$

Relations (A4) provide a link between the coefficients in the recurrence relations pertaining to two consecutive steps in the Gaussian elimination procedure. Using the terminology established above, they relate the recurrence coefficients at two consecutive levels.

Next, we specialize this general procedure to the case considered in this paper. For that purpose, we explain the required steps in the Gaussian elimination procedure as applied to our particular model, where we need to reduce the 6-term recurrence relation (36) to a 3-term recurrence relation.

To begin with, we define the coefficients of the zeroth level $A_n^{(0)}$, $B_n^{(0)}$, $C_n^{(0)}$, $D_n^{(0)}$, $E_n^{(0)}$, $F_n^{(0)}$ to be the coefficients of the initial 6-term recurrence relation (36),

$$A_n^{(0)} \equiv A_n, \; B_n^{(0)} \equiv B_n, \; C_n^{(0)} \equiv C_n,$$
$$D_n^{(0)} \equiv D_n, \; E_n^{(0)} \equiv E_n, \; F_n^{(0)} \equiv F_n, \tag{A5}$$

with $A_n, B_n, C_n, D_n, E_n, F_n$ defined in (37). Similarly, we introduce the coefficients of the j-th level $A_n^{(j)}$, $B_n^{(j)}$, $C_n^{(j)}$, $D_n^{(j)}$, $E_n^{(j)}$, $j = 1, 2, 3$ as the coefficients that appear in the $(6-j)$-term recurrence relation, obtained after the j-th Gaussian elimination.

Applying the first Gaussian elimination to (36) we find the 5-term recurrence relation

$$A_n^{(1)} a_{n+1} + B_n^{(1)} a_n + C_n^{(1)} a_{n-1} + D_n^{(1)} a_{n-2} + E_n^{(1)} a_{n-3} = 0,$$
$$A_2^{(1)} a_3 + B_2^{(1)} a_2 + C_2^{(1)} a_1 + D_2^{(1)} a_0 = 0,$$
$$A_1^{(1)} a_2 + B_1^{(1)} a_1 + C_1^{(1)} a_0 = 0,$$
$$A_0^{(1)} a_1 + B_0^{(1)} a_0 = 0. \tag{A6}$$

The coefficients of the first level are determined as

$$A_n^{(1)} = A_n^{(0)},$$
$$B_n^{(1)} = B_n^{(0)} - \frac{F_n^{(0)}}{E_{n-1}^{(1)}} A_{n-1}^{(1)}, \quad C_n^{(1)} = C_n^{(0)} - \frac{F_n^{(0)}}{E_{n-1}^{(1)}} B_{n-1}^{(1)},$$
$$D_n^{(1)} = D_n^{(0)} - \frac{F_n^{(0)}}{E_{n-1}^{(1)}} C_{n-1}^{(1)}, \quad E_n^{(1)} = E_n^{(0)} - \frac{F_n^{(0)}}{E_{n-1}^{(1)}} D_{n-1}^{(1)}, \tag{A7}$$

for $n \geq 4$, and for $n = 3, 2, 1, 0$, we have

$$A_3^{(1)} = A_3^{(0)}, \quad B_3^{(1)} = B_3^{(0)}, \quad C_3^{(1)} = C_3^{(0)},$$
$$D_3^{(1)} = D_3^{(0)}, \quad E_3^{(1)} = E_3^{(0)},$$
$$A_2^{(1)} = A_2^{(0)}, \quad B_2^{(1)} = B_2^{(0)}, \quad C_2^{(1)} = C_2^{(0)}, \quad D_2^{(1)} = D_2^{(0)},$$
$$A_1^{(1)} = A_1^{(0)}, \quad B_1^{(1)} = B_1^{(0)}, \quad C_1^{(1)} = C_1^{(0)},$$
$$A_0^{(1)} = A_0^{(0)}, \quad B_0^{(1)} = B_0^{(0)}. \tag{A8}$$

The application of the second Gaussian elimination to the recurrence relation (A6) yields the following 4-term recurrence equation

$$A_n^{(2)} a_{n+1} + B_n^{(2)} a_n + C_n^{(2)} a_{n-1} + D_n^{(2)} a_{n-2} = 0,$$
$$A_1^{(2)} a_2 + B_1^{(2)} a_1 + C_1^{(2)} a_0 = 0,$$
$$A_0^{(2)} a_1 + B_0^{(2)} a_0 = 0. \tag{A9}$$

The coefficients of the second level are determined as

$$A_n^{(2)} = A_n^{(1)} = A_n^{(0)}, \qquad B_n^{(2)} = B_n^{(1)} - \frac{E_n^{(1)}}{D_{n-1}^{(2)}} A_{n-1}^{(2)},$$
$$C_n^{(2)} = C_n^{(1)} - \frac{E_n^{(1)}}{D_{n-1}^{(2)}} B_{n-1}^{(2)}, \quad D_n^{(2)} = D_n^{(1)} - \frac{E_n^{(1)}}{D_{n-1}^{(2)}} C_{n-1}^{(2)}, \tag{A10}$$

for $n \geq 3$, and

$$A_2^{(2)} = A_2^{(1)} = A_2^{(0)}, \quad B_2^{(2)} = B_2^{(1)} = B_2^{(0)}, \quad C_2^{(2)} = C_2^{(1)} = C_2^{(0)},$$
$$D_2^{(2)} = D_2^{(1)} = D_2^{(0)},$$
$$A_1^{(2)} = A_1^{(1)} = A_1^{(0)}, \quad B_1^{(2)} = B_1^{(1)} = B_1^{(0)}, \quad C_1^{(2)} = C_1^{(1)} = C_1^{(0)},$$
$$A_0^{(2)} = A_0^{(1)} = A_0^{(0)}, \quad B_0^{(2)} = B_0^{(1)} = B_0^{(0)}, \tag{A11}$$

for $n = 0, 1, 2$. The third and the last Gauss elimination applied to (A9) leads to the recurrence relation

$$A_n^{(3)} a_{n+1} + B_n^{(3)} a_n + C_n^{(3)} a_{n-1} = 0,$$
$$A_0^{(3)} a_1 + B_0^{(3)} a_0 = 0. \tag{A12}$$

The coefficients of the third level, $A_n^{(3)}$, $B_n^{(3)}$, $C_n^{(3)}$, that we have been searching for, are given by

$$A_n^{(3)} = A_n^{(2)} = A_n^{(0)},$$
$$B_n^{(3)} = B_n^{(2)} - \frac{D_n^{(2)}}{C_{n-1}^{(3)}} A_{n-1}^{(3)}, \quad C_n^{(3)} = C_n^{(2)} - \frac{D_n^{(2)}}{C_{n-1}^{(3)}} B_{n-1}^{(3)}, \tag{A13}$$

for $n \geq 2$ and for $n = 0, 1$, we have

$$A_1^{(3)} = A_1^{(2)} = A_1^{(0)}, \quad B_1^{(3)} = B_1^{(2)} = B_1^{(0)}, \quad C_1^{(3)} = C_1^{(2)} = C_1^{(0)},$$
$$A_0^{(3)} = A_0^{(2)} = A_0^{(0)}, \quad B_0^{(3)} = B_0^{(2)} = B_0^{(0)}. \tag{A14}$$

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