Topological defects in Chern-Simons theory

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We construct a new class of topological surface defects in Chern-Simons theory with noncompact, non-Abelian gauge groups. These defects are characterized by isotropic subalgebras defined by solutions of the modified classical Yang-Baxter equation, and their fusion realizes a semigroup structure with noninvertible elements. From a Hamiltonian perspective, we calculate this fusion using the composition of Lagrangian correspondences within the Weinstein symplectic category. Applications include boundary terms and conditions in AdS_3 gravity and higher-spin theories.

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I. INTRODUCTION

A topological defect may separate one phase of a physical system from its symmetry-transformed counterpart. The topological nature of the defects enables fusion that captures the composition law of *generalized symmetries* [1] (see reviews [2,3]). For conventional symmetries, this corresponds to group multiplication. More generally, fusion results in a superposition of defects yielding a noninvertible composition law.

In this work, we exhibit topological defects in three-dimensional (3D) Chern-Simons (CS) theory with a novel and perhaps more elementary kind of noninvertibility: where fusion closes on single elements but no inverse exists, resulting in a *semigroup* composition law. These defects are obtained via the folding trick from topological boundary conditions associated with *Lagrangian* (maximal isotropic) subalgebras of a "doubled" Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}$.

For Abelian CS theory where $\mathfrak{g} = \mathfrak{u}(1)^d$ our characterization of topological defects agrees with that of Kapustin and Saulina [4]. When the inner product κ is positive definite, we make contact with certain defect actions studied by Roumpedakis *et al.* [5] via the introduction of edge modes; also, we show fusion corresponds to the orthogonal group $O(\kappa)$. However, we show that the more general case of indefinite κ admits noninvertible defects forming a semigroup. This is illustrated with the examples $\kappa = \mathrm{diag}(+1, -1)$ and $\kappa = \mathrm{diag}(+1, +1, -1)$.

We then construct defects for non-Abelian Chern-Simons theory, addressing a long-standing open problem identified in [4] (see also [6,7]). Given non-Abelian $\mathfrak{g}$, we construct Lagrangians from $\mathcal{R}$-matrices solving the modified Yang-Baxter equation. These $\mathcal{R}$-defects, specific to noncompact $\mathfrak{g}$, are shown to form a semigroup under fusion using the Hamiltonian approach of [8]. Such constructions have significant applications: CS theory on $\mathfrak{g}^{\mathbb{C}}$ has been extensively studied [9], while theories with $\mathfrak{g}$ as a Drinfel'd double provide a topological quantum field theory framework for T-duality and Poisson-Lie T-duality [10,11].

The Chern-Simons action with $\mathfrak{g} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ provides a rewriting of the 3D Einstein-Hilbert action with negative cosmological constant [12,13]. Augmenting the action with the boundary term for the Drinfel'd-Jimbo $\mathcal{R}$-matrix reproduces the full gravitational boundary contributions used in [14] and readily generalizes to higher-spin theories. We also introduce novel boundary conditions associated with the $\mathcal{R}$-matrix and show these include asymptotically anti–de Sitter (AdS) spacetimes whose asymptotic symmetry algebra is identified as a single Virasoro.

II. FOLDING AND DEFECTS

Chern-Simons theory is defined by the action

$$S_{\mathrm{CS}}[A] = \int_M \left(\langle A, \mathrm{d}A \rangle + \frac{1}{3} \langle A, [A, A] \rangle \right), \qquad (1)$$

where $\kappa_{\mathfrak{g}} = \langle \cdot, \cdot \rangle$ is an appropriately quantized bilinear form on the algebra $\mathfrak{g}$. We bisect M into northern, M_N, and southern, M_S, regions, with a shared boundary D of opposite orientations. Our goal is to define a theory that mediates interactions between gauge fields $A_{N,S}$ living in $M_{N,S}$, respectively, and any degrees of freedom localized on D. If the full system is invariant under displacements of D, the defect is said to be *topological*.

We use the folding trick: since M_N and M_S are diffeomorphic near D we fold them with a parity-odd map $\varphi\colon M_N \to M_S$. This reduces the problem to an analysis of topological boundary conditions in the *doubled* CS action $S_{\mathrm{CS}}[\mathbb{A}]$ on M_N with boundary $\partial M_N = D$ for a connection $\mathbb{A} = (A_N, A_S)$ valued in $\mathfrak{d} = \mathfrak{g}_N \oplus \mathfrak{g}_S$ with the bilinear form $\kappa_{\mathfrak{d}} = \langle\langle \bullet, \bullet \rangle\rangle = \kappa_{\mathfrak{g}_N} - \kappa_{\mathfrak{g}_S}$.

The boundary condition should ensure the vanishing of the surface term $\int_D \langle\langle \delta\mathbb{A}, \mathbb{A} \rangle\rangle$ arising from the variation of the action. A common choice is $\mathbb{A}|_D = \star(\mathbb{A}|_D)$, leading to chiral dynamics on D, but this introduces a Hodge structure, which is not topological. Instead, we require that the gauge field be valued in a *Lagrangian* subalgebra $\mathfrak{h} \subset \mathfrak{d}$ [4] so that $\mathbb{A}|_D \in \Omega^1(D) \otimes \mathfrak{h}$. Here and henceforth we will use the shorthand $\Omega^1(M)$ for the space of one-form tensor fields (i.e., sections of the cotangent bundle $T^\star M$) on a manifold M.

Let P be a projector with complement $P^\perp = \mathrm{id} - P$, with $\mathrm{im}(P) = \mathfrak{h}$ such that $P^\perp \mathbb{A}|_D = 0$. We augment the theory with a boundary term

$$S_{\mathrm{tot}}[\mathbb{A}] = S_{\mathrm{CS}}[\mathbb{A}] + \int_D \langle\langle \mathbb{A}, P^\perp \mathbb{A} \rangle\rangle. \tag{2}$$

This introduces a boundary interaction between the north and south theories while respecting the boundary condition, as the surface term in δS_{tot} is $2\int_D \langle\langle \delta\mathbb{A}, P^\perp \mathbb{A} \rangle\rangle$. The boundary term is included to interpret $P^\perp \mathbb{A}|_D = 0$ as a "boundary equation of motion."

After including this boundary term half of the gauge invariance, $\mathbb{A} \mapsto \mathbb{A}^{\mathbb{g}} = \mathbb{g}^{-1}\mathrm{d}\mathbb{g} + \mathbb{g}^{-1}\mathbb{A}\mathbb{g}$, is preserved. The broken symmetry may be repaired by introducing Stückelberg fields $\mathbb{h} = (h_N, h_S)$ transforming as $\mathbb{h} \mapsto \mathbb{g}^{-1}\mathbb{h}$. Then $\mathbb{A}^{\mathbb{h}}$ is invariant and $\mathbb{h}$ only appears in defect-localized terms outside of a Wess-Zumino term:

$$\begin{aligned} S_{\mathrm{tot}}[\mathbb{A}^{\mathbb{h}}] &= S_{\mathrm{tot}}[\mathbb{A}] + S_{WZ}[\mathbb{h}] + \int_D \langle\langle \mathbb{A}, \mathrm{d}\mathbb{h}\mathbb{h}^{-1} \rangle\rangle \\ &\quad + \int_D \langle\langle \mathbb{A} + \mathrm{d}\mathbb{h}\mathbb{h}^{-1}, (\mathrm{Ad}_{\mathbb{h}} P^\perp \mathrm{Ad}_{\mathbb{h}}^{-1})(\mathbb{A} + \mathrm{d}\mathbb{h}\mathbb{h}^{-1}) \rangle\rangle, \end{aligned} \tag{3}$$

where we use the adjoint action $\mathrm{Ad}_{\mathbb{h}} \cdot = \mathbb{h} \cdot \mathbb{h}^{-1}$. This action generalizes the Lagrangian description of condensation surfaces in Abelian CS theory [[5], Sec. 6.2.2].

We address fusion by dividing M into three regions: north N, south S, and equatorial belt I. The folded fields are $\mathbb{A}_{NI} = (A_N, A_I)$ and $\mathbb{A}_{IS} = (A_I, A_S)$, with defects on $D_{NI} = M_N \cap M_I$ and $D_{IS} = M_I \cap M_S$ prescribed by projectors P_{NI} and P_{IS} into Lagrangians $\mathfrak{h}_{NI}$ and $\mathfrak{h}_{IS}$. Fusion is achieved by shrinking the equatorial belt, merging the defects $D_{NI} = D_{IS} = D$, such that A_I survives only as a nondynamical field on D.

Terms involving A_I are of the form

$$S_I = \int_D \frac{1}{2}\langle A_I, Y A_I \rangle + \langle A_I, B \rangle, \tag{4}$$

with Y a constant skew matrix dependent on the input projectors, and B a one-form depending on the remaining fields. If Y is invertible [15], we can eliminate A_I, yielding

$$S_I = \int_D -\frac{1}{2}\langle B, Y^{-1} B \rangle. \tag{5}$$

We will give an explicit example for Y and B in the following sections. The fused theory takes the form of $S_{\mathrm{tot}}[\mathbb{A}_{NS}]$ of Eq. (2), with a new projector P_{NS}, imposing that $\mathbb{A}_{NS}$ takes values in the Lagrangian $\mathfrak{h}_{NS}$.

Within this treatment, calculating the resulting $\mathfrak{h}_{NS}$ from $\mathfrak{h}_{NI}$ and $\mathfrak{h}_{IS}$ is somewhat opaque. To resolve this we now turn to a Hamiltonian approach.

III. A HAMILTONIAN APPROACH TO FUSION

Recent work [8] by one of us demonstrated that there is a one-to-one relation between lagrangian correspondences and topological defects in Hamiltonian mechanics. We explain how this applies to CS theory and determine a composition rule for defect fusion.

The dynamics of a particle defines a curve $\gamma(\tau)$ on phase space $\mathcal{M} = T^*Q$, with action $\int \gamma^\star\theta$ involving the tautological symplectic potential $\theta = p \cdot dq$. (The expression $\gamma^\star$ is the pullback of differential forms along the map $\gamma\colon \mathbb{R} \to \mathcal{M}$.) A canonical transformation, which we assume is invertible here, defines new coordinates $(\tilde{p}, \tilde{q})$ which we implement via a worldline defect joining two paths γ and $\tilde{\gamma}$ at fixed time τ'. The folding trick in this context yields a path $\Gamma(\tau) = (\gamma(\tau - \tau'), \tilde{\gamma}(\tau' - \tau))$ in the space $\mathcal{M} \times \widetilde{\mathcal{M}}$ with "doubled" action $\int \Gamma^\star(\theta - \tilde{\theta})$ together with an appropriate boundary condition. The defect is topological, i.e., independent of τ', because the map $\mathcal{M} \to \tilde{\mathcal{M}}$ is a symplectomorphism since the canonical transformation is invertible.

This is an example of a *Lagrangian correspondence* in the sense of Weinstein's symplectic "category" [16]: the graph of the map between the symplectic manifolds $(\mathcal{M}, \omega)$ and $(\widetilde{\mathcal{M}}, \tilde{\omega})$ defines a Lagrangian submanifold L in the *correspondence space* $(\mathcal{M} \times \widetilde{\mathcal{M}}, \omega - \tilde{\omega})$, on which $\omega - \tilde{\omega}$ vanishes. Explicitly, if $f\colon \mathcal{M} \to \tilde{\mathcal{M}}$ is the symplectomorphism, and the graph $\mathrm{gr}_f\colon\ \mathcal{M} \to \mathcal{M} \times \tilde{\mathcal{M}}$ is given by $z \to (z, f(z))$, we have $\mathrm{gr}_f^\star(\omega - \tilde{\omega}) = \omega - \omega = 0$. An arbitrary Lagrangian L in correspondence space is viewed as a kind of generalized canonical transformation that may be noninvertible. We refer to [16] for details. The key to the interpretation of Lagrangians as canonical transformations is that they admit a *composition*: if $L_{ij} \subset \mathcal{M}_i \times \mathcal{M}_j$ is a Lagrangian correspondence between $\mathcal{M}_i$ and $\mathcal{M}_j$, we define the composition $L_{ij} \circ L_{jk}$—which is another Lagrangian submanifold whenever smooth—by the formula

$$L_{ij} \circ L_{jk} = \Pi_{ik}[(L_{ij} \times L_{jk}) \cap (\mathcal{M}_i \times \Delta_j \times \mathcal{M}_k)], \quad (6)$$

where Π_{ik} projects onto $\mathcal{M}_i \times \mathcal{M}_k$ and Δ_j is the diagonal embedding $\mathcal{M}_j \hookrightarrow \mathcal{M}_j \times \mathcal{M}_j$. This provides the fusion rules for topological defects in this context [8].

These arguments apply to the folded Chern-Simons theory in the neighborhood of a defect D where the spacetime M takes the form $\mathbb{R}_\tau \times D$ without loss of generality. After splitting $\mathbb{A} = \mathbb{A}_0 \mathrm{d}\tau + \alpha$ with respect to "time" τ normal to the defect, $\mathbb{A}_0$ serves as a Lagrange multiplier enforcing Gauss's law, and the remaining action is that of a particle moving on the symplectic manifold $\mathcal{M}(\mathfrak{d}, D)$ of $\mathfrak{d}$-valued connections:

$$S_{\mathrm{CS}}[\mathbb{A}] = \int \left(\Gamma^* \theta + \int_D \langle\!\langle \mathbb{A}_0 \mathrm{d}\tau, F[\alpha] \rangle\!\rangle \right),$$
$$\theta = \int_D \langle\!\langle \alpha, \delta\alpha \rangle\!\rangle, \qquad \omega = \delta\theta = \int_D \langle\!\langle \delta\alpha, \delta\alpha \rangle\!\rangle. \quad (7)$$

As $\mathfrak{d} = \mathfrak{g}_N \oplus \mathfrak{g}_S$ is equipped with the split signature pairing, we view $\mathcal{M}(\mathfrak{d}, D)$ as the correspondence space $\mathcal{M}(\mathfrak{g}_N, D) \times \mathcal{M}(\mathfrak{g}_S, D)$.

Ignoring the Gauss law constraint initially, defects are identified with Lagrangian submanifolds in $\mathcal{M}(\mathfrak{d}, D)$. Such defects are topological with respect to displacement in the τ-direction chosen above. Defects invariant under arbitrary infinitesimal diffeomorphisms correspond to Lagrangians in $\mathcal{M}(\mathfrak{d}, D)$ of the specific form $\Omega^1(D) \otimes \mathfrak{h}$, where $\mathfrak{h} \subset \mathfrak{d}$ is a maximal isotropic subspace. Compatibility with the Gauss law implies $\mathfrak{h}$ is closed, i.e., a Lagrangian subalgebra [17]. Thus, $\mathbb{A}_{NS}$ restricts, on D, to a gauge field valued in $\mathfrak{h}$, up to gauge transformations.

The fusion of Lagrangian correspondences given by Eq. (6) simplifies to a composition of Lie algebras:

$$\mathfrak{h}_{NI} \circ \mathfrak{h}_{IS} = \Pi_{NS}[(\mathfrak{h}_{NI} \times \mathfrak{h}_{IS}) \cap \mathfrak{g}_N \times \Delta_{\mathfrak{g}_I} \times \mathfrak{g}_S], \quad (8)$$

where $\Delta_{\mathfrak{g}_I}$ is the diagonal embedding $\mathfrak{g}_I \hookrightarrow \mathfrak{g}_I \times \mathfrak{g}_I$. Algorithmically, if $v_{NI} = (v_N, v_I) \in \mathfrak{h}_{NI}$ and $u_{IS} = (u_I, u_S) \in \mathfrak{h}_{IS}$, Eq. (8) tells us to consider elements of the form (v_N, u_S) subject to the relation $v_I = u_I$. This operation is the associative, but non-Abelian, *composition of relations* in set theory.

IV. ABELIAN DEFECTS

Let $\mathfrak{g}_N = \mathfrak{g}_S = \mathfrak{u}(1)^d$ with the definite inner product $\kappa_N = \kappa_S = \mathrm{diag}(+, +, \cdots, +)$. A family of Lagrangians parametrized by a skew-symmetric matrix β is

$$\mathfrak{h}_\beta = \{(Q_+ x, Q_- x) \in \mathfrak{d} | x \in \mathfrak{g}\}, \qquad Q_\pm = \mathrm{id} \pm \beta\kappa. \quad (9)$$

This contains the diagonal Lagrangian $\mathfrak{h}_0 = \{(x, x) \in \mathfrak{d} | x \in \mathfrak{g}\}$. A projector [18] whose image is $\mathfrak{h}_\beta$ is given by

$$P_\beta = \frac{1}{2} \begin{pmatrix} Q_+ & Q_+ \\ Q_- & Q_- \end{pmatrix}. \quad (10)$$

As $Q_\pm$ are invertible, we can use the Cayley transform $M_\beta = Q_+ Q_-^{-1} \in SO(d)$ to write $\mathfrak{h}_\beta = \{(M_\beta x, x)\}$. In fact, the space of Lagrangians is $O(d, \mathbb{R})$.

Consider fusing defects with $P_{NI} = P_\beta$ and $P_{IS} = P_{\tilde{\beta}}$. In the Hamiltonian approach, applying Eq. (8) yields

$$\mathfrak{h}_\beta \circ \mathfrak{h}_{\tilde{\beta}} = \{(M_\beta \cdot M_{\tilde{\beta}} x, x) | x \in \mathfrak{g}\}; \quad (11)$$

i.e., fusion realizes the group multiplication law in $O(d)$. In the Lagrangian approach to fusion, the data defining the intermediate action in Eq. (4) are

$$Y = \beta + \tilde{\beta}, \qquad B = Q_- A_N - \tilde{Q}_+ A_S. \quad (12)$$

Performing the elimination of A_I we recover a defect action defined by a projector P_{NS} whose image matches Eq. (11), and whose kernel is the diagonal $\mathfrak{h}_0$.

A very different feature becomes apparent when κ is *indefinite*. While $\mathfrak{h}_\beta$ of Eq. (9) remains a Lagrangian, $Q_\pm$ need not be invertible. Not coincidentally, such $\mathfrak{h}_\beta$ can never fuse into the diagonal (identity) defect.

Example: $d = 2$ with $\kappa = diag(+, -)$. In this setting Eq. (9) defines a one-parameter family of Lagrangians

$$\mathfrak{h}_\beta = \{(u + \beta v, v + \beta u, u - \beta v, v - \beta u) | u, v \in \mathbb{R}\}. \quad (13)$$

Here, and in the sequel, we denote elements of a Lagrangian by their coordinates in the basis where $\kappa_N - \kappa_S = \mathrm{diag}(+ - - +)$; $\mathfrak{g}_N$ being spanned by the first two.

For $\beta \neq \pm 1$ we use an $O(1, 1)$ Cayley transform to rewrite $\mathfrak{h}_\omega = \{(\cosh\omega u + \sinh\omega v, \sinh\omega u + \cosh\omega v, u, v)\}$, for which fusion acts by addition on the rapidities defined by $\beta = \tanh\omega/2$.

However, at $\beta = \pm 1$ (i.e., in the infinite rapidity limit) we find four Lagrangians of the form

$$\mathfrak{a}^\pm = \{(u, \pm u, v, \mp v)\}, \qquad \mathfrak{b}^\pm = \{(u, \pm u, v, \pm v)\}. \quad (14)$$

These Lagrangians form a direct product of two two-element rectangular band semigroups, $\mathbb{B}_2 = \{\mathfrak{a}^\pm, \mathfrak{b}^\mp\}$, with fusion given by (to be read as $\mathfrak{a}^- \circ \mathfrak{a}^+ = \mathfrak{b}^-$):

$$\begin{array}{c|cccc} \circ & \mathfrak{a}^+ & \mathfrak{a}^- & \mathfrak{b}^+ & \mathfrak{b}^- \\ \hline \mathfrak{a}^+ & \mathfrak{a}^+ & \mathfrak{b}^+ & \mathfrak{b}^+ & \mathfrak{a}^+ \\ \mathfrak{a}^- & \mathfrak{b}^- & \mathfrak{a}^- & \mathfrak{a}^- & \mathfrak{b}^- \\ \mathfrak{b}^+ & \mathfrak{a}^+ & \mathfrak{b}^+ & \mathfrak{b}^+ & \mathfrak{a}^+ \\ \mathfrak{b}^- & \mathfrak{b}^- & \mathfrak{a}^- & \mathfrak{a}^- & \mathfrak{b}^- \end{array} \tag{15}$$

One can never obtain the diagonal/identity defect $\mathfrak{h}_0$ from fusions of $\mathfrak{a}^\pm$ or $\mathfrak{b}^\pm$ with each other or indeed with $\mathfrak{h}_\omega$; they are noninvertible.

Example: $d = 3$ with $\kappa = diag(+,+,-)$. Away from the locus $1 = \beta_1^2 + \beta_2^2 - \beta_3^2$, where $\beta^{ab} = \epsilon^{abc}\beta_c$, we can take $M_\beta = Q_+ Q_-^{-1}$ as an $SO(2,1)$ matrix and construct invertible Lagrangians as before. On that locus we have (β-dependent) elements $t_\pm \in \mathfrak{g}$ obeying

$$Q_\pm t_\pm = 0, \qquad Q_\pm t_\mp = 2t_\mp, \qquad \kappa(t_\pm, t_\pm) = 0. \tag{16}$$

We can express a generic element $x = ut^+ + vt^- + wt^\perp$ where $t^\perp$ is κ-orthogonal to $t^\pm$. This gives rise to noninvertible Lagrangians of the form

$$\mathfrak{h}_\beta = \mathrm{span}\{(ut^+ + wt^\perp, vt^- + wt^\perp)\}. \tag{17}$$

Fusion of these results in a slightly more general type of lagrangian (not necessarily of the form $\mathfrak{h}_\beta$):

$$\mathfrak{h}_\beta \circ \mathfrak{h}_{\tilde\beta} = \mathrm{span}\{(ut^+ + wt^\perp, \tilde v \tilde t^- + w\tilde t^\perp)\}. \tag{18}$$

With the choices of $t^\pm = T_1 \pm T_3$ and $t^\perp = T_2$ (i.e., $\beta_1 = \beta_3 = 0, \beta_2 = 1$) we construct a set of eight distinct noninvertible Lagrangians given in Table I. The fusion of these using formula (8) is given in Table II.

Our examples illustrate that fusion endows the manifold of Lagrangians [which is diffeomorphic to $O(d,\mathbb{R})$] for a split-signature symmetric form on $\mathbb{R}^{2d}$ with a *semigroup structure* depending on the signature of κ. If κ is positive-definite, this semigroup is in fact a group, the group $O(d;\mathbb{R})$. Noninvertibility occurs when κ admits null (light-like, or isotropic) vectors, i.e., for an *indefinite* signature and has a direct physical interpretation: e.g., the Lagrangian $\mathfrak{a}^+$ (14) relevant for defects between $U(1)^2$ CS theories imposes the boundary conditions

$$A_{N1} = A_{N2}, \qquad A_{S1} = -A_{S2}, \tag{19}$$

where the gauge fields A_N and A_S do not glue across D so that the topological boundary condition $\mathfrak{a}^+$ in this folded CS theory unfolds to a pair of topological boundary conditions, one for each side of the defect.

The algebraic structure of the semigroup of Lagrangians under fusion is thus controlled by the signature of κ, i.e., s pluses, $t \leq s$ minuses. The semigroup is a union of sub-semigroups $\mathrm{S}_0, \mathrm{S}_1, \ldots, \mathrm{S}_{s-t}$ whose fusions obey $\mathrm{S}_\mathrm{n} \circ \mathrm{S}_\mathrm{m} \subseteq \mathrm{S}_{\max(\mathrm{m,n})}$, of which $S_0 = O(\kappa)$ is a group of invertibles and $S_{\mathrm{n}>1}$ are semigroups consisting of the noninvertibles.

TABLE I. Some noninvertible Lagrangians (see text).

1 :	$\{ut^+ + wt^\perp, vt^- + wt^\perp\}$	2 :	$\{ut^+ + wt^\perp, vt^+ + wt^\perp\}$
3 :	$\{ut^- + wt^\perp, vt^- + wt^\perp\}$	4 :	$\{ut^- + wt^\perp, vt^+ + wt^\perp\}$
5 :	$\{ut^+ + wt^\perp, vt^- - wt^\perp\}$	6 :	$\{ut^+ + wt^\perp, vt^+ - wt^\perp\}$
7 :	$\{ut^- + wt^\perp, vt^- - wt^\perp\}$	8 :	$\{ut^- + wt^\perp, vt^+ - wt^\perp\}$

TABLE II. Fusion of $\mathcal{R}$-defects. This table describes a direct product $\mathbb{Z}_2 \times \mathbb{B}_2 \times \mathbb{B}_2^{\mathrm{op}}$ where $\mathbb{B}_2$ and $\mathbb{B}_2^{\mathrm{op}}$ denote the two 2-element band semigroups (the subset of greens alone defines $\mathbb{B}_2^{\mathrm{op}}$, while blockwise green and blue yield $\mathbb{B}_2$).

	1	2	3	4	5	6	7	8
1	1	2	1	2	5	6	5	6
2	1	2	1	2	5	6	5	6
3	3	4	3	4	7	8	7	8
4	3	4	3	4	7	8	7	8
5	5	6	5	6	1	2	1	2
6	5	6	5	6	1	2	1	2
7	7	8	7	8	3	4	3	4
8	7	8	7	8	3	4	3	4

V. QUANTIZATION CONDITIONS

For defects between Abelian CS theories with compact gauge groups $G_N = G_S = U(1)^d$, one demands that a Lagrangian $\mathfrak{h}$ exponentiates into a compact [19] subgroup $H = U(1)^d$ of $G_N \times G_S = U(1)^{2d}$. For invertible Lagrangians, which take the form $\{(Mx, x) | x \in \mathfrak{g}_N\}$ for $M \in O(\kappa)$ classically, one would expect that this quantization condition would select Lagrangians in $O(\kappa, \mathbb{Z})$.

It does not. To illustrate, consider $d = 1$, $\kappa = 1$ (we anticipate the following extends to higher dimensions). The allowed Lagrangians are $\mathfrak{h}_{m,\pm} = \{m(x, \pm x) | x \in \mathbb{R}\}$ for $m \in \mathbb{Z}^\times$, where the generators (1,0) and (0,1) are basis vectors for the lattices defining $G_N = G_S = \mathbb{R}/\mathbb{Z}$. $O(\kappa) = O(1)$ does not generate the allowed Lagrangians.

A subset of $O(1,1;\mathbb{Q})$ does instead. In the $(1,\pm 1)$ basis, $SO(1,1;\mathbb{Q})$ is the 2×2 matrix $\mathrm{diag}(r, r^{-1})$ for $r \in \mathbb{Q}$ which acts as $(1,1) \mapsto r(1,1)$ which along with the map $(1,1) \mapsto (1,-1)$ generates the collection of $\mathfrak{h}_{m,\pm}$. However, in general the full action of $O(1,1;\mathbb{Q})$ *does not* respect the quantization. The fusion of Eq. (8) *does not* preserve the quantization condition in general.

VI. $\mathcal{R}$-DEFECTS

Let us now turn to the tools we will need to exhibit topological defects in non-Abelian CS. The additional requirement that the Lagrangians be subalgebras (compatibility with Gauss' law) is restrictive; while the diagonal

$$\mathfrak{g}_\Delta = \{(x,x) \in \mathfrak{d} | x \in \mathfrak{g}\} \tag{20}$$

is a subalgebra of $\mathfrak{d}$, the antidiagonal (though isotropic) is not. To rectify this, we equip $\mathfrak{g}$ with an endomorphism $\mathcal{R}$, skew symmetric with respect to κ, for which the $c^2 = 1$ modified classical Yang-Baxter equation (mCYBE)

$$[\mathcal{R}x, \mathcal{R}y] - \mathcal{R}([\mathcal{R}x, y] + [x, \mathcal{R}y]) = -c^2[x,y] \tag{21}$$

holds for $x, y \in \mathfrak{g}$. We then form a Lagrangian subalgebra

$$\mathfrak{g}_\mathcal{R} = \{((\mathcal{R}+1)X, (\mathcal{R}-1)X) \in \mathfrak{d}\}. \tag{22}$$

Since its intersection with $\mathfrak{g}_\Delta$ is trivial, we have the vector space decomposition $\mathfrak{d} = \mathfrak{g}_\Delta \oplus \mathfrak{g}_\mathcal{R}$. To construct appropriate defect actions we employ the projectors

$$P_\Delta = 1 - P_\mathcal{R} = \frac{1}{2}\begin{pmatrix} 1-\mathcal{R} & 1+\mathcal{R} \\ 1-\mathcal{R} & 1+\mathcal{R} \end{pmatrix}. \tag{23}$$

We let $\bar{\mathcal{R}} = -\mathcal{R}$, which also solves the mCYBE, and denote its corresponding Lagrangian $\mathfrak{g}_{\bar{\mathcal{R}}}$.

For compact semisimple $\mathfrak{g}$ there are no solutions to the $c^2 = 1$ mCYBE, while the *Drinfeld-Jimbo* (DJ) $\mathcal{R}$-matrix provides a canonical solution for the split-signature real form of any complex Lie algebra $\mathfrak{g}^\mathbb{C}$. In a Cartan-Weyl basis for $\mathfrak{g}^\mathbb{C}$ this acts as

$$\mathcal{R}(H_i) = 0, \quad \mathcal{R}(E_\alpha) = E_\alpha, \quad \mathcal{R}(E_{-\alpha}) = -E_{-\alpha}. \tag{24}$$

$\mathcal{R}$ descends to the split real form ($\mathfrak{g}$), and enjoys $\mathcal{R}^3 = \mathcal{R}$.

The Lagrangian $\mathfrak{g}_\mathcal{R}$ has nontrivial fusion properties which we can relate to the defects given in Table I with the following identifications:

1 :	$\mathcal{R}\circ\mathcal{R}$	2 :	$\mathcal{R}\circ\bar{\mathcal{R}}$
3 :	$\bar{\mathcal{R}}\circ\mathcal{R}$	4 :	$\bar{\mathcal{R}}\circ\bar{\mathcal{R}}$
5 :	$\mathcal{R}$	6 :	$\mathcal{R}\circ\mathcal{R}\circ\bar{\mathcal{R}}$
7 :	$\bar{\mathcal{R}}\circ\mathcal{R}\circ\mathcal{R}$	8 :	$\bar{\mathcal{R}}$

It is useful to introduce two involutive automorphisms on $\mathfrak{d}$. The first, $\mathcal{J}$, acts by swapping "north" and "south" i.e., $\mathcal{J}: (x,y) \mapsto (y,x)$. The second, $\mathcal{W}(x,y) \mapsto (x, \mathsf{w}(y))$, uses the Weyl automorphism, $\mathsf{w}: H_i \mapsto -H_i, E_\alpha \mapsto -E_{-\alpha}$. These act as

$$\begin{array}{ccc} \mathcal{R} & \overset{\mathcal{W}}{\longleftrightarrow} & \mathcal{R}\circ\bar{\mathcal{R}} \circlearrowleft \mathcal{J} \\ \updownarrow \mathcal{J} & & \\ \bar{\mathcal{R}} & \overset{\mathcal{W}}{\longleftrightarrow} & \bar{\mathcal{R}}\circ\mathcal{R} \circlearrowleft \mathcal{J} \end{array} \qquad \begin{array}{ccc} \mathcal{R}\circ\mathcal{R} & \overset{\mathcal{W}}{\longleftrightarrow} & \mathcal{R}\circ\mathcal{R}\circ\bar{\mathcal{R}} \circlearrowleft \mathcal{J} \\ \updownarrow \mathcal{J} & & \\ \bar{\mathcal{R}}\circ\bar{\mathcal{R}} & \overset{\mathcal{W}}{\longleftrightarrow} & \bar{\mathcal{R}}\circ\mathcal{R}\circ\mathcal{R} \circlearrowleft \mathcal{J} \end{array}$$

where we use $\mathcal{R} = \mathfrak{g}_\mathcal{R}$, etc. The relations $\mathcal{R}\circ\mathcal{R}\circ\mathcal{R} = \mathcal{R}$ and $\bar{\mathcal{R}}\circ\bar{\mathcal{R}}\circ\bar{\mathcal{R}} = \bar{\mathcal{R}}$ hold, cf. $\mathcal{R}^3 = \mathcal{R}$, and ensure closure of the defects.

VII. APPLICATIONS TO 3D GRAVITY

Three-dimensional gravity admits a formulation in terms of CS theory [12,13]. The Einstein-Hilbert action for the negative cosmological constant is equivalent to

$$S_\mathrm{CS}[A] - S_\mathrm{CS}[\bar{A}] + \int_{\partial M} \mathrm{tr}(A \wedge \bar{A}), \tag{25}$$

in which the $\mathfrak{sl}_2$ connections are related to the dreibein and dualized spin connection according to $A^a = \omega^a + e^a$ and $\bar{A} = \omega^a - e^a$. The boundary term here reproduces the Gibbons-Hawking-York term. To ensure a well-defined variational principle that keeps fixed the metric on the boundary a further contribution is required. In Fefferman-Graham (FG) gauge the combined boundary terms are [14,20,21]

$$S_\mathrm{bdy} = \int_{\partial M} \mathrm{tr}(A \wedge \bar{A}) - \mathrm{tr}((A - \bar{A}) \wedge (A - \bar{A})L_0), \tag{26}$$

with $\mathfrak{sl}_2$ generators obeying $[L_m, L_n] = (m-n)L_{m+n}$.

This can be related to our discussion above by noting that $\mathcal{R}(\bullet) = -[L_0, \bullet]$ provides a DJ $\mathcal{R}$-matrix. Indeed, if we take $\mathbb{A} = (A, \bar{A})$, we have that

$$S_\mathrm{CS}[A] - S_\mathrm{CS}[\bar{A}] + S_\mathrm{bdy}[A, \bar{A}] = S_\mathrm{CS}[\mathbb{A}] + \int \langle\langle \mathbb{A}, P_R^\perp \mathbb{A} \rangle\rangle.$$

We can immediately extend this to higher spins by replacing $\mathfrak{sl}_2$ with $\mathfrak{sl}_N$; the case of $N = 3$ yielding a precise match to the generalization of Eq. (26) presented in [14].

Previously, we described defects in a folded theory. Now, we shift focus to asymptotic boundary conditions in the gravitational theory obtained from $\mathcal{R}$-matrices. From a Chern-Simons perspective, the topological boundary condition $\mathbb{A}|_{\partial M} \in \mathfrak{g}_\mathcal{R}$ translates to

$$(\mathcal{R} - 1)A = (\mathcal{R}+1)\bar{A} \Leftrightarrow [e, L_0] = \omega. \tag{27}$$

These differ from the boundary conditions conventionally applied in gravity since we do not impose conditions on individual components of $A, \bar{A}$ but instead restrict their Lie-algebraic structure. One then asks: is this boundary condition consistent with asymptotically AdS behavior, and what are the associated asymptotic symmetries?

We view AdS_3 as a solid cylinder $\Sigma = \mathbb{R} \times D_2$ with boundary $\partial\Sigma = \mathbb{R} \times S^1$. The real axis $\mathbb{R}$ parametrizes the time t direction while we denote by φ and ρ the angle and radius of the disk D_2, respectively. In the chosen parametrization the boundary of the disk lies at $\rho = \infty$. In these coordinates, and for later use, it will be convenient to express the gauge connection in the so-called radial gauge

$$\begin{aligned} A &= b^{-1}\,\mathrm{d}b + b(\rho)^{-1}a(t,\varphi)b(\rho), \\ \bar{A} &= b\,\mathrm{d}b^{-1} + b(\rho)\bar{a}(t,\varphi)b(\rho)^{-1}, \end{aligned} \tag{28}$$

where $x^i = (t,\varphi)$, $a = a_i \mathrm{d}x^i$ is a one-form on the boundary and the element b is the group element chosen to be $b = \exp(L_0\rho)$, which depends only on the radial coordinate ρ. The one-form a is again an $\mathfrak{sl}(N)$-valued flat gauge connection, specializing for simplicity to $N = 2$, and we will parametrize as $a = \sum_{n=-1}^{1} \ell^n L_n$, and similar for the barred gauge connection. In radial gauge the $\mathcal{R}$-boundary condition then fixes

$$a = \ell^{+1}L_{+1} + \ell^0 L_0, \qquad \bar{a} = \bar{\ell}^{-1}L_{-1} - \ell^0 L_0, \tag{29}$$

where ℓ^0, ℓ^{+1}, and $\bar{\ell}^{-1}$ are one-forms on the boundary. These differ from the general asymptotically AdS_3 boundary conditions of [22], which state that as $\rho \to \infty$,

$$A_\varphi - A_\varphi^{\mathrm{AdS}} = \mathcal{O}(1), \qquad \bar{A}_\varphi - \bar{A}_\varphi^{\mathrm{AdS}} = \mathcal{O}(1). \tag{30}$$

Specializing to the radial gauge, i.e., imposing the constraints $\ell^1_m - \delta_{m+1} \approx 0$ and $\ell^{-1}_m \approx 0$ on the Fourier modes of the filed a, the asymptotically AdS conditions can be expressed in terms of the fields a by

$$a_{\mathrm{AAdS}} = L_{+1} + \ell^0 L_0, \qquad \bar{a}_{\mathrm{AAdS}} = L_{-1} + \bar{\ell}^0 L_0. \tag{31}$$

In [23] it was shown that the asymptotic symmetry algebra corresponding to Eq. (31) is $\mathsf{Vir} \times \mathsf{Vir}$ realized in free-field variables. Setting $\ell^{+1} = \bar{\ell}^{-1} = 1$ in our boundary conditions (29) gives us a special case of the boundary conditions (31). Since in (29) $\ell^0 = -\bar{\ell}^0$, only a diagonal embedding of Vir survives as the asymptotic symmetry algebra of the boundary condition (27).

VIII. DISCUSSION

Our construction provides a novel perspective on topological defects in both Abelian and non-Abelian CS theories. Our methods might point the way toward topological symmetry theories (SymTFTs) for continuous, and even non-Abelian, symmetries (see recent attempts [24–26]). We speculate this would require a map from the Lagrangian subalgebra $\mathfrak{h}$ in the Drinfel'd double $\mathfrak{g} \oplus \mathfrak{g}$, identified here, to Lagrangian objects in a representation category of the quantum group arising from that double.

We note that the noninvertibility in our defect fusions is complementary to that observed in, e.g., [5]. They find, in the context of Abelian CS theories, that some fusions lead to superpositions of defects. We expect that this is related to the quantization condition we outlined in a previous section. One might speculate that noninvertible defects in that sense arise from elements of $O(d,d;\mathbb{Q})$ which would lead to connections $\mathbb{A}_{NS}$ to be such that some integer power of its holonomies are in $U(1)^d$; similar ideas are considered in [4]. In a string-theoretic context the interplay of $O(d,d;\mathbb{Q})$ with quantization of Lagrangians was considered in [27].

With regard to gravity applications, our AdS_3 boundary conditions led to an asymptotic symmetry algebra which is a *single* copy of the Virasoro algebra. This approach might thus be especially interesting for *chiral higher-spin gravity* [28,29] for which one expects a similar structure.

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DATA AVAILABILITY

No data were created or analyzed in this study.

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[15] If dim $\mathfrak{g}$ is odd, Y has a null space. Whilst components of A_I transverse to this can be eliminated, the remainder A_I enforces the projection into the null space as a Lagrange multiplier. This phenomenon also occurs in even dimensions for some P_{NI} and P_{IS}. Despite differences in the resulting defect actions, the on-shell boundary variations after fusion requires $\mathbb{A}_{NS}$ to take values in $\mathfrak{h}_{NS}$.

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[17] What must be true is that $\mathrm{d}\mathbb{A} + \mathbb{A}^2 = 0$ on D where $\mathbb{A}$ takes values in $\mathfrak{h}$. This is because symplectic reduction onto the moduli space of flat connections is a lagrangian correspondence;see [8] for details.

[18] In this presentation we set the kernels of the projectors P_{NI} and P_{IS} to be the anti-diagonal, though the result applies more generally to any bi-vector transformation of the kernel. Both the image and kernel of P_{NS} are generated by fusion of the respective lagrangians Eq. (8).

[19] Non-compact subgroups of $U(1)^{2d}$ are not closed topologically;e.g., a line of irrational slope $\mathbb{R} \to U(1)^2$ which is dense in the torus. This complicates the reduction of structure group from $U(1)^{2d}$ on the defect.

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