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Analog dual to a 2 + 1-dimensional holographic superconductor

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Abstract

We study an analog hydrodynamic model that mimics a 3 + 1 AdS planar BH spacetime dual to a 2 + 1-dimensional superconductor. We demonstrate that the AdS₄ bulk and its holographic dual could be realized in nature in an analog gravity model based on fluid dynamics. In particular we mimic the metric of an O₂ holographic superconductor and calculate the entanglement entropy of a conveniently designed subsystem at the boundary of the analog AdS₄ bulk.

Keywords: analog gravity, superconductor, holographic dual

(Some figures may appear in colour only in the online journal)

1. Introduction

A pseudo-Riemannian geometry of spacetime can be mimicked by fluid dynamics in Minkowski spacetime. The basic idea is the emergence of an effective metric

$$G_{\mu\nu} = a[g_{\mu\nu} - (1 - c_s^2)u_\mu u_\nu], \quad (1)$$

which describes the effective geometry for acoustic perturbations propagating in a fluid potential flow with $u_\mu \propto \partial_\mu \theta$. The quantity c_s is the adiabatic speed of sound, the conformal factor a is related to the equation of state of the fluid, and the background spacetime metric $g_{\mu\nu}$ is usually assumed Minkowski. The metric of the form (1) has been exploited in various contexts including emergent gravity [1, 2], scalar theory of gravity [3], Einstein-aether gravity [4], acoustic geometry [5–8] and Euclidean gravity [9–11].

The work presented here is motivated by recent development of anti-de Sitter/conformal field theory (AdS/CFT) dual theory of 2 + 1-dimensional superconductor [12–21] (for a review

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and additional references see [22]). The AdS/CFT duality in these models is based on a correspondence between gravitational theory and dynamics of quantum field theory on the boundary of asymptotically AdS spacetime. The gravity side can be well described by classical general relativity, while the dual field theory involves the dynamics with strong interaction. This correspondence is often referred to as ‘holography’ since a higher dimensional gravity system is described by a lower dimensional field theory without gravity, which resembles optical holography.

A particularly important work in this context is the minimal model of a holographic superconductor by Bobev *et al* [19] with an abelian gauge field embedded in the truncation of four-dimensional maximal gauged super-gravity. Besides, it is worth mentioning the work on d -wave superconductivity by Benini *et al* [16] in which interesting physical phenomena are demonstrated such as the formation of Fermi arcs.

The AdS₄ spacetime as a solution to Einstein’s equations cannot actually exist in nature due to instability problems. However, it can inspire some configurations where the underlying general gravitational structure can be studied through analogue models. The aim of this paper is to demonstrate that AdS₄ and its holographic dual could be realized in nature in an analog gravity model based on hydrodynamics of a physical fluid. In particular we will mimic the bulk metric of the minimal model of a holographic superconductor consisting of the metric, a charged scalar with a non-trivial potential and an abelian gauge field embedded in the truncation of four-dimensional maximal gauged super-gravity [19]. This model was recently studied in the context of holographic entanglement entropy [20, 21]. The entanglement entropy is an important tool for keeping track of symmetry breaking and phase transition in strong coupling systems. In the context of black-hole thermodynamics the entropy of a black hole is proportional to the area of the horizon in the same way as is the entanglement entropy proportional to the boundary area of between two subsystems of a quantum system.

Our first task is to derive an analog acoustic geometry which mimics a $d + 1$ -dimensional asymptotic AdS geometry with a general planar black hole (BH). Furthermore, we will apply this to a $3 + 1$ -dimensional model and calculate the entanglement entropy for a particular geometry obtained as solution related to the holographic O_2 superconductor. The reason why we are specifically interested in the O_2 type is due to its pronounced first order phase transition at finite temperature.

It is important to stress that analog gravity in general is concerned by curved geometry *per se* without referring to sources of the gravitational field as in general relativity. More specifically, the fluid analog mimics the geometry only and says nothing about the source such as matter and other fields. There are no equations analog to Einstein’s which, as in general relativity, would involve curvature tensor and stress tensor. However, even without Einstein’s equations, the analog BH horizon entropy is realized via quantum entanglement of phonons. This is why the model studied in this paper and analog gravity in general can teach us something about BHs and related phenomena.

We divide the remainder of the paper into three sections and an two appendices. We start with section 2 in which we derive an analog metric for a $d + 1$ -dimensional AdS planar BH hole of the form relevant for a holographic description of the superconductor. In the next section, section 3, we apply our formalism to a $3 + 1$ -dimensional bulk related to the minimal model of the $2 + 1$ -dimensional holographic superconductor. For a particular geometry related to the O_2 superconductor we calculate the entanglement entropy. Concluding remarks are given in section 4. In appendix A we outline a derivation of the relativistic acoustic metric and in appendix B we derive the effective speed of sound in a fluid with an external pressure.

2. Analog planar black hole

Geometric structures in the form of a planar BH may have interesting applications in condensed matter physics [23]. In this section we construct a model of an analog planar BH hole in a general asymptotic AdS_{*d*+1}. A similar model for *d* = 4 was discussed in detail by Hossenfelder [24, 25] and recently in [26, 27]. We will discuss in more detail the case *d* = 3 which is of particular interest for 2 + 1-dimensional superconductor [19, 20, 22]. In our approach we will consider a nonisotropic fluid flow which yields the desired analog metric.

We start from a general form of the AdS planar BH metric in an arbitrary number of space-like dimensions *d*

$$ds^2 = \frac{\ell^2}{z^2} [e^{-\chi(z)}\gamma(z)dt^2 - \gamma(z)^{-1}dz^2 - d\mathbf{x}^2], \quad (2)$$

where ℓ is the curvature radius of AdS_{*d*+1} and

$$d\mathbf{x}^2 = \sum_{i=1}^{d-1} dx^i dx^i. \quad (3)$$

For *d* = 3 we will relate the functions χ and γ to the truncated Lagrangian of the four-dimensional $\mathcal{N} = 8$ super-gravity [28] studied by Bobev *et al* [19] in the context of holographic superconductivity. In order to have an asymptotic AdS for $z \rightarrow 0$ we can always rescale the time coordinate so that, without loss of generality, we may assume

$$\gamma(0) = 1, \quad \chi(0) = 0. \quad (4)$$

Next, the dimensionless functions χ and γ can be thought of as functions of the dimensionless variable z/z_h , where $z = z_h$ is the location of the horizon. In other words

$$\gamma(z_h) = 0 \quad (5)$$

and γ has no zeros on the interval $0 < z < z_h$. Then, the horizon temperature is

$$T = \frac{e^{-\chi/2}}{4\pi} \left. \frac{d\gamma}{dz} \right|_{z=z_h}. \quad (6)$$

This temperature measured in some chosen fixed units, e.g. in units of ℓ^{-1} is ambiguous because the geometry (2) is invariant under rescaling

$$\tau \rightarrow \alpha\tau, \quad z \rightarrow \alpha z, \quad x^i \rightarrow \alpha x^i, \quad z_h \rightarrow \alpha z_h. \quad (7)$$

Thus, the metric (2) has a rescaled horizon z_h/α with the corresponding rescaled horizon temperature

$$\bar{T} = \frac{e^{-\chi/2}}{4\pi} \left. \frac{d\gamma(\alpha z)}{dz} \right|_{z=z_h/\alpha} = \alpha T. \quad (8)$$

However, the temperature T expressed in units of $1/z_h$ is unique, i.e. the quantity Tz_h is invariant under the rescaling (7). Therefore, in the following we will express the temperature and other dimensionfull physical quantities in units of some power of z_h .

Now we seek a fluid analog model which would mimic the induced metric of the form (2). The basic idea is to find a suitable coordinate transformation $t \rightarrow \bar{t}$, $z \rightarrow \bar{z}$ such that the new

metric takes the form of the relativistic acoustic metric (92) derived in appendix A with $g_{\mu\nu}$ replaced by the Minkowski metric $\eta_{\mu\nu}$

$$G_{\mu\nu} = \frac{n}{m^2 c_s w} [\eta_{\mu\nu} - (1 - c_s^2) u_\mu u_\nu]. \quad (9)$$

Here n and w denote the particle number density and specific enthalpy, respectively, and an arbitrary mass scale m is introduced to make $G_{\mu\nu}$ dimensionless. The specific enthalpy is defined as usual

$$w = \frac{p + \rho}{n}, \quad (10)$$

where p and ρ denote the pressure and energy density, respectively. The quantity c_s is the so-called ‘adiabatic’ speed of sound defined by

$$c_s^2 \equiv \left. \frac{\partial p}{\partial \rho} \right|_s = \frac{n}{w} \left(\left. \frac{\partial n}{\partial w} \right|_s \right)^{-1}, \quad (11)$$

where $|_s$ denotes that the specific entropy, i.e. entropy per particle $s = S/N$, is kept fixed. The second equality in (11) follows from the thermodynamic law

$$dw = T ds + \frac{1}{n} dp. \quad (12)$$

Following Hossenfelder [24] we transform the metric (2) by making use of a coordinate transformation

$$t = \bar{t} + h(z), \quad z = z(\bar{z}), \quad (13)$$

where the functions $z(\bar{z})$ and $h(z)$ are determined by the requirement that the transformed metric takes the form (9). By simple algebraic manipulations the line element (2) can be recast into a convenient form

$$ds^2 = \frac{\ell^2}{z^2} \left\{ d\bar{t}^2 - d\bar{z}^2 - d\mathbf{x}^2 - (1 - \tilde{\gamma}) d\bar{t}^2 + 2(1 - \tilde{\gamma})^{1/2} (c_s^2 - \tilde{\gamma})^{1/2} d\bar{t} d\bar{z} - (c_s^2 - \tilde{\gamma}) d\bar{z}^2 \right\}, \quad (14)$$

where we have set

$$\frac{dz}{d\bar{z}} = e^{\chi/2} c_s, \quad (15)$$

$$\frac{dh}{dz} = \frac{(1 - \tilde{\gamma})^{1/2} (c_s^2 - \tilde{\gamma})^{1/2}}{c_s e^{\chi/2} \tilde{\gamma}}, \quad (16)$$

and an abbreviation

$$\tilde{\gamma} = e^{-\chi} \gamma. \quad (17)$$

From (4) and (5) it follows

$$0 \leq \tilde{\gamma} \leq 1; \quad \tilde{\gamma}(z_h) = 0, \quad \tilde{\gamma}(0) = 1. \quad (18)$$

Comparing (14) with the acoustic metric (9) we identify c_s as the speed of sound and the non-vanishing components of the velocity vector $u_{\bar{t}}$ and $u_{\bar{z}}$ in transformed coordinates as

$$u_{\bar{t}} = \frac{(1 - \tilde{\gamma})^{1/2}}{(1 - c_s^2)^{1/2}}, \quad u_{\bar{z}} = -\frac{(c_s^2 - \tilde{\gamma})^{1/2}}{(1 - c_s^2)^{1/2}}. \quad (19)$$

These equations imply

$$\tilde{\gamma} \leq c_s^2 \leq 1. \quad (20)$$

Next, by applying the potential-flow equation (see appendix A)

$$wu_{\mu} = \partial_{\mu}\theta \quad (21)$$

we derive closed expressions for w , n , and c_s in terms of the variable z . Since the metric is stationary, the velocity potential must be of the form

$$\theta = m\bar{t} + g(z), \quad (22)$$

where m is an arbitrary mass parameter which we can identify with the mass scale that appears in (9) and $g(z)$ is a function of \bar{z} through z . Then, from (21) and (22) it follows

$$w = \frac{m}{u_{\bar{t}}} = m \frac{(1 - c_s^2)^{1/2}}{(1 - \tilde{\gamma})^{1/2}}, \quad (23)$$

and the function g in (22) must satisfy

$$\frac{dg}{dz} = wu_{\bar{z}} \left(\frac{dz}{d\bar{z}} \right)^{-1} = -\frac{m}{c_s} \frac{(c_s^2 - \tilde{\gamma})^{1/2}}{e^{x/2} (1 - \tilde{\gamma})^{1/2}}. \quad (24)$$

The particle number density can be obtained from the condition that the conformal factor in (2) must be equal to that of (9), i.e. we require

$$\frac{n}{m^2 c_s w} = \frac{\ell^2}{z^2}. \quad (25)$$

As m is arbitrary it is natural to choose

$$m = \frac{1}{\ell}, \quad (26)$$

so using this and (23) we find

$$n = \frac{c_s}{\ell z^2} \frac{(1 - c_s^2)^{1/2}}{(1 - \tilde{\gamma})^{1/2}}. \quad (27)$$

In this way, both w and n are expressed as functions of z and c_s . However, c_s is not independent since by the definition (11)

$$c_s^2 = \frac{n}{w} \frac{\partial w}{\partial n} \Big|_s = \frac{n}{w} \frac{dw}{dz} \left(\frac{dn}{dz} \right)^{-1}. \quad (28)$$

Using (28) with (23) and (27) we obtain a differential equation for c_s

$$2c_s \frac{dc_s}{dz} - c_s^2 \left[\frac{2}{z} + \frac{1}{2} \frac{d}{dz} \ln(1 - \tilde{\gamma}) \right] + \frac{1}{2} \frac{d}{dz} \ln(1 - \tilde{\gamma}) = 0, \quad (29)$$

with solution

$$c_s^2 = 1 - \frac{z^2}{z_h^2} (1 - \tilde{\gamma})^{1/2} \left(K + 2z_h^2 \int_z^{z_h} \frac{dz}{z^3} (1 - \tilde{\gamma})^{-1/2} \right). \tag{30}$$

The integration constant must satisfy the constraint $1 \geq K \geq 0$ as a consequence of the condition $0 \leq c_s^2 \leq 1$. Dimensionless physical quantities such as ℓw , c_s and the components of the fluid velocity field are functions of z/z_h and are invariant under the rescaling (7).

Plugging (30) into (23) and (25) one obtains w and n as functions of z . Note that explicit functional forms of $z(\bar{z})$, $\gamma(z)$, and $g(z)$ can be obtained by making use of (30) and integrating respectively (15), (16), and (24). However, the precise forms of these functions are not really needed for obtaining a closed expression for the analog metric.

It is of particular interest to discuss the above solution in the asymptotic limit, i.e. in the limit $z \rightarrow 0$. Motivated by the asymptotic behavior of the O_2 holographic superconductor with $d = 3$ (see section 3.1)

$$\tilde{\gamma}(z) = 1 + c_3 \left(\frac{z}{z_h} \right)^3 + \mathcal{O}(z^{3+1}), \tag{31}$$

in the following we assume for general d

$$\tilde{\gamma}(z) = 1 + c_d \left(\frac{z}{z_h} \right)^d + \mathcal{O}(z^{d+1}), \tag{32}$$

with $c_d < 0$. Then, it may be easily shown that in the limit $z \rightarrow 0$ the sound speed squared tends to a constant $c_s^2 \rightarrow d/(d + 4) < 1$. However, in this limit $\tilde{\gamma} \rightarrow 1$ so from equation (19) it follows that the limit $z \rightarrow 0$ cannot be reached since we must have $c_s^2 \geq \tilde{\gamma}$. This puts the constraint as to how close to the boundary is our analog metric applicable. Our analog model breaks down at a point $z = z_{\min}$ which is the maximal root of the equation $c_s^2 = \tilde{\gamma}$. For the minimal value of K , $K_{\min} = 0$, this equation reads

$$(1 - \tilde{\gamma}(z))^{1/2} - 2z^2 \int_z^{z_h} \frac{dy}{y^3} (1 - \tilde{\gamma}(y))^{-1/2} = 0. \tag{33}$$

In the case of a Schwarzschild AdS planar BH, i.e. for $\chi = 0$ and $\gamma = 1 - (z/z_h)^d$, the integration in (29) can be easily performed yielding

$$c_s^2 = \frac{d}{d + 4} + \left(\frac{4}{d + 4} - K \right) \left(\frac{z}{z_h} \right)^{d/2+2}, \tag{34}$$

The condition $c_s^2 - \gamma = 0$ now reads

$$\left(\frac{z}{z_h} \right)^d + \left(\frac{4}{d + 4} - K \right) \left(\frac{z}{z_h} \right)^{d/2+2} - \frac{4}{d + 4} = 0, \tag{35}$$

For example, for $d = 4$, the root z_{\min} is given by

$$\frac{z_{\min}}{z_h} = (3 - 2K)^{-1/4} \geq 3^{-1/4}, \tag{36}$$

and for $d = 3$ we find numerically

$$\frac{z_{\min}}{z_h} = 0.727, \quad \text{for } K = K_{\min} = 0. \tag{37}$$

Hence, the simple prescription for an analog model is only valid from the point z_{\min} up to the location of the horizon at z_h . In principle we could place the boundary of our model at z_{\min} and cut off the section of AdS from $z = 0$ to z_{\min} as it has been done in the Randall–Sundrum model [29, 30]. However, as we aim to make a connection with CFT at the boundary of AdS and calculate the boundary entanglement entropy at $z = 0$, we would like to extend our model all the way down to the AdS boundary at $z = 0$. As we demonstrate in appendix B, such an extension can be achieved by manipulating the equation of state by adding an external pressure. For a fluid with an external pressure of the form

$$p_{\text{ext}} = \alpha(p + \rho), \tag{38}$$

where α is a function of z , one finds the effective speed of sound

$$\tilde{c}_s^2 = \frac{c_s^2 - \alpha}{1 + \alpha}. \tag{39}$$

Depending on the functional form of $\tilde{\gamma}$ we can choose α to make the quantity \tilde{c}_s^2 satisfy equation (20) in the interval $0 \leq z \leq z_h$. For example, if $\tilde{\gamma}$ behaves as in (32) near $z = 0$, we can choose

$$\alpha = \frac{d - (d + 4)\tilde{\gamma}}{(d + 4)(1 + \tilde{\gamma})} \tag{40}$$

to obtain $c_s^2 \geq \tilde{\gamma}$ in the entire interval $0 \leq z \leq z_h$ and

$$\lim_{z \rightarrow 0} \tilde{c}_s^2 = 1. \tag{41}$$

3. Analog bulk for the holographic superconductor

Here we consider a concrete example of the analog metric of the form (2) for $d = 3$ related to the holographic superconductor. Instead of solving the field equations we will implement the already known solutions [19–21] into our analogue setup. Based on the known results we will construct approximate analytic expressions for γ corresponding to a chosen horizon temperature. With this we can calculate the entanglement entropy and by comparison with the results of references [20, 21] we can also find an analytic expression for $\chi(z)$. The analog geometry which we have derived in general form can be used to mimic these analytic expressions.

3.1. Holographic superconductor

Here we briefly review the minimal model of a holographic superconductor following Bobev *et al* [19]. We consider the minimal model of a holographic superconductor realized by an $SO(3) \times SO(3)$ invariant truncation of four-dimensional $\mathcal{N} = 8$ gauged super-gravity [28]. The truncated action is

$$S = \frac{1}{16\pi G_4} \int d^4x \sqrt{-G} (-\mathcal{R} + \mathcal{L}), \tag{42}$$

where \mathcal{L} involves two real dimensionless scalar fields λ and φ coupled to an abelian gauge field A_μ and gravity. The Lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2\partial_\mu \lambda \partial^\mu \lambda + \frac{\sinh^2(2\lambda)}{2} \left(\partial_\mu \varphi - \frac{g}{2} A_\mu \right) \left(\partial^\mu \varphi - \frac{g}{2} A^\mu \right) - \mathcal{P}, \tag{43}$$

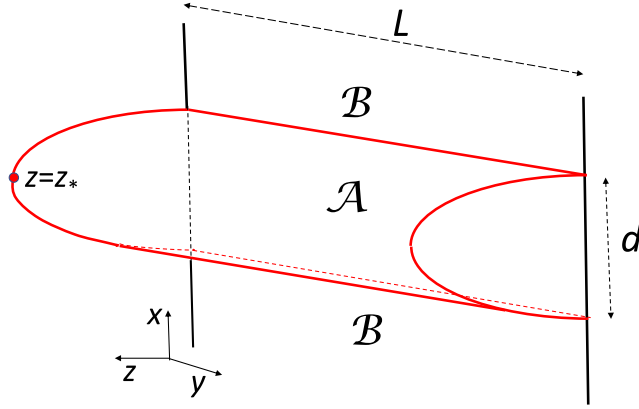


Figure 1. Strip geometry employed to calculate the entanglement entropy.

with potential

$$\mathcal{P} = -g^2 \left(6 \cosh^4 \lambda - 8 \cosh^2 \lambda \sinh^2 \lambda + \frac{3}{2} \sinh^4 \lambda \right). \quad (44)$$

The gauge coupling g sets the scale ℓ of AdS_4 via the relation $\mathcal{P}\ell^2 = -6$ [18] with scalar potential evaluated at a critical point. For the critical point $\lambda = 0$ related to $SO(8)$ global symmetry [19, 28] we obtain the relation $g^2\ell^2 = 1$. The spacetime metric can be parameterized as

$$ds^2 = \frac{\ell^2}{z^2} \left[\gamma(z) e^{-\chi(z)} dt^2 - (dx_1^2 + dx_2^2) - \frac{dz^2}{\gamma(z)} \right], \quad (45)$$

where the functions γ and χ are to be determined by solving the field equations with appropriate boundary conditions. As we have noted in section 2, the value $\chi_0 \equiv \chi(0)$ can be set to zero by rescaling the time coordinate.

The field equations are derived in reference [19] for the gauge choice $\varphi = 0$ and $A_\mu = (\psi(z), 0, 0, 0)$ and solved for two types of superconductors depending on the choice of boundary conditions, with non-trivial gauge fields and scalar condensates below some critical value of the temperature. The solutions are characterized by the vacuum expectation values of the charged operators O_1 and O_2 (see figures 1 and 2 in reference [19]). Depending on the asymptotic behavior of the field λ we distinguish two solutions:

- (a) $\lambda = \lambda_1 \tilde{z} + \mathcal{O}(\tilde{z}^3)$ corresponding to an O_1 superconductor with $O_1 \propto \lambda_1$ and $O_2 = 0$, and
- (b) $\lambda = \lambda_2 \tilde{z}^2 + \mathcal{O}(\tilde{z}^4)$ corresponding to an O_2 superconductor with $O_2 \propto \lambda_2$ and $O_1 = 0$.

Here and from here on we use the dimensionless variable $\tilde{z} = z/\ell$. As functions of temperature, the condensates O_1 and O_2 exhibit the second and first order phase transitions, respectively. The typical behavior of the condensates as functions of temperature is shown in figures 1 and 2 of reference [19]. The quantity ρ_c which was chosen to set the units in these figures appears as a coefficient in the expansion $\psi = \mu\ell - \rho_c \ell z + \dots$ near the AdS boundary. Physically, μ and ρ_c are appropriately normalized chemical potential and charge density, respectively. From the field equations one can derive the following asymptotic expansions near

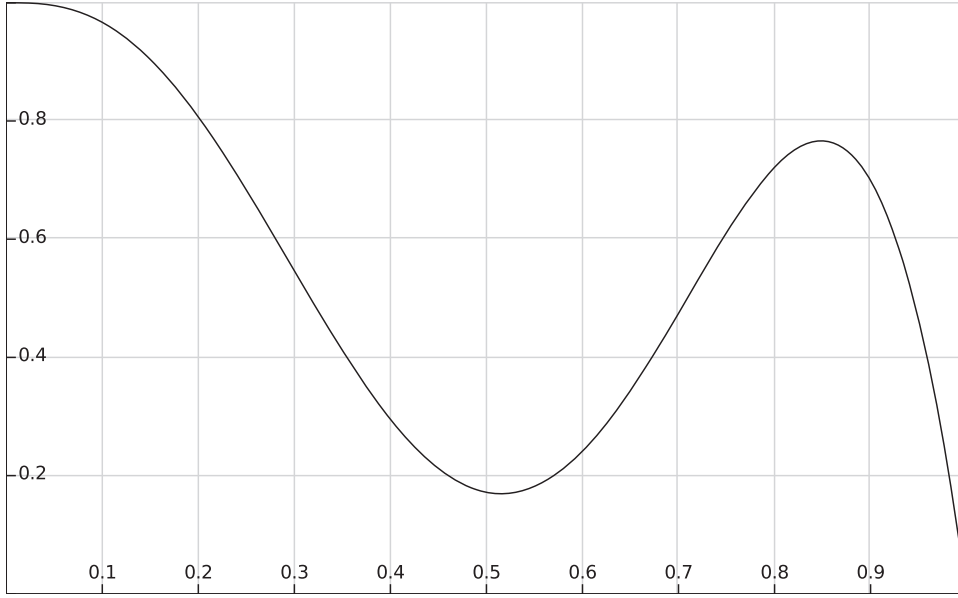


Figure 2. The metric function γ versus z/z_h for $T = 0.61 \times 10^{-2} \sqrt{\rho_c}$.

$z = 0$:

$$\lambda = \lambda_1 \tilde{z} + \lambda_2 \tilde{z}^2 + \frac{\lambda_1}{24} (2\lambda_1^2 - 3 e^{\chi_0} \psi_0^2) \tilde{z}^3 + \mathcal{O}(\tilde{z}^4), \quad (46)$$

$$\psi = \psi_0 + \psi_1 \tilde{z} + \frac{\psi_0}{2} \lambda_1^2 \tilde{z}^2 + \frac{\psi_0}{3} \lambda_1 \lambda_2 \tilde{z}^3 + \mathcal{O}(\tilde{z}^4), \quad (47)$$

$$\gamma = 1 + \lambda_1^2 \tilde{z}^2 + \gamma_3 \tilde{z}^3 + \mathcal{O}(\tilde{z}^4), \quad (48)$$

$$\chi = \chi_0 + \lambda_1^2 \tilde{z}^2 + \frac{8}{3} \lambda_1 \lambda_2 \tilde{z}^3 + \frac{1}{4} (\lambda_1^4 + 8\lambda_2^2 - e^{\chi_0} \lambda_1^2 \psi_0^2) \tilde{z}^4 + \mathcal{O}(\tilde{z}^5). \quad (49)$$

As we have mentioned, χ_0 can be set to 0 and the other coefficients in the expansion are related to physical quantities as follows:

$$\lambda_1 = 4\ell O_1, \quad \lambda_2 = 4\ell^2 O_2, \quad (50)$$

$$\psi_0 = \ell\mu, \quad \psi_1 = -\ell\rho_c. \quad (51)$$

For $\lambda = \chi = 0$ there are no condensates and the solution is just the Reissner–Nordstrom (RN) AdS₄ planar BH with

$$\gamma_{\text{RN}} = 1 - (1 + Q^2) \frac{z^3}{z_{\text{RN}}^3} + Q^2 \frac{z^4}{z_{\text{RN}}^3} \quad (52)$$

and

$$\psi_{\text{RN}} = \frac{2Q\ell}{z_{\text{RN}}} \left(1 - \frac{z}{z_{\text{RN}}} \right). \quad (53)$$

The charge squared Q^2 ranges between 0 and 3 where $Q^2 = 0$ corresponds to a Schwarzschild AdS₄ planar BH and $Q^2 = 3$ to the maximal RN AdS₄ planar BH.

3.2. Entanglement entropy

Here we present the calculation of the holographic entanglement entropy in the analogue model discussed in section 3.1. Before we proceed to do that let us first discuss basic notions related to the entanglement entropy in general.

Suppose we have a quantum system with the density of states matrix $\rho = |\Psi\rangle\langle\Psi|$. If we divide the total system into two subsystems A and B we define the reduced density matrix for the subsystem A by taking a partial trace over the subsystem B . i.e. $\rho_A = \text{tr}_B |\Psi\rangle\langle\Psi|$. Then, the entanglement entropy defined as

$$S_A = -\text{tr}_A \rho_A \log \rho_A \quad (54)$$

is the entropy for an observer who can access information only from the subsystem A and can receive no information from B . The subsystem B is analogous to the interior of a BH horizon for an observer outside of the horizon. However, it is often not easy to compute the entanglement entropy, in particular in field theory in $3 + 1$ or higher dimensions.

A convenient description of the entanglement entropy is derived in a $d + 1$ -dimensional field theory. It has been shown that the leading term of the entanglement entropy can be expressed as the area law [31, 32]

$$S_A = a \frac{\text{Area}(\partial A)}{\ell^{d-1}} + \text{subleading terms}, \quad (55)$$

where ∂A is the boundary of A , ℓ is an ultraviolet cutoff or the minimal length in the theory, and a is a constant which depends on the system. It is not accidental that this area law is of the same form as the Bekenstein–Hawking entropy of BHs in $3 + 1$ dimensions which is proportional to the area of the event horizon, with the constants $d = 3$, $a = 1/4$, and ℓ equal to the Planck length.

It is of particular relevance here that the entropy-area relation arises in the context of AdS/CFT duality. AdS/CFT, or gauge/gravity duality, is a correspondence between string theories in asymptotically AdS bulk spacetimes and certain conformal field theories living on the holographic boundary [33].

According to AdS/CFT, the entanglement entropy being basically tied to the gravity in the bulk, should reflect fundamental features of the boundary gauge theory. In this regard we will study the so called *holographic entanglement entropy* in $3 + 1$ dimensions in the context of holographic superconductivity. There is a subtle difference between the usual entanglement entropy and holographic entanglement entropy: although both obey the area law, in the case of holographic entanglement entropy, as we will shortly demonstrate, for a fixed two-dimensional subsystem on the holographic boundary the area depends on the geometry in the bulk. In particular, we expect that the holographic entanglement entropy in our model should exhibit the phase transition discussed in the previous section as demonstrated by the temperature dependence of the superconductor condensates [19].

The holographic entanglement entropy S in a $2 + 1$ -dimensional boundary CFT for a subsystem \mathcal{A} that has an arbitrary one-dimensional boundary $\partial\mathcal{A}$ is defined by the following area law [34–36]

$$S = \frac{\text{Area}(\Sigma)}{4\ell_{\text{Pl}}^2}, \quad (56)$$

where Σ is the two-dimensional static minimal surface in AdS_4 with boundary $\partial\mathcal{A}$ and ℓ_{Pl} is the Planck length.

As we are dealing with an analog geometry we will assume that there exist a minimal length, typically of the order of the atomic separation, below which the bulk description of the fluid fails. This length is referred to in the condensed matter literature as the coherence length, where the meaning of the word ‘coherence’ is different from that in optics. Since it describes the distance over which the wave function of a BE condensate tends to its bulk value when subjected to a localized perturbation, it is also referred to as the healing length [37]. In analog gravity systems, a healing length ℓ_{hl} plays the role of the Planck length [38–42] and for a BE gas is typically of order $\ell_{\text{hl}} \simeq 1/(mc_s)$ where m is the boson mass. Hence, to calculate the entanglement entropy we use (56) with the Planck length ℓ_{Pl} replaced by the healing length ℓ_{hl} . Furthermore, we will identify the arbitrary scale ℓ with ℓ_{hl} . Next we apply the prescription (56) to the geometry suggested in references [20, 34] illustrated in figure 1 and calculate the entropy S as a function of the strip width d for a fixed temperature.

Consider the bulk metric (2) with $d = 3$ and a surface Σ defined by the equation

$$z - z(x) = 0, \tag{57}$$

where $z(x)$ is a function of x such that Σ extends into the bulk and is bounded by the perimeter of \mathcal{A} as illustrated in figure 1. The induced metric σ_{ij} on Σ defines the line element

$$ds_{\Sigma}^2 = \sigma_{ij} dx^i dx^j = \frac{\ell^2}{z^2} \left[dx^2 \left(1 + \frac{z'^2}{\gamma} \right) + y^2 \right]. \tag{58}$$

The area of Σ can be viewed as a functional

$$I[z, z'] = -\text{Area}(\Sigma)/L = \int_{-d/2}^{d/2} dx \mathcal{L}, \tag{59}$$

where L and d are respectively the length and width of the strip, and

$$\mathcal{L} = -\frac{\ell^2}{z^2} \left(1 + \frac{z'^2}{\gamma} \right)^{1/2}. \tag{60}$$

Next we calculate the maximal area of Σ . Clearly, a maximum of area corresponds to a minimum of I and variation of I yields the equation of motion for z . Instead of solving the equation of motion we will use the Hamiltonian approach. We define the conjugate momentum

$$\pi = \frac{\partial \mathcal{L}}{\partial z'} \tag{61}$$

and construct the Hamiltonian

$$\mathcal{H} = \pi z' - \mathcal{L} = \frac{\ell^2}{z^2} \frac{1}{(1 + z'^2/\gamma)^{1/2}}. \tag{62}$$

It can be easily shown that the equation of motion is satisfied if and only if the Hamiltonian is a constant of motion. In particular, at the bottom of the surface $z = z_*$ we have $z' = 0$ and the Hamiltonian is equal to ℓ^2/z_*^2 . In this way we obtain the equation

$$\frac{\ell^2}{z^2} = \frac{\ell^2}{z_*^2} \frac{1}{(1 + z'^2/\gamma)^{1/2}}, \tag{63}$$

from which we can express z' as

$$z' = \pm \frac{\sqrt{(z_*^4 - z^4)\gamma}}{z^2}. \quad (64)$$

Inserting this into (59) and changing the integration variable from x to z with $dx = dz/z'$ we obtain the area of the extremal surface

$$\text{Area} = 8L \int_0^{z_*} dz \frac{z_*^2}{z^2} \frac{\ell^2}{\sqrt{(z_*^4 - z^4)\gamma}}. \quad (65)$$

Dividing this by $4\ell^2$ we obtain the entanglement entropy expressed as an integral over z

$$S = \frac{\text{Area}}{4\ell^2} = 2L \int_0^{z_*} dz \frac{z_*^2}{z^2} \frac{1}{\sqrt{(z_*^4 - z^4)\gamma}}. \quad (66)$$

The location of the bottom z_* of the extremal surface is related to the strip width

$$d = 2 \int_{-d/2}^{d/2} dx = 2 \int_0^{z_*} dz \frac{z^2}{\sqrt{(z_*^4 - z^4)\gamma}}. \quad (67)$$

The integral in (66) is divergent near $z = 0$ and can be regularized by adding and subtracting a counter-term

$$2L \int_{\epsilon}^{z_*} dz/z^2. \quad (68)$$

The entropy is then expressed as

$$S = S_{\text{fin}} + \frac{2L}{\epsilon}, \quad (69)$$

where the finite part reads

$$S_{\text{fin}} = 2L \int_0^{z_*} dz \left(\frac{z_*^2}{z^2} \frac{1}{\sqrt{(z_*^4 - z^4)\gamma}} - \frac{1}{z^2} \right) - \frac{2L}{z_*}. \quad (70)$$

Next we calculate the entanglement entropy using the bulk profile corresponding to an O_2 superconductor at fixed temperature. The reason why we specifically address the O_2 type is that the O_2 superconductor exhibits a first order phase transition which manifests itself as a discontinuity depicted in figure 2 of reference [19]. To calculate S_{fin} we use a polynomial function

$$\gamma = 1 + \sum_{i=3}^6 c_i \left(\frac{z}{z_h} \right)^i \quad (71)$$

with

$$c_3 = -44, \quad c_4 = 118, \quad c_5 = -98, \quad c_6 = 23. \quad (72)$$

We plot this function in figure 2. This choice is motivated by the superconductor bulk metric profile plotted in figure 7(b) of reference [21] for a fixed horizon temperature $T = 0.61 \times 10^{-2} \sqrt{\rho_c}$ where ρ_c is the charge density of the O_2 superconductor (see section 3.1). The function (71) is an analytic approximation to the bulk metric found by numerically solving the field equations of the holographic superconductor.

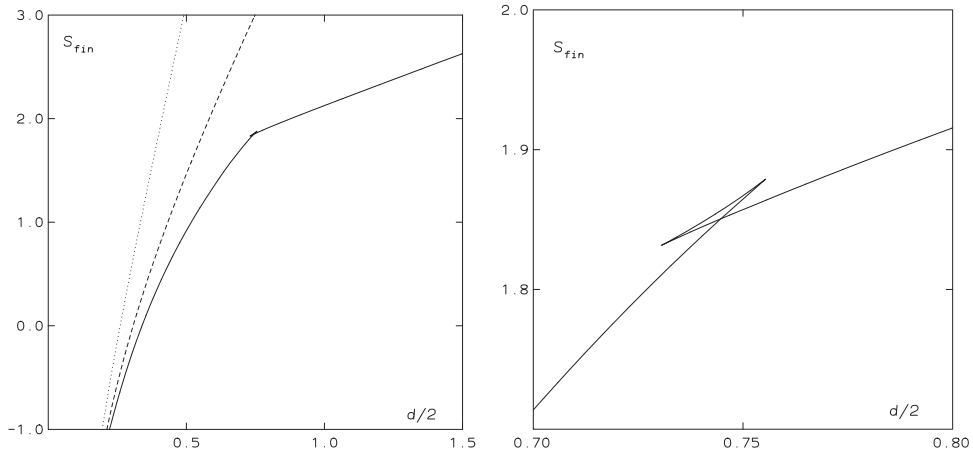


Figure 3. The finite part of the entanglement entropy S_{fin} in units of $2L/z_h$ versus half strip width $d/2$ in units of z_h at fixed temperature. Dotted and dashed lines represent the entanglement entropies of the Schwarzschild AdS planar and maximal RN AdS planar BH, respectively. The right panel shows the zoomed-in crossover region.

In figure 3 we plot S_{fin} as a function of $d/2$. For comparison we plot in the same figure the entanglement entropies of a Schwarzschild AdS planar BH hole and a maximal RN AdS planar BH which have the same asymptotic behavior near $z = 0$. The metric profiles are determined so that the cubic terms are the same as in the O_2 superconductor case. Hence we have

$$\gamma_{AdS} = 1 + c_3 \left(\frac{z}{z_h} \right)^3 \tag{73}$$

for the Schwarzschild AdS planar BH and

$$\gamma_{RN} = 1 + c_3 \left(\frac{z}{z_h} \right)^3 + 3 \left(\frac{c_3}{4} \right)^{4/3} \left(\frac{z}{z_h} \right)^4 \tag{74}$$

for the maximal RN AdS planar BH. The coefficient of the quartic term in (74) was fixed by virtue of (52) and requirement $\gamma_{RN}(z_{RN}) = 0$, where $z_{RN} = (-4/c_3)^{1/3} z_h$ is the location of the RN BH horizon.

To complete our model we still have to determine the function $\chi(z)$. To do this we need to set the scale z_h in relation to the previous works [19–21]. We will make a comparison of the scales at a fixed temperature $T = 0.61 \times 10^{-2} \sqrt{\rho_c}$, where ρ_c is the charge density of dimension of length^{-2} . In reference [19] ρ_c is chosen to set the scale whereas in references [20, 21] the scale is set by the quantity

$$\tilde{\rho}_c = \frac{\rho_c \sqrt{16\pi G_4}}{\ell}. \tag{75}$$

The relation between $\tilde{\rho}_c$ and ρ_c can be fixed by identifying the O_2 phase transition temperature T_{tr} of Albash and Johnson [20] (their figure 2(b)) $T_{tr} = 0.003635 \tilde{\rho}_c^{1/2}$ with that of Bobev *et al* [19] (their figure 2) $T_{tr} = 0.007269 \rho_c^{1/2}$. From this we obtain

$$\tilde{\rho}_c = 4\rho_c. \tag{76}$$

In our approach the scale is set by z_h so we have to find a relation between our z_h and $\tilde{\rho}_c$ or ρ_c . To this end we compare the transition point $d_{tr}/2 = 0.744z_h$ (figure 3) with that of Chakraborty [21] $d_{tr}/2 = 2.56\tilde{\rho}_c^{-1/2}$. This yields

$$\frac{1}{z_h} = 0.2906\tilde{\rho}_c^{1/2} = 0.5812\rho_c^{1/2}. \tag{77}$$

Using this we can express the horizon temperature of our configuration depicted in figure 2 in units of z_h^{-1} ,

$$T_{z_h} \equiv \frac{3}{\pi}e^{-\chi_h/2} = 1.05 \times 10^{-2}, \tag{78}$$

which yields

$$\chi_h \equiv \chi(z_h) = -2 \ln \frac{0.0105\pi}{3} = 9.02. \tag{79}$$

Next, we express χ as a function of z using the expression (49) from section 3.1 in which we set $\chi_0 = 0$, $\lambda_1 = 0$, keep the z^5 term and neglect the higher order terms. Hence we write

$$\chi(z) = 2\lambda_2 \frac{z^4}{\ell^4} + \chi_5 \frac{z^5}{\ell^5}, \tag{80}$$

where the coefficient λ_2 can be fixed from equation (50) with the value of O_2 deduced from figure 2 of reference [19]. At $T = 0.61 \times 10^{-2} \sqrt{\rho_c}$ we find $\lambda_2 = 1.1462$ and using (79) we obtain

$$\chi(z) = 2.63 \left(\frac{z}{z_h}\right)^4 + 6.39 \left(\frac{z}{z_h}\right)^5. \tag{81}$$

This equation together with (71) and (72) can be used to find closed expressions for the hydrodynamic functions and variables of our analog model.

The considerations in this section can as well be carried out for the type O_1 superconductor.

4. Summary and conclusions

We have derived an analog acoustic geometry which mimics a $d + 1$ -dimensional asymptotic AdS geometry with a planar BH. In $3 + 1$ dimensions, this geometry has been exploited as a holographic model for the $2 + 1$ -dimensional superconductor. We have applied this general analog geometry to a $3 + 1$ -dimensional bulk and calculated the entanglement entropy for a particular geometry obtained as solution related to the holographic O_2 superconductor. We have demonstrated that the entanglement entropy in our analog model exhibits the usual first order phase transition which characterizes the O_2 superconductor.

In this way we have confirmed the basic idea that a $3 + 1$ AdS bulk with a planar BH can be realized in nature as a hydrodynamic analog gravity model. Moreover, the analog bulk metric can be parameterized so that the coefficient in the asymptotic expansion in powers of z are such that the dual AdS/CFT boundary field theory corresponds to the type O_2 superconductor. A procedure similar to the one described in section 3.2 can easily be applied to the case of type O_1 superconductor.

It would be of considerable interest to construct a concrete fluid system in the laboratory which would satisfy the properties of the analog geometry described above. It is fare to say that

at this stage we cannot provide a clear proposal of how to prepare an adequate laboratory setup. As we are concerned with fluid velocities close to the speed of light c and sound speed close to c , we would need an essentially relativistic fluid. So far the only known realistic experimental set up for a relativistic-fluid laboratory is provided by high-energy colliders. The study of analog gravity in high energy collisions may in general improve our understanding of the dynamics of general relativistic fluids [43, 44]. Maybe, with the advance of accelerator technology, one day it will be possible, e.g. by choosing appropriate heavy ions and specially designed beam geometry to obtain the desired equation of state and expansion flow of the fluid.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. Acoustic metric

Here we briefly review the derivation of the relativistic acoustic metric. Acoustic metric is the effective metric perceived by acoustic perturbations propagating in a perfect fluid background. Under certain conditions the perturbations satisfy a Klein–Gordon equation in curved geometry with metric of the form (1).

We first derive a propagation equation for linear perturbations of a nonisentropic flow assuming a fixed background geometry. Following Landau and Lifshitz [45] we assume that the enthalpy flow wu_μ is a gradient of a scalar potential, i.e. that there exist a scalar function θ such that the velocity field satisfies

$$wu_\mu = \partial_\mu \theta, \quad (82)$$

where w is the specific enthalpy defined by (10). Then, from the relativistic Euler equation and standard thermodynamic identities it follows [26] that the entropy gradient is also proportional to the gradient of the potential, i.e.

$$s_{,\mu} = w^{-1} u^\nu s_{,\nu} \theta_{,\mu}. \quad (83)$$

Furthermore, instead of the continuity equation $(nu^\mu)_{;\mu} = 0$, one finds

$$(nu^\mu)_{;\mu} = \frac{1}{w} \frac{\partial p}{\partial s} u^\mu s_{,\mu}. \quad (84)$$

In a nonisentropic flow we have $u^\mu s_{,\mu} \neq 0$ and the above equation shows that the particle number is generally not conserved. As demonstrated in reference [26], from equation (83) and Lagrangian description of fluid dynamics it follows that the specific entropy is a function of the velocity potential θ only. Then, using (83) equation (84) can be expressed in the form

$$(nu^\mu)_{;\mu} = \frac{\partial p}{\partial \theta}, \quad (85)$$

where $p = p(w, s(\theta))$ is the pressure of the fluid.

Given some average bulk motion represented by w , n , and u^μ , following the standard procedure [5, 6, 45], we make a replacement

$$w \rightarrow w + \delta w, \quad n \rightarrow n + \delta n, \quad u^\mu \rightarrow u^\mu + \delta u^\mu, \quad (86)$$

where the perturbations δw , δn , and δu^μ are induced by a small perturbation $\delta\theta$ around a background velocity potential θ . From (82) it follows

$$\delta w = u^\mu \delta\theta_{,\mu}, \quad (87)$$

$$w\delta u^\mu = (g^{\mu\nu} - u^\mu u^\nu)\delta\theta_{,\nu}. \quad (88)$$

Using this and (86) equation (85) at linear order yields

$$(f^{\mu\nu}\delta\theta_{,\nu})_{;\mu} + \left[\left(\frac{\partial n}{\partial\theta} u^\mu \right)_{;\mu} - \left(\frac{\partial^2 p}{\partial\theta^2} \right) \right] \delta\theta = 0, \quad (89)$$

where

$$f^{\mu\nu} = \frac{n}{w} \left[g^{\mu\nu} - \left(1 - \frac{w}{n} \frac{\partial n}{\partial w} \right) u^\mu u^\nu \right]. \quad (90)$$

Then, it may be easily shown that equation (89) can be recast into the form

$$\frac{1}{\sqrt{-G}} \partial_\mu \left(\sqrt{-G} G^{\mu\nu} \partial_\nu \delta\theta \right) + m_{\text{eff}}^2 \delta\theta = 0, \quad (91)$$

where the matrix $G^{\mu\nu}$ is the inverse of the acoustic metric tensor

$$G_{\mu\nu} = \frac{n}{m^2 c_s w} [g_{\mu\nu} - (1 - c_s^2) u_\mu u_\nu], \quad (92)$$

with determinant G . Here m is an arbitrary mass parameter introduced to make $G_{\mu\nu}$ dimensionless and c_s is the speed of sound defined by (11).

The effective mass squared is given by

$$m^2 \sqrt{|G|} m_{\text{eff}}^2 = \left[\left(\frac{\partial n}{\partial\theta} u^\mu \right)_{;\mu} - \frac{\partial^2 p}{\partial\theta^2} \right]. \quad (93)$$

Hence, the linear perturbations χ propagate in the effective metric (92) and acquire an effective mass.

In an equivalent field-theoretical description [1, 26, 46] the fluid velocity u_μ is derived from the scalar field as $u_\mu = \partial_\mu \theta / \sqrt{X}$, and n and c_s are expressed in terms of the Lagrangian and its first and second derivatives with respect to the kinetic energy term $X = g^{\mu\nu} \theta_{,\mu} \theta_{,\nu}$. Obviously, the quantity \sqrt{X} in this picture is identified with the specific enthalpy w . Equation (91) with (92) and (11) coincides with that of reference [1] derived in field theory with a general Lagrangian of the form $\mathcal{L} = \mathcal{L}(X, \theta)$.

Appendix B. Effective sound speed with external pressure

Consider a fluid with internal variables p , ρ , and n . Suppose we apply to the fluid an external pressure p_{ext} so that the total pressure is

$$P = p + p_{\text{ext}}. \quad (94)$$

The speed of sound is still defined by

$$c_s^2 = \left. \frac{\partial p}{\partial \rho} \right|_s, \quad (95)$$

but the thermodynamic $T dS$ equation (12) must include the external pressure, i.e.

$$dW = T ds + \frac{1}{n} dP, \quad (96)$$

where

$$W = \frac{P + \rho}{n} = w + \frac{p_{\text{ext}}}{n}. \quad (97)$$

Then the sound speed is given by

$$c_s^2 = \left. \frac{\partial(P - p_{\text{ext}})}{\partial \rho} \right|_s = \left. \frac{n}{W} \frac{\partial W}{\partial n} \right|_s - \frac{\partial p_{\text{ext}}}{\partial \rho}. \quad (98)$$

For an isentropic process from (96) it follows

$$dP = n dW, \quad d\rho = W dn, \quad (99)$$

so by making use of

$$\frac{\partial}{\partial n} = W \frac{\partial}{\partial \rho} \quad (100)$$

we find

$$c_s^2 = \frac{n}{w + p_{\text{ext}}/n} \left(\left. \frac{\partial w}{\partial n} \right|_s - \frac{p_{\text{ext}}}{n^2} \right). \quad (101)$$

Now we make the following ansatz

$$p_{\text{ext}} = \alpha(p + \rho), \quad (102)$$

where $\alpha = \alpha(z)$ will be determined by the requirement that the speed of sound is well defined as $z \rightarrow 0$. With this ansatz we find a modified expression for the sound speed

$$\tilde{c}_s^2 = \frac{1}{1 + \alpha} \left(\left. \frac{n}{w} \frac{\partial w}{\partial n} \right|_s - \alpha \right) = \frac{c_s^2 - \alpha}{1 + \alpha}. \quad (103)$$

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References

- [1] Babichev E, Mukhanov V and Vikman A 2008 *J. High Energy Phys.* **JHEP02(2008)101**
- [2] Novello M and Goulart E 2011 *Class. Quantum Grav.* **28** 145022
- [3] Novello M, Bittencourt E, Moschella U, Goulart E, Salim J M and Toniato J D 2013 *J. Cosmol. Astropart. Phys.* **JCAP06(2013)014**
- [4] Jacobson T 2007 *PoS QG-PH* p 020 arXiv:0801.1547 [gr-qc]
- [5] Visser M 1998 *Class. Quantum Grav.* **15** 1767
- [6] Bilić N 1999 *Class. Quantum Grav.* **16** 3953
- [7] Kinoshita S, Sendouda Y and Takahashi K 2004 *Phys. Rev. D* **70** 123006
- [8] Barceló C, Liberati S and Visser M 2005 *Living Rev. Relativ.* **8** 12
Barceló C, Liberati S and Visser M 2011 *Living Rev. Relativ.* **14** 3
- [9] Barbero J F 1996 *Phys. Rev. D* **54** 1492
- [10] Barbero J F and Villasenor E J S 2003 *Phys. Rev. D* **68** 087501
- [11] Mukohyama S and Uzan J P 2013 *Phys. Rev. D* **87** 065020
- [12] Hartnoll S A, Herzog C P and Horowitz G T 2008 *Phys. Rev. Lett.* **101** 031601
- [13] Hartnoll S A, Herzog C P and Horowitz G T 2008 *J. High Energy Phys.* **JHEP12(2008)015**
- [14] Gubser S S, Herzog C P, Pufu S S and Tesileanu T 2009 *Phys. Rev. Lett.* **103** 141601
- [15] Gauntlett J P, Sonner J and Wiseman T 2009 *Phys. Rev. Lett.* **103** 151601
- [16] Benini F, Herzog C P, Rahman R and Yarom A 2010 *J. High Energy Phys.* **JHEP11(2010)137**
- [17] Horowitz G T 2011 *Lect. Notes Phys.* **828** 313–47
- [18] Aprile F, Roest D and Russo J G 2011 *J. High Energy Phys.* **JHEP06(2011)040**
- [19] Bobev N, Kundu A, Pilch K and Warner N P 2012 *J. High Energy Phys.* **JHEP03(2012)064**
- [20] Albash T and Johnson C V 2012 *J. High Energy Phys.* **JHEP05(2012)079**
- [21] Chakraborty A 2020 *Class. Quantum Grav.* **37** 065021
- [22] Cai R G, Li L, Li L F and Yang R Q 2015 *Sci. China Phys. Mech. Astron.* **58** 060401
- [23] Hartnoll S A 2009 *Class. Quantum Grav.* **26** 224002
- [24] Hossenfelder S 2016 *Phys. Lett. B* **752** 13
- [25] Hossenfelder S 2015 *Phys. Rev. D* **91** 124064
- [26] Bilić N and Nikolic H 2018 *Class. Quantum Grav.* **35** 135008
- [27] Bilić N and Zingg T 2019 arXiv:1903.03401 [gr-qc]
- [28] Fischbacher T, Pilch K and Warner N P 2010 arXiv:1010.4910 [hep-th]
- [29] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 3370
- [30] Randall L and Sundrum R 1999 *Phys. Rev. Lett.* **83** 4690
- [31] Bombelli L, Koul R K, Lee J and Sorkin R D 1986 *Phys. Rev. D* **34** 373
- [32] Srednicki M 1993 *Phys. Rev. Lett.* **71** 666
- [33] Maldacena J 1998 *Adv. Theor. Math. Phys.* **2** 231–52
- [34] Ryu S and Takayanagi T 2006 *Phys. Rev. Lett.* **96** 181602
Ryu S and Takayanagi T 2006 *J. High Energy Phys.* **JHEP08(2006)045**
- [35] Lewkowycz A and Maldacena J 2013 *J. High Energy Phys.* **JHEP08(2013)090**
- [36] Engelhardt N and Wall A C 2015 *J. High Energy Phys.* **JHEP01(2015)073**
- [37] Pethick C J and Smith H 2006 *Bose–Einstein Condensation in Diluted Gases* (Cambridge: Cambridge University Press)
- [38] Uhlmann M, Xu Y and Schützhold R 2005 *New J. Phys.* **7** 248
- [39] Girelli F, Liberati S and Sindoni L 2008 *Phys. Rev. D* **78** 084013
- [40] Fleurov V and Schilling R 2012 *Phys. Rev. A* **85** 045602
- [41] Rinaldi M 2011 *Phys. Rev. D* **84** 124009
- [42] Anderson P R, Balbinot R, Fabbri A and Parentani R 2013 *Phys. Rev. D* **87** 124018
- [43] Bilić N and Tolić D 2013 *Phys. Rev. D* **87** 044033
- [44] Bilić N and Tolić D 2013 *Phys. Rev. D* **88** 105002
- [45] Landau L D and Lifshitz E M 1993 *Fluid Mechanics* (Oxford: Pergamon) p 507
- [46] Piattella O F, Fabris J C and Bilić N 2014 *Class. Quantum Grav.* **31** 055006