Noncommutative double geometry

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We construct noncommutative theories with the Moyal-Weyl product in the double field theory (DFT) framework. We deform the infinitesimal generalized diffeomorphisms and the Leibniz rule in a consistent way. The prescription requires a generalized star metric, which can be thought of as the fundamental double metric, in order to construct the action. Finally we use the generalized scalar field dynamics and the generalized scalar field-perfect fluid correspondence to construct the generalized energy-momentum tensor of a perfect fluid in the noncommutative double geometry. The present formalism paves the way to the study of string cosmologies scenarios including the Moyal-Weyl product in a T-duality invariant way.

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I. INTRODUCTION

T-duality is an exact symmetry of closed string theory [1] which establishes that toroidal backgrounds T^d related via the noncompact group $O(d,d,Z)$ are physically equivalent. When one compactifies the low-energy limit of string theory considering Kaluza-Klein compactifications, the continuous form of the duality group, $O(d,d,R)$, appears as a symmetry. Interestingly enough, the effect of T-duality is producing a noncommutative [2–14] relation in the commutator of the target coordinates and/or its dual coordinates [15–16]. This effect can be thought of as a central extension of the zero-mode operator algebra, an effect set by the string length scale even in cases where the backgrounds are trivial.

At the effective level (supergravity level) it is possible to construct commutative theories where the T-duality group appears as a symmetry before compactification by doubling the geometry and (re)writing all the fundamental fields and parameters via multiplets of the duality group. This framework is known as double field theory (DFT) [17–24]. One interesting aspect about DFT is the existence of a strong constraint or section condition which can be performed in order to get rid of the extra coordinates required by $O(D,D,R)$ invariance, where $D=n+d$ and n is the dimension of the external space. Applying the strong constraint only on the external coordinates it is possible to arrive to a hybrid theory with an n-dimensional exterior space-time and a d-dimensional double internal space [25]. While this is a promising formulation constructed on an arbitrary double internal space, the formulation is purely classical. While this setup can be used to effectively describe theories and its duals, the relation between noncommutativity and gravity has been studied in the double geometry. Another missing ingredient in this type of formulation is the possibility that both the external and internal backgrounds are partially/fully non-Riemannian [26–32].

If we focus on a model which could be reduced to the Einstein-Hilbert action, one possibility is to define a noncommutative geometry considering that the commutator of the coordinates is given by [33]

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

with θ a generic antisymmetric tensor which is also used to defined an algebra $\mathcal{A}_\theta$ through the following star product,

$$f \star g(x) = e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\otimes\frac{\partial}{\partial y^\sigma}} f(x)\otimes g(y)|_{y\to x}, \tag{2}$$

$\mu = 1,\ldots,D-1$. In this work we will focus on the $\theta = const.$ case (Moyal-Weyl product), where Lorentz invariance is broken from the very beginning [34]. Since the ordinary relation

$$g_{\mu\nu}\star g^{\nu\rho} = \delta_\mu^\rho + \mathcal{O}(\theta), \tag{3}$$

receives θ-corrections, one is forced to construct an inverse metric compatible with the star product, $g^{\star\mu\nu}$, which satisfies

$$g_{\mu\nu}\star g^{\star\nu\rho} = \delta_\mu^\rho. \tag{4}$$

While the natural expectation is to deform both symmetries and action using the star product, the natural proposal

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$$\delta_\xi v^\mu = \xi^\nu \star \partial_\nu v^\mu - \partial_\nu \xi^\mu \star v^\nu \tag{5}$$

does not satisfy closure

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{\xi_{21}}. \tag{6}$$

In [35], the authors keep diffeomorphism transformations untouched (with a modification to the Leibniz rule) and they define a consistent action as

$$S = \int d^D x \sqrt{-\mathbf{g}} \star R + \mathrm{cc} = \int d^D x \sqrt{-g} R + \mathcal{O}(\theta^2), \tag{7}$$

where both the determinant of the metric and the Ricci scalar contain θ-contributions [33] and the c.c. (complex conjugate) terms guarantee that the action is real. Therefore the previous action is invariant under infinitesimal diffeomorphisms, paying the cost of deforming the Leibniz rule [35]. In this work we deform the transformation rules using a generic commutative and associative product, so the Leibniz rule is consistently modified but the action remains unchanged. Moreover, the closure of the transformations is also fulfilled. For clarity's sake, we present our main results in the following part.

A. Main results

Our formulation consists of a systematic procedure to construct θ-deformed theories and we mainly focus on the DFT framework. In this work we provide a simple way to generalize this construction defining the notion of DFT[1] on noncommutative double spaces.

We start by considering the noncommutative relation at the level of double coordinates,

$$[X^M, X^N] = i\theta^{MN}, \tag{8}$$

where $M = 0, \ldots, 2D-1$. We define the generalized star metric through

$$\mathcal{H}^\star_{MP} \star \mathcal{H}^{\star PN} = \delta^N_M, \tag{9}$$

since there is a notion of an inverse metric in DFT. The generalized star metric can be expanded in powers of θ as

$$\mathcal{H}^{\star PN} = \mathcal{H}^{PN} - \frac{i}{4}\mathcal{H}^{PR}\theta^{QS}\partial_Q \mathcal{H}_{RT}\partial_S \mathcal{H}^{TN} + \mathcal{O}(\theta^2). \tag{10}$$

Moreover, we consider this new metric as the fundamental field of the theory, so the construction of the noncommutative version of the DFT projectors is given by

$$P^\star_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{H}^\star_{MN}) \quad \text{and} \quad \bar{P}^\star_{MN} = \frac{1}{2}(\eta_{MN} + \mathcal{H}^\star_{MN}), \tag{11}$$

which satisfy the following properties:

$$\bar{P}^\star_{MQ} \star \bar{P}^{\star Q}{}_N = \bar{P}^\star_{MN}, \tag{12}$$

$$P^\star_{MQ} \star P^{\star Q}{}_N = P^\star_{MN},$$
$$P^\star_{MQ} \star \bar{P}^{\star Q}{}_N = \bar{P}^\star_{MQ} \star P^{\star Q}{}_N = 0. \tag{13}$$

Mimicking the procedure given in [44], we define a connection Γ_{MNP} and a covariant Riemann tensor as

$$\mathcal{R}_{MNKL} = R_{MNKL} + R_{KLMN} + \Gamma_{QMN} \star \Gamma^Q{}_{KL}, \tag{14}$$

$$R_{MNKL} = 2\partial_{[M}\Gamma_{N]KL} + 2\Gamma_{[M|QL} \star \Gamma_{N]K}{}^Q. \tag{15}$$

The action is given by

$$\int d^{2D}X\, e^{-2\mathbf{d}} \star P^{\star MN} \star P^{\star QR} \star \mathcal{R}_{MQNR} + \mathrm{c.c.} \tag{16}$$

Finally, we have checked that the inclusion of matter through a generalized and massless scalar field is also consistent with our formalism

$$\mathcal{L}_{\mathrm{matter}}[\mathcal{H}, \Phi] = \frac{1}{2}\partial_M \Phi \star \mathcal{H}^{\star MN} \star \partial_N \Phi - V(\Phi), \tag{17}$$

which is compatible with the free scalar action to all orders in θ after parametrization and imposing the strong constraint. We consider the generalized perfect fluid-scalar field correspondence in order to construct the generalized energy-momentum tensor,

$$\mathcal{T}_{MN} = 4[\bar{P}^\star_{[M|K} \star P^\star_{N]L}](\sqrt{\tilde{e}+\tilde{p}}U_M \star \sqrt{\tilde{e}+\tilde{p}}U_N)$$
$$- \frac{1}{2}\eta_{MN}\sqrt{\tilde{e}+\tilde{p}}U_P \star \mathcal{H}^{\star PQ} \star \sqrt{\tilde{e}+\tilde{p}}U_Q, \tag{18}$$

with U_M a generalized velocity [45] and $\tilde{e}$, $\tilde{p}$ the generalized energy density and pressure respectively. The previous tensor can be used in order to construct the generalized Einstein equation for these geometries, which pave the way to the study of noncommutativity string cosmologies in a T-duality invariant way.

II. NONCOMMUTATIVE GEOMETRY WITH MOYAL-WEYL PRODUCT

A. Symmetry transformations

The algebra $\mathcal{A}_\theta$ is based on the relation,

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{19}$$

[1]See [36–38] for pedagogical reviews and [39–43] for related works.

with $\theta^{\mu\nu}$ constant and real. This algebra can be realized on the linear space $\mathcal{F}$ of complex functions $f(x)$ of commuting variables: The elements of the algebra $\mathcal{A}_\theta$ are represented by functions of the commuting variables $f(x)$, their product by the Moyal-Weyl star product, i.e.,

$$f \star g(x) = e^{\frac{i}{2}\theta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\otimes\frac{\partial}{\partial y^\sigma}} f(x) \otimes g(y)|_{y\to x}. \tag{20}$$

The $\star$-derivative, $\partial^\star_\mu f \coloneqq \partial_\mu f$, satisfies

$$\partial_\mu x^\rho = \delta^\rho_\mu, \tag{21}$$

and the usual product rule with respect to the $\star$-product (from now on we use the standard notation for the star derivative),

$$\partial_\mu(f \star g) = (\partial_\mu f) \star g + f \star (\partial_\mu g). \tag{22}$$

Finally let us construct the Leibniz rule. It is possible to define the infinitesimal diffeomorphisms in the following form for scalars fields:

$$\hat{\delta}_\xi \phi = X^\star_\xi \rhd \phi, \tag{23}$$

and for vector fields,

$$\hat{\delta}_\xi V^\mu = X^\star_\xi \rhd V^\mu + X^\star_{(\partial_\rho \xi^\mu)} \rhd V^\rho. \tag{24}$$

The operators $X^\star_\xi$ and $X^\star_{(\partial_\mu \xi^\lambda)}$ are given by

$$\begin{aligned} X^\star_\xi &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{2}\right)^n \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}(\partial_{\rho_1}\ldots\partial_{\rho_n}\xi^\lambda) \\ &\quad \star\, \partial^\star_{\sigma_1}\ldots\partial^\star_{\sigma_n}\partial^\star_\lambda, \\ X^\star_{(\partial_\mu\xi^\lambda)} &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{i}{2}\right)^n \theta^{\rho_1\sigma_1}\ldots\theta^{\rho_n\sigma_n}(\partial_{\rho_1}\ldots\partial_{\rho_n}\partial_\mu\xi^\lambda) \\ &\quad \star\, \partial^\star_{\sigma_1}\ldots\partial^\star_{\sigma_n}. \end{aligned}$$

The deformed Leibniz rule takes the following form:

$$\begin{aligned} X^\star_\xi \rhd (f \star g) = \mu_\star\{&e^{-\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma}(X^\star_\xi \otimes 1 \\ &+ 1 \otimes X^\star_\xi) e^{\frac{i}{2}\theta^{\rho\sigma}\partial^\star_\rho\otimes\partial^\star_\sigma} \rhd (f \otimes g)\}. \end{aligned} \tag{25}$$

So far we have reviewed the basis of noncommutativity in flat space. It proves illustrative to focus first on the physics of a free massless scalar field in noncommutative geometry. The action in this case is given by

$$S_\phi = \frac{1}{2}\int d^D x \partial_\mu \phi \star \partial^\mu \phi, \tag{26}$$

where the indices are contracted with a flat inverse metric $\eta^{\mu\nu}$.

From the previous Lagrangian, it is simple to observe that the terms created for the θ-expansion are even and real. Moreover, in this case the squared contribution is a total derivative due to the constant and antisymmetric nature of θ. The most interesting point here is that the ϕ-field is no longer a scalar field when the star transformations are expanded. Therefore, we can analyze covariance with respect to the star product or with respect to the ordinary product. While ϕ is a scalar for the former, for the latter its transformation is noncovariant. However, it is straightforward to prove that the action (26) is invariant. In these geometries, Lorentz invariance is broken from the very beginning. For instance, taking a four-dimensional manifold, if the only nonvanishing component of $\theta^{\mu\nu}$ is the θ^{12}-component then Lorentz boosts in the 3-direction and rotations in the 1–2 plane are still preserved as are translations in any direction.

In the next section we will review the basic steps to generalize these concepts to the gravitational case.

B. Noncommutative gravity

For more general scenarios we need the notion of a covariant derivative. The introduction of this operator follows as in the commutative case, in the sense that we introduce a connection $\Gamma^\rho_{\mu\nu}$ in order to define the covariant derivative as

$$\nabla_\mu V_\nu \coloneqq \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} \star V_\rho. \tag{27}$$

The metric of a noncommutative $g_{\mu\nu}$ is defined as a symmetric tensor under infinitesimal diffeomorphisms, and its inverse $g^{\star\mu\nu}$ is constructed such that

$$g_{\mu\nu} \star g^{\star\nu\rho} = \delta^\rho_\mu. \tag{28}$$

Therefore $g^{\star\mu\nu}$ is not symmetric and the noncovariant contributions compensate the noncovariant terms arising from the star product,

$$g^{\star\mu\nu} = g^{\mu\nu} - \frac{i}{2} g^{\mu\xi}\theta^{\rho\sigma}\partial_\rho g_{\xi\epsilon}\partial_\sigma g^{\epsilon\nu} + \mathcal{O}(\theta^2). \tag{29}$$

Using the previous metrics it is possible to determine the torsionless star Levi-Civita connection, defined as

$$\Gamma^\sigma_{\nu\beta} = \frac{1}{2}\Gamma_{\gamma\nu\beta} \star g^{\star\gamma\sigma} = \frac{1}{2}(\partial_\nu g_{\beta\gamma} + \partial_\beta g_{\nu\gamma} - \partial_\gamma g_{\nu\beta}) \star g^{\star\gamma\sigma}, \tag{30}$$

while the Riemann tensor is given by

$$R_{\mu\nu\sigma}{}^\rho = 2\partial_{[\mu}\Gamma^\rho_{\nu]\sigma} + 2\Gamma^\rho_{\kappa[\mu} \star \Gamma^\kappa_{\nu]\sigma}. \tag{31}$$

From the previous expression we can extract the Ricci scalar for the noncommutative geometries as

$$R = g^{\star\mu\nu} \star R_{\nu\sigma\mu}{}^{\sigma}, \tag{32}$$

which transforms as a scalar. Finally, the action can be defined as

$$S = \int d^D x \sqrt{-\mathbf{g}} \star R + \text{c.c.}, \tag{33}$$

where the determinant of the metric also includes star contributions [35].

III. DOUBLE FIELD THEORY: THE COMMUTATIVE CASE

The geometry of DFT is based on a double space equipped with η_{MN}, an invariant metric of $O(D,D)$ and a dynamical metric, the generalized metric $\mathcal{H}_{MN}$. The indices $M, N, \ldots$ are in the fundamental representation of the duality group and are raised and lowered with η^{MN} and η_{MN}, respectively. The generalized metric $\mathcal{H}_{MN}$ encodes the bosonic tensors of the universal NS-NS sector of string theory, namely, a metric tensor $g_{\mu\nu}$ and a b-field $b_{\mu\nu}$. This metric is a tensor under $O(D,D)$ transformations. Infinitesimal $O(D,D)$-transformations acting on an arbitrary double vector read,

$$\delta_h V^M = V^N h_N{}^M, \tag{34}$$

where $h \in O(D,D)$ is an arbitrary parameter. Another symmetry of the theory is given by generalized diffeomorphisms, generated infinitesimally by ξ^M through the generalized Lie derivative, defined by

$$\mathcal{L}_\xi V_M = \xi^N \partial_N V_M + (\partial_M \xi^N - \partial^N \xi_M) V_N, \tag{35}$$

where V_M is an arbitrary generalized vector. The closure of the gauge transformations

$$[\delta_{\xi_1}, \delta_{\xi_2}] = \delta_{\xi_{21}}, \tag{36}$$

is given by the C-bracket

$$\xi_{12}^M(X) = \xi_1^P \frac{\partial \xi_2^M}{\partial X^P} - \frac{1}{2} \xi_1^P \frac{\partial \xi_{2P}}{\partial X_M} - (1 \leftrightarrow 2). \tag{37}$$

The DFT Jacobiator is not trivial (but it is given by a trivial parameter) and therefore the algebraic structure of DFT is given by an L_∞-algebra with a nontrivial l_3 product [46–49], which measures the failure of the Jacobi identity in the double geometry.

On the other hand, the generalized metric is an element of $O(D,D)$ and therefore satisfies

$$\mathcal{H}_{MP} \mathcal{H}^{PN} = \delta_M^N. \tag{38}$$

Using both DFT metrics one can construct the DFT projectors in the following way:

$$P_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{H}_{MN}), \quad \bar{P}_{MN} = \frac{1}{2}(\eta_{MN} + \mathcal{H}_{MN}). \tag{39}$$

The previous projectors satisfy

$$\bar{P}_{MQ} \bar{P}^Q{}_N = \bar{P}_{MN}, \qquad P_{MQ} P^Q{}_N = P_{MN}, \\ P_{MQ} \bar{P}^Q{}_N = \bar{P}_{MQ} P^Q{}_N = 0. \tag{40}$$

One of the purposes of DFT is to define a theory manifestly invariant under $O(D,D)$, which is a symmetry of string theory. Because of that, all the DFT fields and parameters are $O(D,D)$ multiplets or group-invariant objects. Since the dimension of the fundamental representation of $O(D,D)$ is $2D$, the coordinates of DFT are $X^M = (x^\mu, \tilde{x}_\mu)$. The coordinates $\tilde{x}_\mu$ are known as the dual coordinates and are taken away imposing the strong constraint,

$$\partial_M(\partial^M V) = (\partial_M V)(\partial^M W) = 0, \tag{41}$$

where V and W can be products of arbitrary generalized fields or parameters. Solving the previous constraint with $\tilde{\partial}^\mu = 0$, the components of the fields of DFT depend only on x^μ. The parametrization of the invariant metric is given by

$$\eta_{MN} = \begin{pmatrix} 0 & \delta^\mu_\nu \\ \delta^\nu_\mu & 0 \end{pmatrix}, \tag{42}$$

while the parametrization of the generalized metric is given by

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\sigma} b_{\sigma\nu} \\ b_{\mu\sigma} g^{\sigma\nu} & g_{\mu\nu} - b_{\mu\sigma} g^{\sigma\rho} b_{\rho\nu} \end{pmatrix}, \tag{43}$$

where $b_{\mu\nu}$ is the Kalb-Ramond field. It is straightforward to check that the previous parametrization satisfies (38).

The action of DFT is constructed from the following Lagrangian:

$$\mathcal{L} = \frac{1}{8} \mathcal{H}^{MN} \partial_M \mathcal{H}^{KL} \partial_N \mathcal{H}_{KL} - \frac{1}{2} \mathcal{H}^{MN} \partial_N \mathcal{H}^{KL} \partial_L \mathcal{H}_{MK} \\ + 4 \mathcal{H}^{MN} \partial_M \partial_N d + 4 \partial_M \mathcal{H}^{MN} \partial_N d \\ - 4 \mathcal{H}^{MN} \partial_M d \partial_N d - \partial_M \partial_N \mathcal{H}^{MN}, \tag{44}$$

where d is known as the generalized dilaton and it is parametrized as $\partial_\nu d = \partial_\nu \phi - \frac{1}{4} g^{\sigma\rho} \partial_\nu g_{\sigma\rho}$. The full action is given by $S = \int e^{-2d} \mathcal{L} d^{2D} X$ and after parametrization and using the strong constraint, the resulting action coincides with the low energy limit of the universal NS-NS sector of string theory up to total derivatives,

$$S = \int d^d X e^{-2\phi} \sqrt{g} \left(R + 4 \partial_\mu \phi \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right), \tag{45}$$

where $H_{\mu\nu\rho} = 3 \partial_{[\mu} b_{\nu\rho]}$ is the curvature of the Kalb-Ramond field.

IV. NONCOMMUTATIVE DOUBLE FIELD THEORY

A. Vacuum fields and action

The fundamental noncommutative relation between double coordinates is defined as

$$[X^M, X^N] = i\theta^{MN} \tag{46}$$

with θ^{MN} an $O(D,D)$ real multiplet. This algebra can be realized on the linear space $\mathcal{F}$ of complex functions $F(x)$ of commuting variables: The elements of the algebra $\mathcal{A}_\theta$ are represented by functions of the commuting variables $F(X)$, their product by the Moyal-Weyl star product, i.e.,

$$F \star G(X) = e^{\frac{i}{2}\theta^{PQ}\frac{\partial}{\partial X^P}\otimes\frac{\partial}{\partial y^Q}} F(X) \otimes G(Y)|_{Y\to X}, \tag{47}$$

where θ^{PQ} is constant.

The previous product reproduces the ordinary star product if

$$\frac{\partial}{\partial X^P}\theta^{PQ}\frac{\partial}{\partial y^Q}\bigg|_{Y\to X} \to \frac{\partial}{\partial x^\rho}\theta^{\rho\sigma}\frac{\partial}{\partial y^\sigma}\bigg|_{y\to x}, \tag{48}$$

and therefore the relevant component of θ^{MN} is given by $M =^\mu$ and $N =^\nu$ due to the strong constraint.

On the other hand, the $\star$-derivatives satisfies

$$\partial_M X^P = \delta^P_M, \tag{49}$$

and the usual product rule with respect to the $\star$-product,

$$\partial_M(F \star G) = (\partial_M F) \star G + F \star (\partial_M G). \tag{50}$$

We impose a deformed Leibniz rule considering that the generalized transformations can be defined as

$$\hat{\delta}_\xi \Phi = X^\star_\xi \rhd \Phi, \tag{51}$$

for a generalizied scalar field and

$$\hat{\delta}_\xi V^M = X^\star_\xi \rhd V^M + X^\star_{(\partial_P \xi^M)} \rhd V^P - X^\star_{(\partial^P \xi_M)} \rhd V^P, \tag{52}$$

for a generalized vector field.

The generalized metric for noncommutative DFT is given by

$$\mathcal{H}_{MN} = \begin{pmatrix} g^{\mu\nu} & -g^{\mu\sigma}b_{\sigma\nu} \\ b_{\mu\sigma}g^{\sigma\nu} & g_{\mu\nu} - b_{\mu\sigma}g^{\sigma\rho}b_{\rho\nu} \end{pmatrix}, \tag{53}$$

and we define the generalized star metric through

$$\mathcal{H}^\star_{MP} \star \mathcal{H}^{\star PN} = \delta^N_M, \tag{54}$$

which reduces to the condition (38) for $\theta = 0$. The $\mathcal{H}^{\star PN}$ metric now contains its own noncovariant θ-expansion as

$$\mathcal{H}^{\star PN} = \mathcal{H}^{PN} - \frac{i}{4}\mathcal{H}^{PR}\theta^{QS}\partial_Q\mathcal{H}_{RT}\partial_S\mathcal{H}^{TN} + \mathcal{O}(\theta^2). \tag{55}$$

The covariant derivative acting on a generic double vector is defined as

$$\nabla_M V_N = \partial_M V_N - \Gamma_{MN}{}^P \star V_P, \tag{56}$$

where we have introduced a generalized affine connection $\Gamma_{MN}{}^P$ whose transformation properties must compensate the failure of the partial derivative of a tensor to transform covariantly under generalized diffeomorphisms.

We can now demand some properties on the connection, namely:

(i) *Compatibility with* η_{MN}:

$$\nabla_M \eta_{NP} = 0, \tag{57}$$

and then the generalized affine connection is antisymmetric in its last two indices, i.e.,

$$\Gamma_{MNP} = -\Gamma_{MPN}. \tag{58}$$

(ii) *Compatibility with* $\mathcal{H}^\star_{MN}$:

$$\nabla_M \mathcal{H}^\star_{NP} = 0. \tag{59}$$

In order to discuss this item, it is convenient to define the star projectors,

$$P^\star_{MN} = \frac{1}{2}(\eta_{MN} - \mathcal{H}^\star_{MN}),$$
$$\bar{P}^\star_{MN} = \frac{1}{2}(\eta_{MN} + \mathcal{H}^\star_{MN}), \tag{60}$$

which satisfy the following properties:

$$\bar{P}^\star_{MQ} \star \bar{P}^{\star Q}{}_N = \bar{P}^\star_{MN},$$
$$P^\star_{MQ} \star P^{\star Q}{}_N = P^\star_{MN}, \tag{61}$$

$$P^\star_{MQ} \star \bar{P}^{\star Q}{}_N = \bar{P}^\star_{MQ} \star P^{\star Q}{}_N = 0,$$
$$\bar{P}^\star_{MN} + P^\star_{MN} = \eta_{MN}. \tag{62}$$

The projections $\Gamma_{\overline{MNP}}$ and $\Gamma_{\underline{MNP}}$ remain undetermined after imposing $\nabla_M P^\star_{NP} = 0$ and $\nabla_M \bar{P}^\star_{NP} = 0$ as in the noncommutative case.

(iii) *Vanishing torsion*:

$$\Gamma_{[MNP]} = \frac{1}{3}T_{MNP} = 0. \tag{63}$$

Let us observe that the generalized torsion T_{MNP} is antisymmetric in all its indices and transforms as a tensor (unlike $\Gamma_{[MN]P}$).

The noncommutative version of the DFT Lagrangian is given by the following scalar,

$$\mathcal{R} = P^{\star MN} P^{\star QR} \star \mathcal{R}_{MQNR}, \tag{64}$$

where

$$\mathcal{R}_{MNKL} = R_{MNKL} + R_{KLMN} + \Gamma_{QMN} \star \Gamma^{Q}{}_{KL} \tag{65}$$

$$R_{MNKL} = 2\partial_{[M}\Gamma_{N]KL} + 2\Gamma_{[M|QL} \star \Gamma_{N]K}{}^{Q}, \tag{66}$$

while the full action can be written as

$$S = \int d^{2D}X e^{-2d} \star \mathcal{R} + \text{c.c.}, \tag{67}$$

where $e^{-2d} = e^{-2\phi}\sqrt{\mathbf{g}}$. On the other hand the components of the generalized star metric can be easily computed order by order. For example, the first order contributions are given by

$$\mathcal{H}^{\star\rho\nu} = g^{\rho\nu} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu g_{\tau\lambda}\partial_\sigma g^{\nu\tau}g^{\rho\lambda} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\tau\lambda}\partial_\sigma b_{\xi\delta}g^{\nu\tau}g^{\rho\xi}g^{\lambda\delta} + \mathcal{O}(\theta^2), \tag{68}$$

$$\begin{aligned}\mathcal{H}^{\star\rho}{}_{\nu} = {} & -g^{\rho\alpha}b_{\alpha\nu} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu g_{\tau\lambda}\partial_\sigma g^{\tau\xi}b_{\nu\xi}g^{\rho\lambda} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\nu\tau}\partial_\sigma g^{\tau\rho} \\ & - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\tau\lambda}\partial_\sigma g_{\nu\xi}g^{\rho\tau}g^{\lambda\xi} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\tau\lambda}\partial_\sigma b_{\xi\delta}b_{\nu\alpha}g^{\rho\tau}g^{\lambda\xi}g^{\delta\alpha} + \mathcal{O}(\theta^2),\end{aligned} \tag{69}$$

$$\begin{aligned}\mathcal{H}^{\star}_{\rho\nu} = {} & g_{\rho\nu} - b_{\rho\alpha}g^{\alpha\beta}b_{\beta\nu} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu g_{\tau\lambda}\partial_\sigma g^{\tau\xi}b_{\nu\xi}b_{\rho\chi}g^{\lambda\chi} + \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\nu\tau}\partial_\sigma g_{\lambda\xi}b_{\rho\chi}g^{\tau\lambda}g^{\xi\chi} \\ & + \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\tau\lambda}\partial_\sigma g_{\nu\xi}b_{\rho\chi}g^{\tau\xi}g^{\lambda\chi} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\tau\lambda}\partial_\sigma b_{\xi\chi}b_{\nu\delta}b_{\rho\alpha}g^{\tau\xi}g^{\lambda\delta}g^{\chi\alpha} - \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\tau\rho}\partial_\sigma g^{\tau\xi}b_{\nu\xi} \\ & + \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\nu\tau}\partial_\sigma b_{\lambda\rho}g^{\tau\lambda} + \frac{i}{4}\theta^{\mu\sigma}\partial_\mu g_{\nu\tau}\partial_\sigma g^{\tau\lambda}g_{\rho\lambda} + \frac{i}{4}\theta^{\mu\sigma}\partial_\mu b_{\tau\lambda}\partial_\sigma g^{\tau\xi}b_{\nu\chi}g_{\rho\xi}g^{\lambda\chi} + \mathcal{O}(\theta^2).\end{aligned} \tag{70}$$

So far we have presented the construction of the vacuum Lagrangian in terms of the fundamental degrees of freedom, namely, the generalized dilaton and the generalized star metric. In the next part we will briefly discuss the inclusion of matter using a generalized scalar field.

B. Inclusion of matter

Now we focus on an $O(D,D)$ invariant free scalar field Φ coupled to the background content of a noncommutative DFT. For simplicity we consider the massless case, the parametrization of which is given by the ordinary scalar field ϕ. The matter Lagrangian is given by

$$\mathcal{L}_{\text{matter}}[\mathcal{H},\Phi] = \frac{1}{2}\partial_M\Phi \star \mathcal{H}^{\star MN} \star \partial_N\Phi - V(\Phi). \tag{71}$$

To first order in θ, the generalized star metric $\mathcal{H}^{\star\mu\nu}$ contains a b-field contribution. However, it is easy to see that the term depending on the b-field vanishes because of the symmetric contraction given by the derivatives of the scalar field. This effect happens order by order and we recover Eq. (26) exactly. The inclusion of scalar field dynamics in double noncommutative geometry is very promising because of the correspondence between these dynamics and the dynamics of a perfect fluid (see for example [50–51]), the latter also studied in noncommutative scenarios [52–54]. Imposing the generalized version of the correspondence given by

$$U_M = \frac{\partial_M\Phi}{\sqrt{|\partial_P\Phi \star \mathcal{H}^{\star PQ} \star \partial_Q\Phi|}}, \tag{72}$$

$$\tilde{p} = -\frac{1}{2}\partial_P\Phi \star \mathcal{H}^{\star PQ} \star \partial_Q\Phi - V(\Phi), \tag{73}$$

$$\tilde{e} + \tilde{p} = |\partial_P\Phi \star \mathcal{H}^{\star PQ} \star \partial_Q\Phi|, \tag{74}$$

in the energy-momentum tensor of the generalized scalar field,

$$\mathcal{T}_{MN} = \eta_{MN}\mathcal{L}_m + 4\bar{P}^{\star}_{[MK} \star P^{\star}_{N]L}\left(\frac{\delta\mathcal{L}_m}{\delta P^{\star}_{KL}} - \frac{\delta\mathcal{L}_m}{\delta\bar{P}^{\star}_{KL}}\right),$$

we find that the generalized energy-momentum tensor for a perfect fluid coupled to the noncommutative double geometry is given by

$$\begin{aligned}\mathcal{T}_{MN} = {} & 4\bar{P}^{\star}_{[M|K} \star P^{\star}_{N]L}(\sqrt{\tilde{e}+\tilde{p}}U_M \star \sqrt{\tilde{e}+\tilde{p}}U_N) \\ & - \frac{1}{2}\eta_{MN}\sqrt{\tilde{e}+\tilde{p}}U_P \star \mathcal{H}^{\star PQ} \star \sqrt{\tilde{e}+\tilde{p}}U_Q.\end{aligned} \tag{75}$$

The previous generalized energy-momentum tensor describes statistical matter (perfect fluid dynamics) in the double geometry, and contains a tower of higher-derivative terms related to the θ contributions. This object can be used in the RHS of the generalized Einstein equation, which in turn corresponds with the EOM of $\mathcal{H}^{\star}_{MN}$. We leave this issue for future work.

V. DISCUSSION AND OUTLOOK

In this work we have extended the DFT construction in order to include noncommutative effects through a Moyal-Weyl product. We have deformed generalized diffeomorphism transformations in a consistent way; expanding explicitly the θ-contributions gives the transformation rules of the fundamental fields. We expect $\theta \propto \hbar$, and therefore the symmetries are preserved at the classical level. We catalog here the main differences between our construction and the standard (or classical) construction of DFT:

(i) The fundamental fields of the theory are given by the generalized star metric and the generalized dilaton. While both fields might have their own θ-expansion, the former has to include contributions in order to guarantee Eq. (54).

(ii) The projectors $P^{\star}$ and $\bar{P}^{\star}$ are no longer symmetric due to the θ-expansion of the generalized star metric. This is similar to the antisymmetric deformation of the inverse metric in the noncommutative GR context.

(iii) Both the duality transformations and the generalized diffeomorphisms are deformed, and as such they introduce noncovariant terms in the action from the point of view of ordinary DFT. Since the b-field is part of the components of the generalized star-metric, the Abelian gauge transformations are not preserved when $\theta \neq 0$.

(iv) The construction of (53) can be done considering the generalized frame formalism of DFT [55–56]. In that case, the form of (53) remains the same, but $g^{\mu\nu} = e^{\mu}{}_{a} \star e^{\nu a} + e^{\nu}{}_{a} \star e^{\mu a}$ and $g_{\mu\nu} = e_{\mu a} \star e_{\nu}{}^{a} + e_{\nu a} \star e_{\mu}{}^{a}$. This means that $\mathcal{H}_{MN}$ might contain its own θ-expansion.

We finish this section discussing some future directions:

(i) Covariant deformations: While the present analysis is based on a deformation of the generalized symmetries from the point of view of the commutative DFT, one can try to preserve the covariance of the star product using the covariant derivative as [34]

$$F \star G(X) = e^{\frac{i}{2}\theta \nabla_X \otimes \nabla_Y} F(X) \otimes G(Y)|_{Y \to X}. \quad (76)$$

The main difficulty of these kinds of deformations is that on the one hand the resulting product is not associative anymore and, on the other hand, one has the problem of the undetermined connection Γ_{MNP}. However it would be interesting to study if these undetermined projections vanish from the resulting action.

(ii) L_{∞} structure and Hopf algebras: The L_{∞} structure of DFT is well-known [44] in the commutative case. Extending these studies to include the noncommutative case might be adequate to find straightforward generalizations to braided symmetries [57] from the L_{∞} algebra [58–61]. Furthermore, from the rewriting of the diffeomorphisms in the double geometry given in (52) it is possible to study generalizations of the standard Hopf algebra from a T-duality invariant perspective [33].

(iii) α'-corrections: The terms produced by the $\circ$- and $\star$-product expansion are higher-derivative corrections. In case of covariant deformations, these terms could be related to the structure of higher-derivative corrections of DFT (see [62] for a review) as in [63–65]. Exploring the interplay between noncommutativity and covariant higher corrections might be an alternative way of avoiding some present obstructions of the double geometry as noticed in [66].

(iv) Non-Riemannian geometries: One possibility is to extend our formalism in order to describe non-Riemannian geometries such as stringy Newton-Cartan geometry [67–68]. This means that the generalized metric now is parametrized considering an $(n, \bar{n})$ decomposition as in [26–27]. Since T-duality along the longitudinal direction of this theory describes a relativistic string theory on a Lorentzian geometry with a compact light-like isometry [69–70], it is to be expected to find within a DFT a universal encoding of both Riemannian and non-Riemannian geometries.

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[1] A. Sen, $O(d) \times O(d)$ symmetry of the space of cosmological solutions in string theory, scale factor duality and two-dimensional black holes, Phys. Lett. B **271**, 295 (1991).
[2] A. Connes, M. R. Douglas, and A. S. Schwarz, Noncommutative geometry and matrix theory: Compactification on Tori, J. High Energy Phys. 02 (1998) 003.
[3] J. Madore, S. Schraml, P. Schupp, and J. Wess, Gauge theory on noncommutative spaces, Eur. Phys. J. C **16**, 161 (2000).
[4] M. R. Douglas and N. A. Nekrasov, Noncommutative field theory, Rev. Mod. Phys. **73**, 977 (2001).
[5] R. J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rep. **378**, 207 (2003).
[6] G. Veneziano, A stringy nature needs just two constants, Europhys. Lett. **2**, 199 (1986).
[7] D. J. Gross and P. F. Mende, String theory beyond the Planck scale, Nucl. Phys. **B303**, 407 (1988).
[8] D. Amati, M. Ciafaloni, and G. Veneziano, Superstring collisions at Planckian energy, Phys. Lett. B **197**, 81 (1987).
[9] D. Amati, M. Ciafaloni, and G. Veneziano, Can space-time be probed below the string size?, Phys. Lett. B **216**, 41 (1989).
[10] D. Amati, M. Ciafaloni, and G. Veneziano, Classical and quantum gravity effects from Planckian energy superstring collisions, Int. J. Mod. Phys. A **03**, 1615 (1988).
[11] D. Amati, M. Ciafaloni, and G. Veneziano, Higher order gravitational deflection and soft bremsstrahlung in Planckian energy superstring collisions, Nucl. Phys. **B347**, 530 (1990).
[12] K. Konishi, G. Paffuti, and P. Provero, Minimal physical length and the generalized uncertainty principle in string theory, Phys. Lett. B **234**, 276 (1990).
[13] N. Seiberg and E. Witten, String theory and noncommutative geometry, J. High Energy Phys. 09 (1999) 032.
[14] G. Landi, F. Lizzi, and R. J. Szabo, String geometry and the noncommutative torus, Commun. Math. Phys. **206**, 603 (1999).
[15] L. Freidel, R. G. Leigh, and D. Minic, Intrinsic noncommutativity of closed string theory, J. High Energy Phys. 09 (2017) 060.
[16] L. Freidel, R. G. Leigh, and D. Minic, Noncommutativity of closed string zero modes, Phys. Rev. D **96**, 066003 (2017).
[17] W. Siegel, Two vierbein formalism for string inspired axionic gravity, Phys. Rev. D **47**, 5453 (1993).
[18] W. Siegel, Superspace duality in low-energy superstrings, Phys. Rev. D **48**, 2826 (1993).
[19] C. Hull and B. Zwiebach, Double field theory, J. High Energy Phys. 09 (2009) 099.
[20] O. Hohm, C. Hull, and B. Zwiebach, Generalized metric formulation of double field theory, J. High Energy Phys. 08 (2010) 008.
[21] J. H. Park, Comments on double field theory and diffeomorphisms, J. High Energy Phys. 06 (2013) 098.
[22] D. S. Berman, M. Cederwall, and M. J. Perry, Global aspects of double geometry, J. High Energy Phys. 09 (2014) 066.
[23] I. Jeon, K. Lee, and J.-H. Park, Stringy differential geometry, beyond Riemann, Phys. Rev. D **84**, 044022 (2011).
[24] I. Jeon, K. Lee, and J.-H. Park, Differential geometry with a projection: Application to double field theory, J. High Energy Phys. 04 (2011) 014.
[25] O. Hohm and H. Samtleben, Gauge theory of Kaluza-Klein and winding modes, Phys. Rev. D **88**, 085005 (2013).
[26] K. Morand and J.-H. Park, Classification of non-Riemannian doubled-yet-gauged spacetime, Eur. Phys. J. C **77**, 685 (2017); **78**, 901(E) (2018).
[27] K. Cho and J.-H. Park, Remarks on the non-Riemannian sector in double field theory, Eur. Phys. J. C **80**, 101 (2020).
[28] C. D. A. Blair, A worldsheet supersymmetric Newton-Cartan string, J. High Energy Phys. 10 (2019) 266.
[29] J.-H. Park and S. Sugimoto, String theory and non-Riemannian geometry, Phys. Rev. Lett. **125**, 211601 (2020).
[30] A. D. Gallegos, U. Gürsoy, S. Verma, and N. Zinnato, Non-Riemannian gravity actions from double field theory, J. High Energy Phys. 06 (2021) 173.
[31] C. Blair, G. Oling, and J.-H. Park, Non-Riemannian isometries from double field theory, J. High Energy Phys. 04 (2021) 072.
[32] E. Lescano, $\mathrm{N}=1$ supersymmetry and non-Riemannian double field theory, arXiv:2212.13853.
[33] P. Aschieri, M. Dimitrijevic, P. Kulish, F. Lizzi, and J. Wess, *Noncommutative Spacetimes: Symmetries in Noncommutative Geometry and Field Theory* (Springer Berlin, Heidelberg, 2009).
[34] E. Harikumar and Victor O. Rivelles, Noncommutative gravity, Classical Quantum Gravity **23**, 7551 (2006).
[35] P. Aschieri, M. Dimitrijevic, F. Meyer, and J. Wess, Noncommutative geometry and gravity, Classical Quantum Gravity **23**, 1883 (2006),
[36] G. Aldazabal, D. Marques, and C. Nuñez, Double field theory: A pedagogical review, Classical Quantum Gravity **30**, 163001 (2013).
[37] O. Hohm, D. Lust, and B. Zwiebach, The spacetime of double field theory: Review, remarks, and outlook, Fortschr. Phys. **61**, 926 (2013).
[38] D. S. Berman and D. C. Thompson, Duality symmetric string and M-theory, Phys. Rep. **566**, 1 (2014).
[39] D. Lust, T-duality and closed string non-commutative (doubled) geometry, J. High Energy Phys. 12 (2010) 084.
[40] C. D. A. Blair, Non-commutativity and non-associativity of the doubled string in non-geometric backgrounds, J. High Energy Phys. 06 (2015) 091.
[41] A. Chatzistavrakidis, L. Jonke, F. S. Khoo, and R. J. Szabo, Double field theory and membrane sigma-models, J. High Energy Phys. 07 (2018) 015.
[42] G. Aldazabal, E. Andrés, M. Mayo, and V. Penas, Symmetry enhancement interpolation, non-commutativity and double field theory, J. High Energy Phys. 03 (2019) 012.
[43] L. Alfonsi and D. S. Berman, Double field theory and geometric quantisation, J. High Energy Phys. 06 (2021) 059.
[44] O. Hohm and B. Zwiebach, On the Riemann tensor in double field theory, J. High Energy Phys. 05 (2012) 126.
[45] E. Lescano and N. Miron-Granese, Double field theory with matter and its cosmological application, Phys. Rev. D **107**, 046016 (2023).
[46] O. Hohm and B. Zwiebach, L_∞ algebras and field theory, Fortschr. Phys. **65**, 1700014 (2017).
[47] Y. Cagnacci, T. Codina, and D. Marques, L_∞ algebras and tensor hierarchies in exceptional field theory and gauged supergravity, J. High Energy Phys. 01 (2019) 117.

[48] C. J. Grewcoe and L. Jonke, Double field theory algebroid and curved L_∞, J. Math. Phys. (N.Y.) **62**, 5 (2021).

[49] E. Lescano and M. Mayo, Gauged double field theory as an L_∞ algebra, J. High Energy Phys. 06 (2021) 058.

[50] E. Lescano and N. Miron-Granese, On the phase space in double field theory, J. High Energy Phys. 07 (2020) 239.

[51] E. Lescano and N. Miron-Granese, Double field theory with matter and the generalized Bergshoeff-de Roo identification, Phys. Rev. D **107**, 086008 (2023).

[52] M. Saitou, K. Bamba, and A. Sugamoto, Hydrodynamics on non-commutative space: A step toward hydrodynamics of granular materials, Prog. Theor. Exp. Phys. **2014**, 103B03 (2014).

[53] P. Das and S. Ghosh, Noncommutative geometry and fluid dynamics, Eur. Phys. J. C **76**, 627 (2016).

[54] K. Ma, Y.-J. Ren, Y.-H. Wang, and K.-Q. Liu, Probing noncommutativities of phase space by using persistent charged current and its asymmetry, Phys. Rev. D **97**, 115011 (2018).

[55] O. Hohm and S. K. Kwak, Frame-like geometry of double field theory, J. Phys. A **44**, 085404 (2011).

[56] O. Hohm and S. K. Kwak, Double field theory formulation of heterotic strings, J. High Energy Phys. 06 (2011) 096.

[57] M. Dimitrijević Ćirić, G. Giotopoulos, V. Radovanović, and R. J. Szabo, Braided L_∞-algebras, braided field theory and noncommutative gravity, Lett. Math. Phys. **111**, 148 (2021).

[58] C. J. Grewcoe, L. Jonke, T. Kodzoman, and G. Manolakos, From Hopf algebra to braided L_∞-algebra, Universe **8**, 222 (2022).

[59] R. Blumenhagen, M. Fuchs, and M. Traube, On the structure of quantum L_∞ algebras, J. High Energy Phys. 10 (2017) 163.

[60] R. Blumenhagen, I. Brunner, V. Kupriyanov, and D. Lust, Bootstrapping non-commutative gauge theories from L_∞ algebras, J. High Energy Phys. 05 (2018) 097.

[61] O. Hohm, V. Kupriyanov, D. Lust, and M. Traube, Constructions of L_∞ algebras and their field theory realizations, Adv. Theor. Math. Phys. **2018**, 9282905 (2018).

[62] E. Lescano, α'–corrections and their double formulation, J. Phys. A **55**, 053002 (2022).

[63] O. Hohm, W. Siegel, and B. Zwiebach, Doubled α'-geometry, J. High Energy Phys. 02 (2014) 065.

[64] O. Hohm and B. Zwiebach, Double metric, generalized metric, and α'-deformed double field theory, Phys. Rev. D **93**, 064035 (2016).

[65] E. Lescano and D. Marques, Second order higher-derivative corrections in double field theory, J. High Energy Phys. 06 (2017) 104.

[66] S. Hronek and L. Wulff, $O(D,D)$ and the string α' expansion: An obstruction, J. High Energy Phys. 04 (2021) 013.

[67] E. A. Bergshoeff, K. T. Grosvenor, C. Simsek, and Z. Yan, An action for extended string Newton-Cartan gravity, J. High Energy Phys. 01 (2019) 178.

[68] E. A. Bergshoeff, J. Gomis, J. Rosseel, C. Simsek, and Z. Yan, String theory and string Newton-Cartan geometry, J. Phys. A **53**, 014001 (2020).

[69] E. Bergshoeff, J. Gomis, and Z. Yan, Nonrelativistic string theory and T-duality, J. High Energy Phys. 11 (2018) 133.

[70] J. Kluson, Remark about non-relativistic string in Newton-Cartan background and null reduction, J. High Energy Phys. 05 (2018) 041.