

Gauge Invariance at Large Charge

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Quantum field theories with global symmetries simplify considerably in the large-charge limit allowing us to compute correlators via a semiclassical expansion in the inverse powers of the conserved charges. A generalization of the approach to gauge symmetries has faced the problem of defining gauge-independent observables and, therefore, has not been developed so far. We employ the large-charge expansion to calculate the scaling dimension of the lowest-lying operators carrying $U(1)$ charge Q in the critical Abelian Higgs model in $D = 4 - \epsilon$ dimensions to leading and next-to-leading orders in the charge and all orders in the ϵ expansion. Remarkably, the results match our independent diagrammatic computation of the three-loop scaling dimension of the operator $\phi^Q(x)$ in the Landau gauge. We argue that this matching is a consequence of the equivalence between the *gauge-independent* dressed two-point function of Dirac type with the *gauge-dependent* two-point function of $\phi^Q(x)$ in the Landau gauge. We, therefore, shed new light on the problem of defining gauge-independent exponents which has been controversial in the literature on critical superconductors, as well as lay the foundation for large-charge methods in gauge theories.

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Introduction and overview.—The field theoretical description of condensed matter systems is one of the pillars of contemporary physics. In this context, superconductors are defined as the materials with $U(1)$ gauge invariance in the broken phase with the relevant field theory taking the name of the Abelian Higgs model. This is a very well-known textbook example of the Higgs mechanism and is relevant to the description of a plethora of physical systems such as the aforementioned superconductors [1–4], liquid crystals [5], cosmic strings [6], and vortex lines in superfluids [7]. However, one of the main issues regarding this model is the definition of gauge invariant correlation functions describing physical quantities. Consider as a starting point the two-point function of a complex scalar field $G_\phi(x_f - x_i) \equiv \langle \bar{\phi}(x_f)\phi(x_i) \rangle$; while this correlator is invariant under global $U(1)$ transformations, it violates $U(1)$ gauge symmetry and thus it vanishes identically due to Elitzur’s theorem [8]. To make progress, one is led to define a gauge invariant generalization of this correlator which, however, is not unique and different approaches were shown to lead to different physical results. In particular,

the two main proposals considered in the literature date back to the works of Dirac [9] and Schwinger [10,11]. Dirac’s approach introduces a Wilson line as

$$G_D(x_f - x_i) = \left\langle \bar{\phi}(x_f) \exp\left(ie \int d^D x J^\mu(x) A_\mu(x)\right) \phi(x_i) \right\rangle, \quad (1)$$

where $\partial^\mu J_\mu = \delta(x - x_f) - \delta(x - x_i)$ and $\partial^2 J_\mu = 0$. The explicit form of the nonlocal current is $J_\mu = J'_\mu(z - x_f) - J'_\mu(z - x_i)$ with

$$J'_\mu(z) = -i \int \frac{d^D k}{(2\pi)^D} \frac{k_\mu}{k^2} e^{ik \cdot z} = -\frac{\Gamma(D/2 - 1)}{4\pi^{D/2}} \partial_\mu \frac{1}{z^{D-2}}. \quad (2)$$

Based on the definitions above $G_D(x_f - x_i) = \langle \bar{\phi}_{nl}(x_f)\phi_{nl}(x_i) \rangle$, where

$$\phi_{nl}(x) \equiv e^{-ie \int d^D z J'_\mu(z-x) A^\mu(z)} \phi(x) \quad (3)$$

has been proposed as the *nonlocal order parameter* for the superconducting phase transition. Noticeably, in the Landau gauge $\partial^\mu A_\mu = 0$, $\phi_{nl}(x)$ reduces to the *local order parameter* $\phi(x)$ since J'_μ is a total derivative [12]. Therefore, we expect that the result for a *gauge-independent* Dirac correlator will coincide with the result obtained for the *gauge-dependent*

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$G_\phi(x_f - x_i)$ correlator in the Landau gauge. Physically, $\phi_{nl}(x)$ can be interpreted as the creation operator of a charged scalar particle dressed with a coherent state of photons describing its Coulomb field. On the other hand, in the Schwinger approach the gauge invariant correlator reads

$$G_S(x_f - x_i) = \left\langle \bar{\phi}(x_f) \exp\left(-ie \int dx^\mu A_\mu(x)\right) \phi(x_i) \right\rangle \quad (4)$$

and, analogously to its Dirac counterpart, $G_S(x_f - x_i)$ reduces to $G_\phi(x_f - x_i)$ in the *traceless gauge* [13].

Of special interest are the correlators computed in the critical model, where they take the form

$$G_{\phi,D,S}(x_f - x_i) = \frac{1}{|x_f - x_i|^{D-2+2\eta_{\phi,D,S}}}. \quad (5)$$

In the context of the $O(n)$ model, it has been verified in [14] ([13]) that η_D (η_S) coincide with η_ϕ in the Landau (traceless) gauge to leading order in the ϵ expansion and to leading order in $1/n$. Moreover, it has been pointed out in [15], that $G_S(x_f - x_i)$ cannot provide a correct description of the model in the broken phase since it does not lead to long-range order.

In this Letter, we study the issue of extracting gauge-invariant critical exponents from the novel perspective of the large-charge expansion which has attracted a lot of attention recently [16–24]. There, one starts with a critical theory invariant under a set of global symmetries and evaluates scaling exponents via an expansion in inverse powers of the conserved charges. In practice, this is achieved by computing the ground state energy at fixed charge Q of the theory defined on the unit cylinder $\mathbb{R} \times S^{D-1}$, which, by virtue of the state-operator correspondence [25,26] is equal to the scaling dimension Δ_Q of the lowest-lying operators with charge Q . In perturbative models, the approach amounts to a semiclassical evaluation of the expectation value of the evolution operator e^{-HT} in an arbitrary state with fixed charge Q which allows to resum an infinite number of Feynman diagrams at every order of the semiclassical expansion [18,19]. At first sight, a generalization of the approach to gauge theories looks impossible due to the lack of gauge invariance of correlators of matter fields such as $G_\phi(x_f - x_i)$. Remarkably, we instead show by explicit calculation that our results for Δ_Q in the large-charge expansion are gauge-independent and correspond to the conformal dimensions of $\phi^Q(x)$ computed in Landau gauge. We interpret the result as the consequence of the equivalence in the Landau gauge of $\phi^Q(x)$ to the non-local operator $\phi_{nl}^Q(x)$ generalizing the Dirac construction (3) to arbitrary values of Q .

Ultimately, our results confirm that the critical exponent η_S is not adequate to describe the broken phase. In fact, η_S

is more negative than η_D at the leading order in the ϵ expansion [13,14]. Therefore, it leads to a lower conformal dimension which, in turn, would be computed in our approach by construction.

Highlighting our findings: (i) We show that the large-charge expansion can be applied also to gauge theories where the relevant gauge-invariant observables are in general nonlocal. (ii) We explicitly show that the nonlocal operators ϕ_{nl}^Q are the lowest-lying operators with charge Q well defined at criticality. In particular, this signals that ϕ_{nl} is the relevant order parameter for long-range order in superconductors and it is automatically selected by the large charge approach. (iii) We compute Δ_Q to the next-to-leading order in the large-charge expansion and all orders in the loop expansion. Moreover, we explicitly calculate the full three-loop Δ_Q in perturbation theory and find perfect agreement with our semiclassical result.

Large charge operators in Scalar QED.—Classical contribution: Our starting point is the massless gauged $(\bar{\phi}\phi)^2$ theory in $D = 4 - \epsilon$ dimensions given by the action

$$S = \int d^D x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D_\mu \phi + \frac{\lambda(4\pi)^2}{6} (\bar{\phi}\phi)^2 \right), \quad (6)$$

where $D_\mu \phi = (\partial_\mu + ieA_\mu)\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The above action is invariant under the local $U(1)$ gauge transformations $\phi \rightarrow e^{i\alpha(x)}\phi$ and $A_\mu \rightarrow A_\mu - e^{-1}\partial_\mu\alpha(x)$. The equations of motion (EOM) are given by

$$-D^\mu D_\mu \phi + \frac{\lambda(4\pi)^2}{3} (\bar{\phi}\phi)\bar{\phi} = 0, \quad \partial_\mu F^{\mu\nu} = J^\nu, \quad (7)$$

where J_μ is the $U(1)$ electromagnetic current. The associated charge is defined as $Q = \int d^{D-1}x J^0$. Note that since the $U(1)$ symmetry is gauged, in order to avoid long-range electric fields causing infrared divergences the system needs to be electrically neutral. This can be achieved by introducing a neutralizing background J_b^ν such that $J_{\text{total}}^0 = J^0 - J_b^0 = 0$. This background current may be seen as the one used to define the nonlocal operators ϕ_{nl}^Q . However, being nondynamical, it does not affect the scaling dimension Δ_Q . Throughout the text we use the same notation for bare and renormalized couplings. The theory features a Wilson-Fisher fixed point which, at the one-loop level, reads [2,27]

$$\lambda^* = \frac{3}{20} \left(19\epsilon \pm i\sqrt{719}\epsilon \right), \quad a_e^* = \frac{3}{2}\epsilon, \quad (8)$$

with $a_e = [e^2/(4\pi)^2]$. Note that the fixed point occurs at complex values of λ . However, once Δ_Q is calculated at the fixed point, one can rewrite the result in terms of the renormalized couplings and obtain an expression valid for arbitrary values of λ and a_e [28]. Mapping the theory to the

cylinder $\mathbb{R} \times S^{D-1}$ with unit radius, the Weyl invariant action becomes

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi + m^2 \bar{\phi} \phi + \frac{\lambda(4\pi)^2}{6} (\bar{\phi} \phi)^2 \right), \quad (9)$$

The scalar mass arises from the conformal coupling between ϕ and the Ricci scalar of the S^{D-1} sphere and is given by $m^2 = (D-2)^2/4$ [29]. Due to the state-operator correspondence, after parametrizing the scalar field as $\phi(x) = [\rho(x)/\sqrt{2}] e^{i\chi(x)}$, the calculation of the scaling dimensions reduces to the evaluation of the following matrix element

$$\langle Q | e^{-HT} | Q \rangle = \mathcal{Z}^{-1} \int_{\rho=f}^{\rho=f} \mathcal{D}\rho \mathcal{D}\chi \mathcal{D}A e^{-S_{\text{eff}}}, \quad (10)$$

where the effective action S_{eff} is succinctly written as

$$S_{\text{eff}} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{D-1} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (\partial\rho)^2 + \frac{1}{2} \rho^2 (\partial\chi)^2 + \frac{1}{2} m^2 \rho^2 + e\rho^2 A_\mu \partial^\mu \chi + \frac{1}{2} e^2 \rho^2 A_\mu A^\mu + \frac{\lambda(4\pi)^2}{24} \rho^4 + \frac{iQ}{\Omega_{D-1}} \dot{\chi} \right). \quad (11)$$

Here $\mathcal{Z} = \int \mathcal{D}\phi \mathcal{D}\bar{\phi} \mathcal{D}A e^{-S}$ is the partition function, $|Q\rangle$ is an arbitrary state with charge Q , and f is a fixed value for the $\rho(x)$ field. The last term fixes the charge of initial and final states. More details on the approach can be found in, e.g., [18,24]. The above matrix element is related to Δ_Q as

$$\langle Q | e^{-HT} | Q \rangle_{T \rightarrow \infty} = \mathcal{N} e^{-\Delta_Q T}, \quad (12)$$

where \mathcal{N} is an unimportant T -independent normalization coefficient. By rescaling the fields as $\rho \rightarrow \sqrt{Q}\rho$, $A_\mu \rightarrow \sqrt{Q}A_\mu$ to exhibit Q as the loop counting parameter, one sees that (10) can be calculated via a semiclassical expansion around the saddle points of S_{eff} . Accordingly, Δ_Q takes the following form

$$\Delta_Q = \sum_{j=-1}^{\infty} \frac{1}{Q^j} \Delta_j(Qe^2, Q\lambda). \quad (13)$$

Every Δ_j resums an infinite series of Feynman diagrams of the conventional perturbative expansion which can be recovered by expanding the Δ_j for small values of the 't Hooft-like couplings Qe^2 and $Q\lambda$. The lowest energy solution of the EOM corresponds to a homogeneous ground state

$$\rho(x) = f, \quad \chi(x) = -i\mu\tau, \quad A_\mu = 0. \quad (14)$$

The vev of the radial mode f and the chemical potential μ are determined in terms of Q and the couplings via the EOM as

$$\mu^3 - \mu = \frac{4}{3} \lambda Q, \quad f^2 = \frac{6}{(4\pi)^2 \lambda} (\mu^2 - m^2). \quad (15)$$

The classical contribution to Δ_Q is then obtained by plugging the solution (14) into S_{eff} . Since at the classical level $A_\mu = 0$, the leading order of the semiclassical expansion is equivalent to the one obtained in [18] for the ungauged $(\bar{\phi}\phi)^2$ model, reading

$$4\Delta_{-1} = \frac{3^{2/3} (x + \sqrt{-3 + x^2})^{1/3}}{3^{1/3} + (x + \sqrt{-3 + x^2})^{2/3}} + \frac{3^{1/3} (3^{1/3} + (x + \sqrt{-3 + x^2})^{2/3})}{(x + \sqrt{-3 + x^2})^{1/3}}, \quad (16)$$

where $x \equiv 6\lambda Q$.

Leading quantum correction: The next-to-leading order Δ_0 is given by the functional determinant of the fluctuations around the classical solution and can be expressed as a sum of zero-point energies

$$\Delta_0 = \frac{1}{2} \sum_{\ell=\ell_0}^{\infty} \sum_i d_\ell \omega_i(\ell), \quad (17)$$

where the innermost sum runs over all the dispersion relations of the spectrum, d_ℓ is the degeneracy of the corresponding eigenvalues of the momentum and ℓ_0 is given in Table I. To find the spectrum, we expand the fields around the classical configuration (14) as $\rho(x) = f + r(x)$ and $\chi(x) = -i\mu\tau + f^{-1}\pi(x)$. The action at the quadratic order in the fluctuations takes the form

$$S_{\text{eff}}^{(2)} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{D-1} \left(\frac{1}{4} F_{\mu\nu}^2 + \frac{1}{2} (\partial_\mu r)^2 + \frac{1}{2} (\partial_\mu \pi)^2 - \frac{1}{2} 2(m^2 - \mu^2)r^2 - 2i\mu r \partial_\tau \pi + ef \partial_\mu \pi A^\mu - 2ie\mu f r A_0 + \frac{1}{2} (ef)^2 A_\mu A^\mu \right). \quad (18)$$

It is clear that the Higgs mechanism has occurred with the gauge field acquiring mass $m_A^2 = (ef)^2$ and $\pi(x)$ being the massless Goldstone mode of the spontaneously broken $U(1)$ symmetry. Nevertheless, due to Elitzur's theorem [8] the local part of a gauge symmetry of a compact group cannot be spontaneously broken and the action (18) is invariant under the residual gauge symmetry

$$\delta r = 0, \quad \delta \pi = f\alpha(x), \quad \delta A_\mu = -\frac{1}{e} \partial_\mu \alpha(x), \quad (19)$$

TABLE I. The fluctuations spectrum of the model. The second column shows the scalar and vector fields degeneracies, where $n_s(\ell) = (2\ell + D - 2)\Gamma(\ell + D - 2)/\Gamma(D - 1)\Gamma(\ell + 1)$ and $n_v(\ell) = \ell(\ell + D - 2)(2\ell + D - 2)\Gamma(\ell + D - 3)/\Gamma(\ell + 2)\Gamma(D - 2)$. The third column contains the dispersions entering (17), where $J_{\ell(s)}^2 = \ell(\ell + D - 2)$ and $J_{\ell(v)}^2 = \ell(\ell + D - 2) - 1$ are the Laplacian eigenvalues on S^{D-1} .

Field	d_ℓ	$\omega_i(\ell)$	ℓ_0
B_i	$n_v(\ell)$	$\sqrt{J_{\ell(v)}^2 + (D - 2) + e^2 f^2}$	1
C_i	$n_s(\ell)$	$\sqrt{J_{\ell(s)}^2 + e^2 f^2}$	1
(c, \bar{c})	$-2n_s(\ell)$	$\sqrt{J_{\ell(s)}^2 + e^2 f^2}$	0
A_0	$n_s(\ell)$	$\sqrt{J_{\ell(s)}^2 + e^2 f^2}$	0
ϕ	$n_s(\ell)$	$\sqrt{J_{\ell(s)}^2 + 3\mu^2 - m^2 + \frac{1}{2}e^2 f^2 \pm \sqrt{(3\mu^2 - m^2 - \frac{1}{2}e^2 f^2)^2 + 4J_{\ell(s)}^2 \mu^2}}$	0

where $\alpha(x)$ is the phase of the original $U(1)$ gauge transformations. To evaluate the path integral we employ the R_ξ gauge and the Faddeev-Popov method. Then the action (18) is replaced as $S_{\text{eff}}^{(2)} \rightarrow S_{\text{eff}}^{(2)} + \frac{1}{2} \int d^D x G^2$, with

$$G = \frac{1}{\sqrt{\xi}} (\nabla_\mu A^\mu + ef\pi), \quad \frac{\delta G}{\delta \alpha} = \frac{1}{\sqrt{\xi}} \left(-\frac{1}{e} \nabla^2 + ef^2 \right), \quad (20)$$

where we used (19) to take the variation with respect to α . Rescaling G with e , the determinant of $\delta G/\delta \alpha$ can be represented using a set of Fadeev-Popov ghosts \bar{c}, c with $\mathcal{L}_{\text{ghost}} = \bar{c}[-\nabla^2 + (ef)^2]c$. Consequently, the quadratic Lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\text{eff}}^{(2)} = & \frac{1}{2} A_\mu \left(-g^{\mu\nu} \nabla^2 + \mathcal{R}^{\mu\nu} + \left(1 - \frac{1}{\xi} \right) \nabla^\mu \nabla^\nu + (ef)^2 g^{\mu\nu} \right) A_\nu \\ & + \frac{1}{2} (\partial_\mu r)^2 - \frac{1}{2} 2(m^2 - \mu^2) r^2 + \frac{1}{2} (\partial_\mu \pi)^2 - \frac{1}{2\xi} (ef)^2 \pi^2 \\ & - 2i\mu r \partial_\tau \pi - 2if\mu r A^0 + ef \left(1 - \frac{1}{\xi} \right) A_\mu \partial^\mu \pi + \mathcal{L}_{\text{ghost}}, \end{aligned} \quad (21)$$

where we used that $[\nabla^\mu, \nabla^\nu] A_\mu = \mathcal{R}_\nu^\mu A_\mu$ with $\mathcal{R}_\nu^\mu = (D - 2)\delta_\nu^\mu$ the Ricci tensor on S^{D-1} . With these results at our disposal, we are ready to evaluate the

determinant of the partition function using $-\nabla^2 = -\partial_\tau^2 + (-\nabla_{S^{D-1}}^2)$ on $\mathbb{R} \times S^{D-1}$ space. The sphere Laplacian eigenvalues and degeneracies are shown in Table I.

Let us separately comment on the first line of (21). When the operator in the brackets acts on A_0 , which is a scalar on S^{D-1} , the Ricci tensor does not contribute and $-\nabla^2$ is equivalent to the scalar Laplacian. On the other hand, the vector field A^i is decomposed as the kernel plus the image of the nabla operator. In mathematical terms $A^i = B^i + C^i$, where $\nabla_i B^i = 0$ and $C^i = \nabla^i h$ with h an arbitrary function. These fields are orthogonal to each other and terms containing products of them vanish. This implies that B^i is a vector while the C^i are a set of scalars. As a consequence, the A_0 and C_i fields can be organized in the same scalar multiplet while the Gaussian path integration over the B^i evaluates to

$$\int \frac{d\omega}{2\pi} \sum_\ell n_v(\ell) \det(-\partial_\tau^2 + J_{\ell(v)}^2 + (D - 2) + (ef)^2)^{-1/2}. \quad (22)$$

In addition, using (21) along with the scalar components of A^μ , we obtain the inverse propagator matrix for the (r, π, A_0, C_i) fields

$$\mathcal{B} = \begin{pmatrix} -\omega^2 + J_{\ell(s)}^2 + 2(\mu^2 - m^2) & -2i\mu\omega & -2ie\mu f & 0 \\ 2i\mu\omega & -\omega^2 + J_{\ell(s)}^2 + \frac{1}{\xi} e^2 f^2 & -ef \left(1 - \frac{1}{\xi} \right) \omega & -ief \left(1 - \frac{1}{\xi} \right) |J_{\ell(s)}| \\ -2ie\mu f & ef \left(1 - \frac{1}{\xi} \right) \omega & -\frac{1}{\xi} \omega^2 + J_{\ell(s)}^2 + (ef)^2 & i \left(1 - \frac{1}{\xi} \right) \omega |J_{\ell(s)}| \\ 0 & ief \left(1 - \frac{1}{\xi} \right) |J_{\ell(s)}| & i \left(1 - \frac{1}{\xi} \right) \omega |J_{\ell(s)}| & -\omega^2 + \frac{1}{\xi} J_{\ell(s)}^2 + (ef)^2 \end{pmatrix}. \quad (23)$$

The determinant of (23) factorizes as

$$\xi \det \mathcal{B} = (\omega^2 + \omega_+^2)(\omega^2 + \omega_-^2)(\omega^2 + \omega_1^2)^2. \quad (24)$$

Table I includes the dispersion relations ω_{\pm} , ω_1 along with the ones from the B^i fields in (22) and the ghosts. The functional determinant of the ghosts cancels against the contribution stemming from C_i and A_0 , leaving a single ghost zero mode ($\ell = 0$) contribution which, in turn, cancels one of the scalar zero modes. Also, since in (10) the quadratic part of the partition function \mathcal{Z} in the denominator contains the gauge field, we should gauge fix it as well. Following the same gauge fixing procedure as above and performing the Gaussian integration, the ξ dependence factorizes as in (24) and cancels out in the final result. The obtained expression (17) arising from the Gaussian integrals in the numerator and the denominator is

divergent and needs to be renormalized. To this end, we follow [18] and regularize the sum over ℓ by subtracting the divergent terms in the $\ell \rightarrow \infty$ limit. Then we use dimensional regularization to isolate the $(1/\epsilon)$ pole which is canceled by performing renormalization in the usual $\overline{\text{MS}}$ scheme. The renormalized result reads

$$\begin{aligned} \Delta_0 = & \frac{1}{16} \left(-15\mu^4 - 6\mu^2 + 8\sqrt{6\mu^2 - 2 + 5} \right) \\ & + \frac{1}{2} \sum_{\ell=1} \sigma(\ell) - \frac{3a_e}{8\lambda} (\mu^2 - 1) \\ & \times \left(\frac{3a_e}{\lambda} (7\mu^2 + 5) - 9\mu^2 + 5 \right), \end{aligned} \quad (25)$$

where the summand $\sigma(\ell)$ is given by

$$\begin{aligned} \sigma(\ell) = & \frac{9a_e}{2\lambda\ell} (\mu^2 - 1) \left[\left(\frac{3a_e}{\lambda} - 1 \right) (\mu^2 - 1) - 2\ell(\ell + 1) \right] + \frac{5}{4\ell} (\mu^2 - 1)^2 - 2(\ell + 1)(2\ell(\ell + 2) + \mu^2) \\ & + (\ell + 1)^2 (R_+ + R_-) + 2\ell(\ell + 2)R, \end{aligned} \quad (26)$$

with

$$\begin{aligned} R_{\pm} = & \sqrt{\frac{3a_e}{\lambda} (\mu^2 - 1) + 3\mu^2 + \ell(\ell + 2) - 1 \pm \sqrt{\left(\frac{3a_e}{\lambda} (\mu^2 - 1) - 3\mu^2 + 1 \right)^2 + 4\ell(\ell + 2)\mu^2}}, \\ R = & \sqrt{\frac{6a_e}{\lambda} (\mu^2 - 1) + \ell(\ell + 2) + 1}. \end{aligned} \quad (27)$$

Equation (25) is our main result which, combined with (16), gives Δ_Q to the next-to-leading order in the large-charge expansion and all orders in the loop expansion at the fixed point (8).

Explicit three-loop calculation.—In perturbation theory we are able to compute the anomalous dimension of ϕ^Q operator as a series in small couplings a_e and λ ,

$$\gamma_Q(\lambda, a_e, \xi) = \sum_{l=1}^{\infty} \gamma_Q^{(l\text{-loop})}(\lambda, a_e, \xi), \quad \gamma_Q^{(l\text{-loop})} \equiv \sum_{k=0}^l C_{kl} Q^{l+1-k}, \quad (28)$$

where we use the linear R_{ξ} gauge. The l -loop contribution $\gamma_Q^{(l\text{-loop})}$ is a polynomial of degree $l + 1$ in Q and to find the coefficients C_{kl} at l loops one can explicitly compute the anomalous dimensions of ϕ^Q operators for fixed $Q = 1, \dots, l + 1$.

The case $Q = 1$ corresponds to the well-known field anomalous dimension. In addition, we considered one-particle irreducible Green's functions with ϕ^Q operator insertions (see Fig. 1) for $Q = 2, 3, 4$. We used an infrared rearrangement trick [30] together with the MATAD [31] package to find the required renormalization constants Z_Q in the $\overline{\text{MS}}$ scheme at three loops. This allows us to derive the following expression for the anomalous dimension at arbitrary Q :

$$\begin{aligned} \gamma_Q^{(1)}(\lambda, a_e, \xi) = & \underbrace{\frac{\lambda}{3} Q^2}_{\text{leading}} - \underbrace{Q \left(3a_e + \frac{\lambda}{3} \right)}_{\text{sub-leading}} + a_e Q^2 \xi, \\ \gamma_Q^{(2)}(\lambda, a_e) = & \underbrace{-\frac{2\lambda^2}{9} Q^3}_{\text{leading}} + \underbrace{\left(a_e^2 - \frac{4a_e\lambda}{3} + \frac{2\lambda^2}{9} \right) Q^2}_{\text{sub-leading}} + \left(\frac{7a_e^2}{3} + \frac{4a_e\lambda}{3} + \frac{\lambda^2}{9} \right) Q, \end{aligned} \quad (29)$$

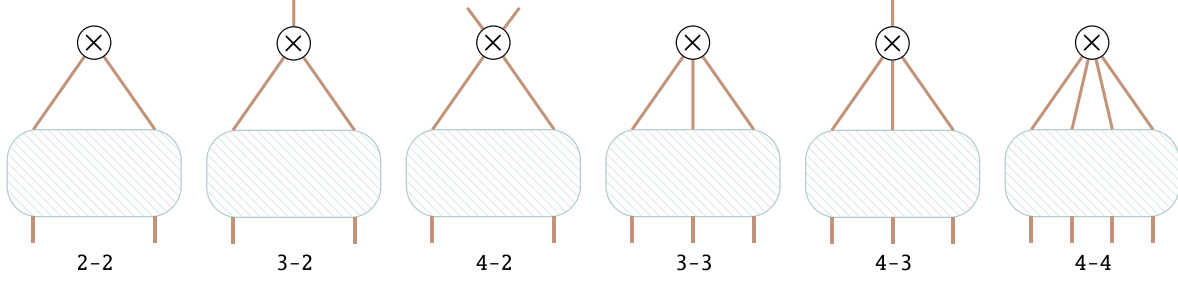


FIG. 1. Operator insertions into various Green's functions considered up to the three-loop order. Abbreviation $Q - M$ means insertion of the operator ϕ^Q with M legs attached to the loop diagram. $M = 2$ contribution starts from the one-loop order, $M = 3$ from two-loop and $M = 4$ from three-loop order, respectively. Hatched blobs include loop corrections calculated with standard scalar QED Feynman rules.

$$\begin{aligned}
 \gamma_Q^{(3)}(\lambda, a_e) = & \underbrace{\frac{8\lambda^3}{27} Q^4}_{\text{leading}} + \underbrace{Q^3 \left[\frac{4a_e\lambda^2}{3} (3 - 2\zeta_3) - \frac{8a_e^2\lambda}{3} (1 + 3\zeta_3) + 4a_e^3(9\zeta_3 - 1) + \frac{2\lambda^3}{27} (16\zeta_3 - 17) \right]}_{\text{sub-leading}} \\
 & + Q^2 \left[\frac{29a_e^2\lambda}{6} + a_e^3(95 - 108\zeta_3) + \frac{\lambda^3}{18} (57 - 64\zeta_3) - \frac{4a_e\lambda^2}{9} (31 - 30\zeta_3) \right] \\
 & + Q \left[\frac{13a_e^2\lambda}{6} + \frac{2a_e\lambda^2}{9} (49 - 48\zeta_3) - \frac{2\lambda^3}{27} (31 - 32\zeta_3) - a_e^3 \left(\frac{3251}{54} - 72\zeta_3 \right) \right]. \quad (30)
 \end{aligned}$$

As we argued above, we expect that once we fix the Landau gauge ($\xi = 0$), which corresponds to the fixed point of ξ [32], and evaluate dimensions of operators ϕ^Q at $\lambda = \lambda^*$, and $a_e = a_e^*$ (8), we should reproduce (13). We computed these scaling dimensions

$$\Delta_Q = Q \left(\frac{D-2}{2} \right) + \gamma_Q(\lambda^*, a_e^*, \xi^* = 0), \quad (31)$$

and indeed find perfect agreement with the corresponding terms from (16) and (25) which, for the reader's convenience, we highlighted with a brace below each term. In fact, as it was noticed in Refs. [19,33], at *two loops and higher* the expansion of (13) in small $a_e Q$ and λQ precisely coincides with the perturbative result, even away from the fixed point which we confirmed explicitly. In particular, the coefficients C_{kl} appear in the small $a_e Q$ and λQ expansion of Δ_{k-1} . As for the one-loop contribution to Δ_Q (31), we find perfect agreement with Eq. (13) only at the fixed point (8). Let us also note that the case $a_e = 0$ corresponds to the $O(2)$ -symmetric model, for which the full six-loop result is available [34].

To facilitate further comparison with perturbative calculations, we provide an explicit expression for the coefficients C_{0l} and C_{1l} appearing in (28) from $l = 4$ to $l = 6$:

$$C_{04} = -\frac{14}{27}\lambda^4, \quad C_{05} = \frac{256\lambda^5}{243}, \quad C_{06} = -\frac{572}{243}\lambda^6, \quad (32)$$

$$\begin{aligned}
 C_{14} = & 2a_e^4(41\zeta_3 - 125\zeta_5 + 17) - \frac{4}{3}a_e^3\lambda(134\zeta_3 - 140\zeta_5 + 25) + \frac{2}{3}a_e^2\lambda^2(80\zeta_3 - 80\zeta_5 + 33) - \frac{8}{27}a_e\lambda^3(7\zeta_3 - 40\zeta_5 + 37) \\
 & - \frac{2}{81}\lambda^4(77\zeta_3 + 80\zeta_5 - 142), \quad (33)
 \end{aligned}$$

$$\begin{aligned}
 C_{15} = & -4a_e^5(224\zeta_3 - 50\zeta_5 - 441\zeta_7 + 95) + \frac{8}{3}a_e^4\lambda(413\zeta_3 - 50\zeta_5 - 525\zeta_7 + 167) - \frac{8}{9}a_e^3\lambda^2(164\zeta_3 - 315\zeta_5 - 140\zeta_7 + 152) \\
 & - \frac{4}{27}a_e^2\lambda^3(368\zeta_3 + 420\zeta_5 - 560\zeta_7 + 217) + \frac{4}{81}a_e\lambda^4(156\zeta_3 - 130\zeta_5 - 560\zeta_7 + 559) \\
 & + \frac{2}{243}\lambda^5(476\zeta_3 + 480\zeta_5 + 448\zeta_7 - 1179), \quad (34)
 \end{aligned}$$

$$\begin{aligned}
 C_{16} = & 2a_e^6(5662\zeta_3 + 462\zeta_5 - 3514\zeta_7 - 7434\zeta_9 + 2487) - \frac{16}{3}a_e^5\lambda(2956\zeta_3 - 319\zeta_5 - 1680\zeta_7 - 2583\zeta_9 + 1270) \\
 & + \frac{4}{9}a_e^4\lambda^2(14932\zeta_3 - 8350\zeta_5 - 10185\zeta_7 - 6300\zeta_9 + 6286) - \frac{16}{27}a_e^3\lambda^3(2413\zeta_3 - 1335\zeta_5 - 910\zeta_7 - 840\zeta_9 + 650) \\
 & + \frac{8}{81}a_e^2\lambda^4(2645\zeta_3 + 515\zeta_5 - 490\zeta_7 - 2520\zeta_9 + 1421) - \frac{4}{243}a_e\lambda^5(1886\zeta_3 - 46\zeta_5 - 2464\zeta_7 - 4032\zeta_9 + 4633) \\
 & - \frac{2}{729}\lambda^6(3294\zeta_3 + 3202\zeta_5 + 3360\zeta_7 + 2688\zeta_9 - 10063). \tag{35}
 \end{aligned}$$

Conclusions.—In the present Letter we took the first step towards the application of the large-charge expansion in gauge theories. We demonstrated that our semiclassical *gauge-independent* results for Δ_Q can be interpreted as scaling dimensions of nonlocal operators $\phi_{nl}^Q(x)$ that extend the Dirac construction to arbitrary Q . As a by-product of our analysis, we confirmed that the adequate critical exponent to describe the broken phase is η_D . Our Letter opens new avenues for the large-charge methods in application to the most general Gauge-Yukawa theories such as the standard model of particle physics and its extensions.

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