Lepton phenomenology of Stueckelberg portal to dark sector

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We propose an extension of the Standard Model (SM) with a $U_{A'}(1)$ gauge-invariant dark sector connected to the SM via a new portal—the Stueckelberg portal, arising in the framework of dark photon A' mass generation via the Stueckelberg mechanism. This portal opens through the effective $\dim = 5$ operators constructed from the covariant term of the auxiliary Stueckelberg scalar field σ providing flavor nondiagonal renormalizable couplings of both σ and A' to the SM fermions ψ. The Stueckelberg scalar plays a role of Goldstone boson in the generation of mass of the dark photon. Contrary to the conventional kinetic mixing portal, in our scenario, flavor diagonal $A' - \psi$ couplings are not proportional to the fermion charges and are, in general, flavor nondiagonal. These features drastically change the phenomenology of dark photon A' relaxing or avoiding some previously established experimental constraints. We focus on the phenomenology of the described scenario of the Stueckelberg portal in the lepton sector and analyze the contribution of the dark sector fields A' to the anomalous magnetic moment of muon $(g-2)_\mu$, lepton flavor–violating decays $l_i \to l_k \gamma$, and $\mu - e$ conversion in nuclei. We obtain limits on the model parameters from the existing data on the corresponding observables.

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I. INTRODUCTION

The idea of the dark sector (DS) of the Universe, existing almost independently of the Standard Model (SM) sector, has attracted growing interest in recent years. Originally, DS was thought to be populated by only one dark species, necessary to make up for the lack of matter in the Universe with dark matter (DM). In particular, extensions of DS were motivated by the popular scenario of light sub-GeV DM. It was realized that in this case a dark boson, known as the dark photon, would need to be introduced to prevent the Universe from overclosing. An extended DS can have not only cosmological but also interesting phenomenological consequences. This DS physics beyond the SM can manifest itself in the phenomena observable experimentally (for a status report see, e.g., Ref. [1]).

Presently, there are a number of experiments to search for DS physics, and others are planned for the near future. Among them, we mention CERN-based experiments NA64 [2,3], NA62 [4], SHiP [5–7], LHCb [8], ATLAS [9], and CMS [10] and the *BABAR* experiment at SLAC [11], HPS at JLab [12], and Belle at KEK [13]. So far, no signal of DS or another kind of new physics beyond the SM (BSM) is observed.

An encouraging indication of new physics has recently come from measurements of the anomalous magnetic moment (AMM) of the muon $(g-2)_\mu$. The Fermilab Muon $g-2$ Collaboration published [14] the observation of 4.2σ deviation of the $(g-2)_\mu$ from its SM value and stimulated an explosion of the BSM literature. As is known, measurements of $(g-2)_\mu$ are a very sensitive probe of BSM physics. The Fermilab Muon $g-2$ result with such unprecedented precision can severely limit or refute many BSM models.

On the other hand, there is no doubt that the SM is an incomplete theory, requiring some physics beyond its scope to explain a number of problems that cannot be addressed in the SM. Among them, the DM problem is one of the most obvious. As we already mentioned, DM hints at the existence of a DS of the Universe, which not only provides DM particle candidates but is also populated by other particles involved in interactions governed by some dark symmetries. The DS with possible nontrivial physics could have a phenomenological impact on the SM sector through portals such as the well-known kinetic mixing of dark and normal photons. Other hypothetical DS particles can have access to the SM sector through different portals and contribute to various observables, in particular, to $(g-2)_\mu$, allowing one to probe the DS.

We should stress that there is much evidence of deviation from SM. Besides $(g-2)$ of muon [15,16], evidence includes the strong CP problem and rare meson decays [17–21], flavor nonuniversality [22,23], the $b-s$ quark anomaly, and others. This motivates theoretical study/construction of effective Lagrangians beyond Standard Model trying to involve new particles/portals, like the axion, dark photon, vectors, pseudoscalars, scalars, axials, etc., [1,5,24–56].

Here, we propose an extension of the SM by inclusion of DS with $U_{A'}(1)$ symmetry. The corresponding gauge boson A', also known as the dark photon, acquires a nonzero gauge-invariant mass via the Stueckelberg mechanism [57,58], which implies the existence of a scalar Stueckelberg field σ, which is unphysical.

This field opens a new portal from the SM to the dark sector via the effective dimension-5 operator with the covariant derivative of the σ field. We call it the Stueckelberg portal. In our setup, this portal coexists with the conventional kinetic mixing portal and leads to new phenomenological effects in the SM sector, in particular, flavor violation both in the lepton and quark sectors. In the present work, we focus on the lepton flavor violation (LFV) and the corresponding experimental observables.

We also introduce one dark fermion, χ, charged under $U(1)_D$, which is a viable light DM particle candidate. We postulate that the dark scalar boson (DSB) plays an important role in this model: (i) generating mass of the dark gauge boson (DGB) or dark photon, via the Stueckelberg mechanism [57,58]; (ii) generating a mixing of DS with the SM fermion including couplings preserving and violating symmetries of SM (e.g., LFV) [interaction of DGB and DSB with fermions is based on the idea of a familon (or flavons)] [35,45,59]; (iii) playing the role auxiliary field for DGB and reducing degrees of freedom of one of something.

The paper is organized as follows. In Sec. II, we describe our theoretical setup. In Secs. III, IV, and V, we consider application of the Stueckelberg portal to phenomenological aspects of the $g-2$ lepton anomaly, lepton conversion, and rare lepton-flavor-violating decays $l_i \to l_k\gamma$ which were used to derive limits for couplings occurring in the Stueckelberg portal. In Sec. VI, we discuss the boundary to DGB couplings and a possible contribution to the $g-2$ lepton from obtained restrictions for different channels. Section VII is the conclusion. Technical details of our calculations are placed in Appendixes.

II. THEORETICAL SETUP

We consider an extension the SM with a $U_{A'}(1)$ gauge invariant dark sector described by the Lagrangian

$$\mathcal{L}_{\rm SM+DS} = \mathcal{L}_{\rm SM} + \mathcal{L}_{\rm DS} + \mathcal{L}_{\rm int}, \tag{1}$$

where $\mathcal{L}_{\rm SM}$ and $\mathcal{L}_{\rm DS}$ are the SM and dark sector Lagrangians. Communication between these two sectors takes place through a portal interaction Lagrangian $\mathcal{L}_{\rm int}$.

We suppose that the DS, blind to the SM interactions, is populated with Dirac fermions χ_i charged under $U_{A'}(1)$, the lightest of which is stable and plays the role of dark matter. By definition, all the SM fields are neutral with respect to this group. The gauge boson, A', of the dark sector $U_{A'}(1)$ group is called the dark photon. In the conventional dark photon scenario, A' acquires its mass $m_{A'}$ from spontaneous breaking of the $U_{A'}(1)$ group. In contrast, in our approach, its mass is a gauge-invariant quantity generated by the Stueckelberg mechanism. This requires the introduction of a scalar Stueckelberg field σ, which plays a role of Goldstone boson in the generation of mass of the dark photon. The $U_{A'}(1)$ gauge-invariant Stueckelberg Lagrangian $\mathcal{L}_{\rm DS}$ with one specie of dark fermion, χ, reads

$$\mathcal{L}_{\rm DS} = -\frac{1}{4}A'_{\mu\nu}A'^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \bar{\chi}(i\not{D}_\chi - m_\chi)\chi, \tag{2}$$

where, as usual, $A'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ and $(iD_\chi)_\mu = i\partial_\mu - g_{A'}A'_\mu$ is the conventional covariant derivative.

The Stueckelberg covariant derivative is defined as

$$D_\mu\sigma = \partial_\mu\sigma - m_{A'}A'_\mu. \tag{3}$$

The $U_{A'}(1)$ symmetry is realized on the dark sector fields according to the transformations

$$A'_\mu \to A'_\mu + \frac{i}{g_{A'}}\partial_\mu\Omega_{A'}\Omega_{A'}^{-1}, \quad A'_{\mu\nu} \to A'_{\mu\nu}, \tag{4}$$

$$\sigma \to \sigma - \frac{m_{A'}}{g_{A'}}\theta_{A'}, \quad \partial_\mu\sigma \to \partial_\mu\sigma + \frac{im_{A'}}{g_{A'}}\partial_\mu\Omega_{A'}\Omega_{A'}^{-1},$$
$$D_\mu\sigma \to D_\mu\sigma, \tag{5}$$

$$\chi_D \to \Omega_{A'}\chi_D, \quad i\not{D}_{\chi_D}\chi_D \to \Omega_{A'} i\not{D}_{\chi_D}\chi_D, \tag{6}$$

where

$$\Omega_{A'}(x) = \exp[i\theta_{A'}(x)]. \tag{7}$$

As seen from (5), the σ is an axionlike field shift transformed under the $U_{A'}(1)$. Quantization of the $U(1)_{A'}$ dark sector requires adding to the Lagrangian (2) a gauge-fixing term [32]. We choose it in the form

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial_\mu A'^\mu + \xi m_{A'}\sigma)^2, \tag{8}$$

where ξ is a gauge parameter. Then, the dark sector Lagrangian can be written as

$$\mathcal{L}'_{\rm DS} = \mathcal{L}_{\rm DS} + \mathcal{L}_{gf} = -\frac{1}{4}A'_{\mu\nu}A'^{\mu\nu} + \frac{m_{A'}^2}{2}A'_\mu A'^\mu + \bar{\chi}(i\not{D}_\chi - m_\chi)\chi - \frac{1}{2\xi}(\partial_\mu A'^\mu)^2 + \frac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \xi\frac{m_{A'}^2}{2}\sigma^2.$$

In this gauge, the Stueckelberg field is decoupled from other fields, making the theory manifestly unitary and renormalizable. Note that the mass of the σ field is proportional to the gauge parameter ξ, signaling that this field is unphysical.

In the gauge (8), the dark photon propagator takes the form

$$D_A^{\mu\nu}(k;\xi) = \frac{1}{k^2 - m_{A'}^2}\left[g^{\mu\nu} - \frac{k^\mu k^\nu(1-\xi)}{k^2 - \xi m_{A'}^2}\right]. \tag{9}$$

Let us turn to the SM-DS portals $\mathcal{L}_{\rm port}$ possible in the present model. The well-known example of these is the generic renormalizable portal given by kinetic mixing of the dark and the SM photons, $A - A'$, according to

$$\mathcal{L}_{\rm mix} = -\frac{\epsilon_A}{2}F_{\mu\nu}A'^{\mu\nu}, \tag{10}$$

where ϵ_A is the mixing parameter. It has a rather particular phenomenology, which we comment on latter.

In the Stueckelberg framework, there exists another specific portal, which relies on the $SU_c(3)\times SU_L(2)\times U_Y(1)\times U_{A'}(1)$ gauge-invariant effective operator

$$\mathcal{L}_{\rm int} = \frac{1}{\Lambda}D_\mu\sigma\sum_{ij}[\bar{Q}^i\chi^{ij}\gamma^\mu Q^j + \bar{u}_R^i\chi_u^{ij}\gamma^\mu u_R^j + \bar{d}_R^i\chi_d^{ij}\gamma^\mu d_R^j + \bar{L}^{im}\kappa_L^{ij}\gamma^\mu L^{jm} + \bar{\ell}_R^i\kappa_R^{ij}\gamma^\mu \ell_R^j]. \tag{11}$$

The fields in the expression (11) belong to the following $Q(3;2;1/3;0)$, $u_R(3;1;4/3;0)$, $d_R(3;1;-2/3;0)$, $L(1;2;-1;0)$, $l_R(1;1;-2;0)$ representations of the $SU_c(3)\times SU_L(2)\times U_Y(1)\times U_{A'}(1)$ group. The parameter Λ is the characteristic scale of this effective operator, defining when it opens up in terms of renormalizable interactions of a UV completion. The parameters χ^{ij} and κ^{ij} form $3\otimes 3$ Hermitian matrices leading to the neutral current flavor violation both in quark and lepton sectors. In the present work, we focus only on the lepton sector and make an *ad hoc* assumption $\chi = \chi_u = \chi_d = 0$.

Let us look closely at the effective operator (11). At first glance, it looks like a nonrenormalizable operator of dimension 5. However, after the substitution of the expression (3), we find that the gauge-invariant operator (11) generates dimension-4 interactions of the dark photon with the SM fermions ψ in the form

$$\mathcal{L}^{A'-\psi} = A'_\mu\sum_{ij}\bar{\psi}_i\gamma^\mu(g_{ij}^V + g_{ij}^A\gamma_5)\psi_j, \tag{12}$$

where vector g^V and axial-vector g^A dimensionless couplings are defined as

$$g_{ij}^V = \frac{m_{A'}}{\Lambda}v_{ij} \text{ and } g_{ij}^A = \frac{m_{A'}}{\Lambda}a_{ij}, \quad v_{ij} = \frac{1}{2}(\kappa_R + \kappa_L)_{ij} \text{ and } a_{ij} = \frac{1}{2}(\kappa_R - \kappa_L)_{ij}. \tag{13}$$

As seen, these couplings are linearly scaled with the dark photon mass $m_{A'}$, which is crucial for our analysis of the A' contribution to the lepton sector observables and setting limits on the corresponding couplings in function of the intermediate-state mass.

The operator (11) also contains the interaction of the unphysical Goldstone-like field with the SM fermions of the form $j^\mu\partial_\mu\sigma$. In principle, these interactions should be taken into account in calculations made in the R_ξ gauge (8) with the arbitrary parameter ξ. To avoid this complication, we select from now on the unitary-type gauge $\xi\to\infty$ in which, as seen from (9), the σ becomes infinitely heavy and decouples completely from the observable sector. Therefore, the only physical interactions generated by (11) in this gauge are due to the renormalizable couplings of the dark photon to the SM fermions (12).

These interactions also absorb the kinetic portal (10). In fact, the latter can be removed from the Lagrangian by the conventional field redefinition converting its effect to the flavor diagonal vector interactions of the form (12). As usual, we shift the SM photon field

$$A_\mu \to A_\mu - \epsilon_A A'_\mu \tag{14}$$

and, then, rescale the dark photon field

$$A'_\mu \to A'_\mu(1-\epsilon_A^2)^{-1/2}. \tag{15}$$

These redefinitions generate flavor diagonal couplings of the SM fermions ψ to the dark photon,

$$\mathcal{L}_{\rm mix}^{A'-\psi} = e\epsilon_A A'_\mu \bar{\psi}_i \gamma^\mu T_Q \psi_i, \qquad (16)$$

originating from the kinetic mixing term (10). Here, T_Q is the charge matrix of SM fermions.These interactions feature the characteristic property of the kinetic portal requiring the A' couplings to the SM fermions to be proportional to their electric charges. It is clear that terms (16) are completely absorbed by redefinition of the flavor diagonal vector couplings g_{ii}^V in (12).

Thus, our model contains the following free parameters: g_{ij}^V, g_{ij}^A, and $m_{A'}$ with $g_{ij}^{V,A} = g_{ji}^{V,A}$. Let us highlight two principal differences between the conventional kinetic portal and the Stueckelberg portal scenarios of the dark sector. First, in the latter case, contrary to the former one, the A' couplings to the SM fermions are not proportional to the SM fermion electric charges. Second, these couplings are flavor nondiagonal, leading to reach LFV phenomenology. Note that the first point can significantly affect the conclusions following from the existing searches of dark photon. In particular, the conventional dark photon from the kinetic portal scenario has been strongly constrained from the data of the NA64 experiment at SPS CERN [2,3]. For the Stueckelberg dark photon, these constraints can be significantly relaxed.

In the subsequent sections, we will study contributions of the dark photon A' to muon anomalous magnetic moment $(g-2)_\mu$ and LFV decays $l_i \to l_k \gamma$ as well as $\mu - e$ conversion in nuclei.

III. ANOMALOUS MAGNETIC MOMENT

In the Stueckelberg portal scenario, the SM leptons l receive the dark photon A' one-loop contributions to their $(g-2)_l$ shown in Fig. 1. The loop involves A' due to its couplings to the l and f SM fermions according to Eqs. (12). We calculate the corresponding A' loop contribution in the unitary gauge, setting $\xi \to \infty$ in (9). In this case, the σ field is decoupled from low-energy theory, as commented in the previous section.

The first calculation of the vector and axial contributions to the lepton anomalous moments in the R_ξ gauge (9), taking into account LFV, was made in Ref. [37]. As it was noted in Ref. [37], the axial contributions δa_l^A can obtained from the vector ones δa_l^V by inverting the sign in front of the mass of the internal lepton $m_f \to -m_f$. In particular, the δa_l^V and δa_l^A contributions due to exchange of the dark photon, A', read [37]

$$\delta a_l^V = \frac{(g_{lf}^V)^2}{4\pi^2} y_l \int_0^1 dx \frac{1-x}{\Delta(x, y_A, y_l)} \left[x(2 - y_l(1+x)) + \frac{(1-y_l)^2}{2y_A^2}(1+y_l x)(1-x) \right], \qquad (17)$$

$$\delta a_l^A = -\frac{(g_{lf}^A)^2}{4\pi^2} y_l \int_0^1 dx \frac{1-x}{\Delta(x, y_A, y_l)} \left[x(2 + y_l(1+x)) + \frac{(1+y_l)^2}{2y_A^2}(1-y_l x)(1-x) \right], \qquad (18)$$

where we defined $y_l = m_l/m_f$, $y_A = m_{A'}/m_f$, and $\Delta(x, a, b) = a^2 x + (1-x)(1-b^2 x)$. The dimensionless couplings are defined in Eq. (13). For convenience, we present details of the calculations of these integrals in Appendix A.

The recent experimental measurements of the anomalous magnetic dipole moments of muon and electron $a_{\mu,e} = (g_{\mu,e} - 2)/2$ demonstrate conspicuous deviation from the predictions of the SM. In particular, defining $\Delta a_\ell = a_\ell^{\rm exp} - a_\ell^{\rm SM}$, one gets

$$\Delta a_\mu = (2.51 \pm 0.59) \times 10^{-9}\, [14, 15, 60\text{–}63] \qquad (19)$$

$$\Delta a_e = (8.7 \pm 0.5) \times 10^{-13}\, [64],$$
$$\Delta a_e = (4.8 \pm 3.0) \times 10^{-13}\, [65]. \qquad (20)$$

In the case of the muon, the value $a_\mu^{\rm exp}$ was extracted from the combined data of the E821 experiment at BNL [66] and recent Muon $(g-2)$ measurements at FNAL [14]. This experimental result shows 4.2σ deviation from the SM prediction. The value for Δa_e was derived from the recent measurement of the fine-structure constant [65]. For completeness, we also include in our analysis the same observable for the τ lepton,

$$\Delta a_\tau = (2.79) \times 10^{-4}\, [67, 68]. \qquad (21)$$

Its precision is significantly worse than for the case of e and μ. This is due to the experimental difficulties in measuring the properties of such a short-lived particle as the τ lepton. For rough estimations, we will use the central value of Δa_τ in (21).

We compare our theoretical predictions (17) and (18) with the experimental data (19), (20), and (21). First, we extract upper limits on the coupling constants $g_{ij}^{V,A}$ of the

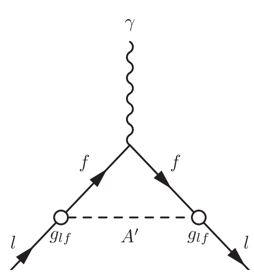

FIG. 1. Feynman diagrams describing the contributions of the dark sector vector A' to the anomalous magnetic moments δa_l of the leptons, taking into account flavor nondiagonal $l - f$ couplings, where $l, f = e, \mu, \tau$.

dark photon to the SM fermions. Then, in Sec. VI, we will discuss the possibility of simultaneous explanation of the muon and the electron $(g-2)$ in our model, taking into account the limits from LFV processes.

Extracting limits on $g_{ij}^{V,A}$, we apply the conventional simplifying assumption about the presence of only one nonvanishing coupling constant at a time. In Figs. 2 and 3, we show the resulting upper limits for the couplings $g_{le}^{V,A}$, which are significantly more stringent than for other combinations of flavor indices $g_{lf}^{V,A}$ with $f \neq e$ due to the factor m_l/m_f in Eqs. (17) and (18). These latter limits can be approximately obtained from $g_{le}^{V,A}$ with the corresponding rescaling using the mentioned factor.

To estimate the effect of the combined contribution of g^V and g^A couplings, we also studied the upper limits on the coupling g^V for different values of the ratio g^A/g^V. The results are shown in Fig. 3.

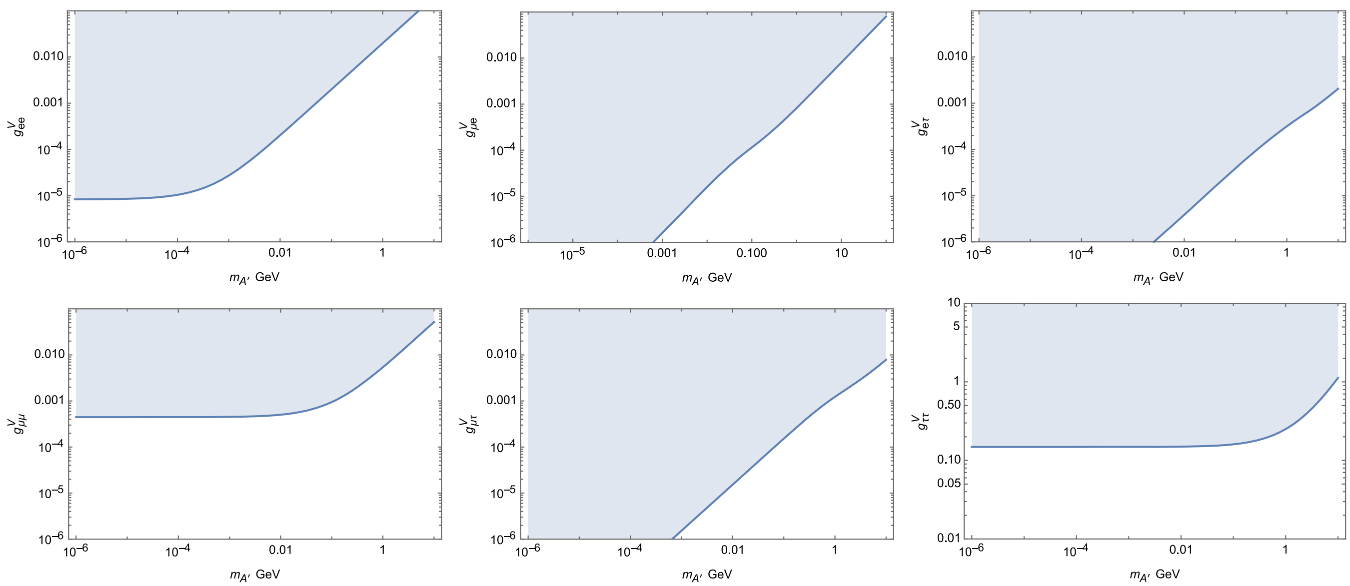

FIG. 2. Upper bounds on the couplings g^V of the dark photon A' to the SM fermions as a function of its mass m_A' derived from the data on leptonic $(g-2)$. The shaded area is excluded.

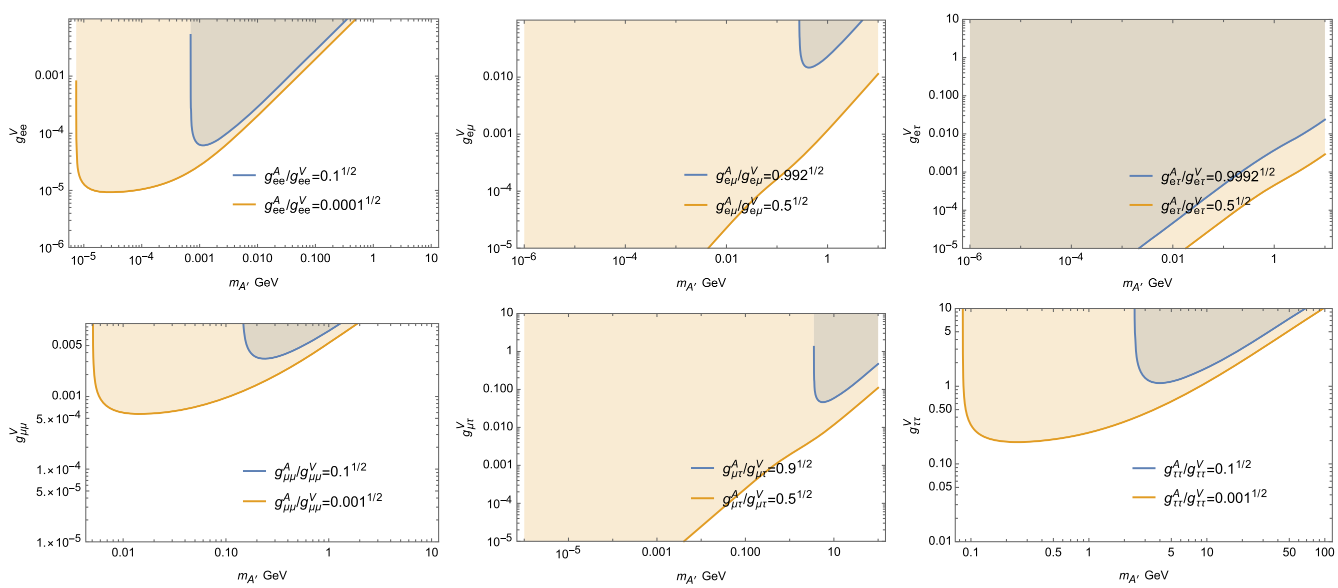

FIG. 3. Upper bounds on the couplings g_{ij}^V of the dark photon A' to the SM fermions in the vector+axial-vector channel as a function of its mass $m_{A'}$, derived from the experimental data on $(g-2)_l$. The shaded area is excluded. We show the plots for different values of the ratio g_{ij}^A/g_{ij}^V.

Let us to note that the combination of vector and negative axial-vector contributions to $(g-2)_\mu$ makes limits less stringent in comparison with the case of the pure vectorial term. This can significantly extend a window in the mass-coupling parameter space for the light dark vector particles. Finally, we note that, if the LFV effects in the $(g-2)$ were not taken into account, then the upper limits for the couplings g^V_{ll} would be almost the same as for g^A_{ll}.

IV. NUCLEAR LEPTON-FLAVOR CONVERSION

We recall again that the Stueckelberg portal model inherently features LFV couplings of the dark sector field A' to the SM leptons, described by (12). These LFV couplings can contribute to both flavor-conserving observables [e.g., $(g-2)_l$ studied in the previous section] and to the LFV ones, some of which we consider in what follows. We start with nuclear $\mu^- - e^-$ conversion, which is a LFV process with the participation of a nucleus,

$$\ell_1^- + (A,Z) \to \ell_2^- + X. \tag{22}$$

It was recently advocated that deep inelastic lepton conversion on nuclei with X denoting all the possible final-state particles has good prospects for setting limits on the effective couplings of ℓ_1 and ℓ_2 in the $e-\mu$, τ and $\mu-\tau$ channels at the fixed-target NA64 experiment [25].

However, the process most studied experimentally is coherent $\mu^- - e^-$ conversion in muonic atoms, in which one electron is replaced by a muon. In this case, $\ell_1 = \mu$, $\ell_2 = e$, and $X = (A,Z)$. This process has not yet been discovered experimentally. Presently, the best upper limits on its rate $R_{\mu e}$ have been set by the SINDRUM II experiment on $\mu - e$ conversion in ^{198}Au [69]:

$$R^{\mathrm{Au}}_{\mu e} \le 4.3 \times 10^{-12}. \tag{23}$$

In the near future, the PRISM/PRIME experiment [70] with a titanium ^{48}Ti target is going to reach the limit

$$R^{\mathrm{Ti}}_{\mu e} \lesssim 10^{-18}. \tag{24}$$

In Ref. [71], on the basis of nucleon-meson effective field theory, the above experimental limits were translated into the upper limits on the effective couplings $\alpha^{V,A}$ of the LFV $\mu - e$ current to nucleons defined by Lagrangian

$$\mathcal{L}^{\ell N}_{\mathrm{eff}} = \frac{1}{\Lambda^2_{LFV}} \bar{N}\gamma^\mu N \bar{e}[\alpha^V \gamma_\mu + \alpha^A \gamma_\mu\gamma_5]\mu + \mathrm{H.c.} \tag{25}$$

These limits are

$$\alpha^{V,A}_{A'} \left(\frac{1\ \mathrm{GeV}}{\Lambda_{\mathrm{LFV}}}\right)^2 \le 8.5 \times 10^{-13}, \quad \text{from SINDRUM II [69]}, \tag{26}$$

$$\alpha^{V,A}_{A'} \left(\frac{1\ \mathrm{GeV}}{\Lambda_{\mathrm{LFV}}}\right)^2 \le 1.6 \times 10^{-15}, \quad \text{from PRISM/PRIME [70]}. \tag{27}$$

In our approach, the Lagrangian (25) is generated at tree level by the t-channel exchange with A' between the $\mu - e$ and qq currents, where $q = u, d$ are valence quarks of the nucleon. Note that transitions with $q_1 \ne q_2$ do not contribute to coherent $\mu - e$ nuclear conversion. Starting from our Lagrangian (12) and matching at a certain scale the quark currents with nucleon ones (see for details Ref. [71]), we find the relations

$$\alpha^{V(A)}_{A'} \simeq z g^{V(A)}_{qq} g^{V(A)}_{e\mu} \frac{m^2_{A'}}{m^2_{A'} + m^2_\mu}, \tag{28}$$

where z is a dimensionless constant of order $\mathcal{O}(1)$. Thus, from Eqs. (26)–(28), we find the following upper limits:

$$|g^V_{e\mu} g^V_{qq}| \lesssim \begin{cases} 8.5 \times 10^{-13} \left(\frac{m^2_{A'}}{m^2_{A'} + m^2_\mu}\right)^{-1} & \text{SINDRUM} \\ 1.6 \times 10^{-15} \left(\frac{m^2_{A'}}{m^2_{A'} + m^2_\mu}\right)^{-1} & \text{PRISM/PRIME.} \end{cases} \tag{29}$$

We will use these limits in Sec. VI for our combined analysis of the $(g-2)_l$ and the LFV experimental data.

V. LFV DECAYS $l_i \to l_k \gamma$

The matrix element of this LFV process can be parametrized as

$$iM_{ik} = ie\epsilon^\mu(q)\bar{u}_k(p_2, m_k)\left[\frac{i}{2m_i}\sigma_{\mu\nu}q^\nu F_M + \frac{i}{2m_i}\sigma_{\mu\nu}q^\nu\gamma_5 F_D\right] \times u_i(p_1, m_i). \tag{30}$$

Then,

$$|M_{ik}|^2 = m_i^2\left[1 - \frac{m_k^2}{m_i^2}\right]^2 (|F_M|^2 + |F_D|^2). \tag{31}$$

Here, we used the Gordon identities listed in Appendix B. Taking into account that $m_e \ll m_\mu \ll m_\tau$, we have for the decay width of this process in a very good approximation the expression [72]

$$\Gamma(l_i \to l_k\gamma) = \frac{1}{2m_i}\int \frac{d^3p_2 d^3q}{4E_2E_q(2\pi)^6}(2\pi)^4 \times \delta^{(4)}(p_1 - p_2 - q)|M_{ik}|^2 = \frac{\alpha}{2} m_i (|F_M|^2 + |F_D|^2), \tag{32}$$

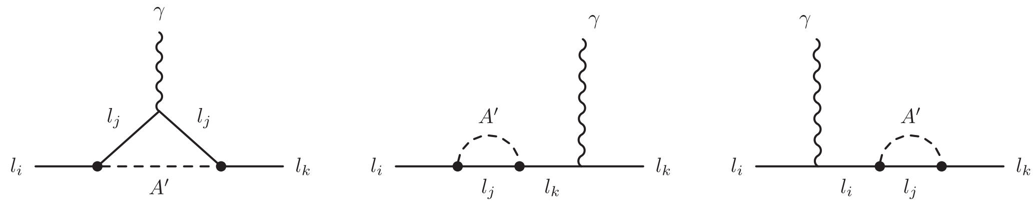

FIG. 4. Feynman diagrams of gauge invariant matrix elements of the interaction lepton with external electromagnetic field accounting for the LFV effect generated by the A' dark photon.

where F_M and F_D are the magnetic and dipole form factors and $\alpha = e^2/(4\pi) = 1/137.036$ is the fine-structure constant. The dark sector contributes to $l_i \to l_k\gamma$ decay only with the dark photon A' according to the diagrams in Fig. 4.

The corresponding analytical expressions for single LFV coupling read

$$F_M = -\frac{1}{16\pi^2}\Big[\Big(g^V_{ik}g^V_{ii} + g^A_{ik}g^A_{ii}\Big)h^V_2(x_\mu) + \Big(g^V_{ik}g^V_{kk} + g^A_{ik}g^A_{kk}\Big)h^V_3(x_\mu)\Big], \tag{33}$$

$$F_D = \frac{1}{16\pi^2}\Big[\Big(g^V_{ik}g^A_{ii} + g^A_{ik}g^V_{ii}\Big)h^V_2(x_\mu) + \Big(g^V_{ik}g^A_{kk} + g^A_{ik}g^V_{kk}\Big)h^V_3(x_\mu)\Big]. \tag{34}$$

For $\mu \to e\gamma$ process with the τ lepton in loop with double LFV coupling, we have

$$F_M = -\frac{1}{16\pi^2}\left(\frac{m_\mu}{m_\tau}\right)\Big[g^V_{\mu\tau}g^V_{e\tau}h^V_1(x_\tau) + g^A_{\mu\tau}g^A_{e\tau}h^V_1(x_\tau)\Big],$$
$$F_D = \frac{1}{16\pi^2}\left(\frac{m_\mu}{m_\tau}\right)\Big[g^V_{\mu\tau}g^A_{e\tau}h^V_1(x_\tau) + g^A_{\mu\tau}g^V_{e\tau}h^V_1(x_\tau)\Big], \tag{35}$$

where $x_i = m^2_{A'}/m^2_i$. Expressions for the loop functions $h^V_i(x_i)$ in the approximation $m_e \ll m_\mu \ll m_\tau$ are shown in Appendix D.

Let us note that the diagrams in Figs. 4(b) and 4(c) are needed to guarantee gauge invariance of the photon interactions with leptons through loop diagrams induced by the A' dark photon. This simultaneously leads to cancellation of a divergence arising from the diagram in Fig. 4(a).

Similarly, we can write the A' contributions to F_M and F_D form factors of the for $\tau \to \mu\gamma$ and $\tau \to e\gamma$ LFV rare decays for the case when the initial or final leptons are different from the lepton in the loop. We have for $\tau \to \mu\gamma$ decay

$$F_M = -\frac{1}{16\pi^2}\Big[g^V_{e\tau}g^V_{e\mu} + g^A_{e\tau}g^A_{e\mu}\Big]h^V_3(x_\tau),$$
$$F_D = \frac{1}{16\pi^2}\Big[g^V_{e\tau}g^A_{e\mu} + g^A_{e\tau}g^V_{e\mu}\Big]h^V_3(x_\tau) \tag{36}$$

and for $\tau \to e\gamma$ decay

$$F_M = -\frac{1}{16\pi^2}\Big[g^V_{\mu\tau}g^V_{e\mu} + g^A_{\mu\tau}g^A_{e\mu}\Big]h^V_3(x_\tau),$$
$$F_D = \frac{1}{16\pi^2}\Big[g^V_{\mu\tau}g^A_{e\mu} + g^A_{\mu\tau}g^V_{e\mu}\Big]h^V_3(x_\tau). \tag{37}$$

VI. ANALYSIS OF CURRENT LIMITS

In this section, we derive experimental bounds on the A' couplings $g^{V,A}_{ij}$ for several benchmark scenarios.

The current limits for the branchings of the LFV lepton decays $l_i \to l_k\gamma$ are [68]

$$\mathrm{Br}(\mu \to e\gamma) < 4.2 \times 10^{-13},$$
$$\mathrm{Br}(\tau \to e\gamma) < 3.3 \times 10^{-8},$$
$$\mathrm{Br}(\tau \to \mu\gamma) < 4.4 \times 10^{-8}. \tag{38}$$

First, we analyze the dimensionless LFV couplings g_{ij} by focusing on the scenario of lepton-flavor universality, assuming equal values of their diagonal elements, i.e., $g^V_{ii} = g^V$ and $g^A_{ii} = g^A$ for $i = e, \mu, \tau$. In this scenario, we calculated and estimate coupling from $\mu \to e\gamma$ and $\tau \to e\gamma$ LFV decays. The results are presented in Fig. 5 for the particular value $g_{ii} = 1$. Bounds from the lepton $(g-2)_l$ (see Fig. 5) are shown for the case of vanishing flavor-conserving couplings g_{ii}. Note that bounds from $\tau \to \mu\gamma$ are the same as for $\tau \to e\gamma$ because in the approximation $m_e \ll m_\mu \ll m_\tau$ the contributions from the loops are same. Also, we omit in our analysis doubly LFV suppressed diagrams with heavy leptons propagating in the loop.

The peaks in Fig. 5 are induced by behavior of the loop integrals $h_i(x)$ near the point $x = 1$ located in the vicinity of the vector boson production threshold. For resolving this problem, one needs to include in our analysis finite width $\Gamma_{A'}$ of the decay of dark vector boson to the leptonic pair with $\Gamma_{A'} \sim \tau^{-1}_{A'} \sim g^2_{ij}$ in the Breit-Wigner propagator.

Limits on the LFV couplings in Fig. 5 include constraints from the lepton $(g-2)$ and rare LFV $l_i \to l_k\gamma$ lepton decays. In the case of $e-\mu$ LFV transition (see left pictures in Fig. 5) we add bound from $e-\mu$ conversion. Suppression for $e-\mu$ conversion at heavy masses is induced by heavy bosons exchange in the t channel. The

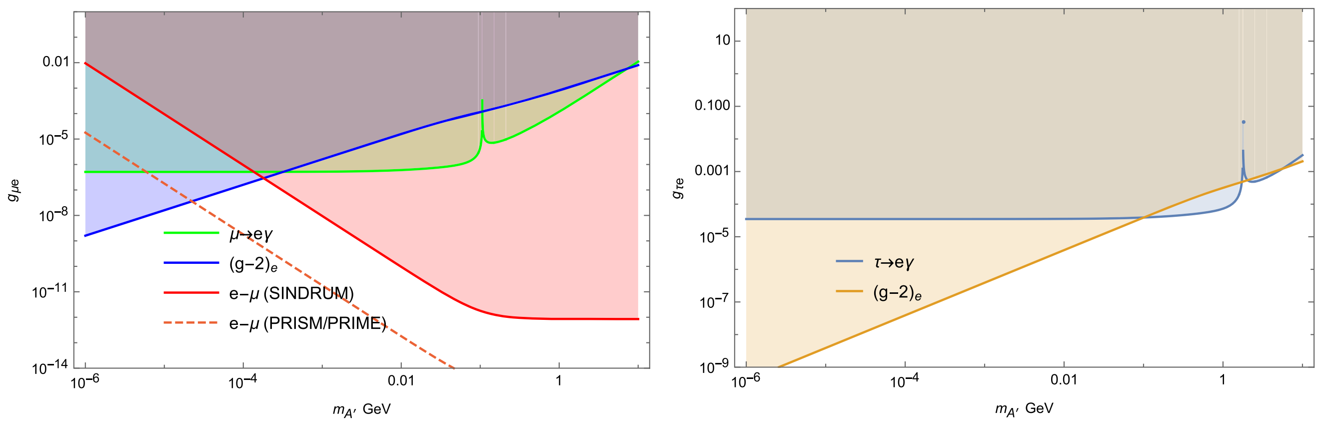

FIG. 5. Limits on vector g_{ij} couplings in dependence on masses $M_{A'}$ are deduced from an analysis of the following phenomena: $g-2$ ratios of leptons, widths of LFV decays $\mu \to e\gamma$, and $\tau \to e\gamma$ and lepton conversion. The shaded area is excluded by the data.

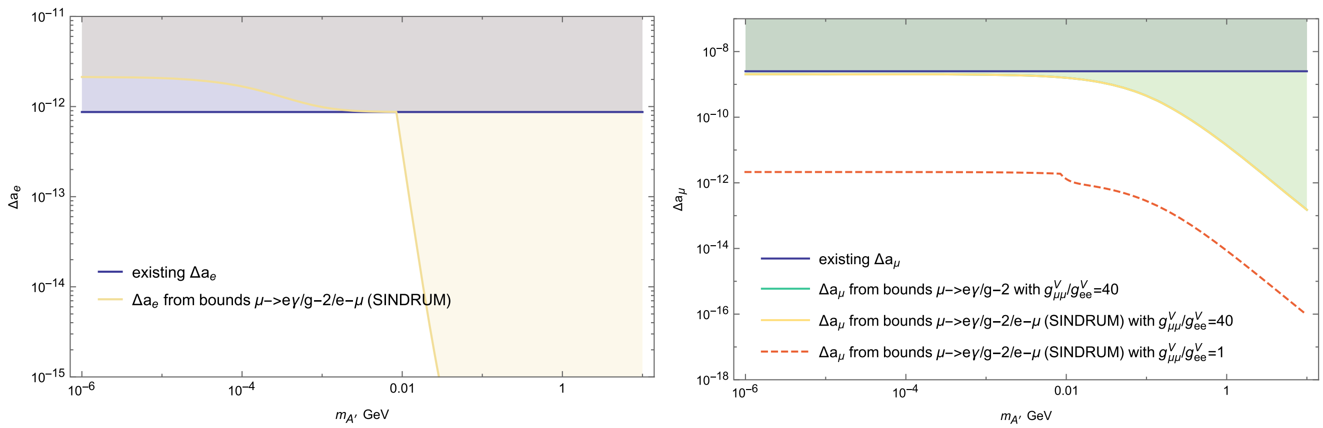

FIG. 6. Estimate of contribution to lepton AMM is made in dependence on masses of dark photon. The limit is established for the benchmark case $g_{ee} = 10^{-5}$ with taking into account the restriction for LFV couplings.

dark A' photon keeps a window for possible huge LFV couplings g_{ij} at light masses of leptons. We also would like to note that couplings g_{ij} are different from one pushing the limits from LFV processes up.

Using the limit from $l_i \to l_k\gamma$ decay in the scenario of universal lepton flavor–conserving (LFC) couplings g_{ii}^V or g_{ii}^A, we can deduce the possible lepton contribution to $(g-2)$ in function of the mass of dark vector photon A'. We have estimated the contribution to $(g-2)_l$ of the sum of loops with light leptons e and μ, taking into account the contribution of LFC and LFV couplings, which are constrained by $l_i \to l_k\gamma$ decay. In this way, we write down

$$\Delta a_{l_i} = (\Delta a_{l_i})_{\rm LFC} + (\Delta a_{l_i})_{\rm LFV}. \tag{39}$$

For the vector contribution, we also included constraints from $\mu - e$ conversion. The results are shown in Fig. 6. As can be seen, the dark photon A' contribution to $(g-2)_e$ through the vector channel explains the electron anomaly Δa_e for $m_{A'} < 10^{-2}$ GeV. Attempting to explain both Δa_e and Δa_μ anomalies on account of the A' contribution, we find that it is not possible at least in the lepton universal benchmark scenario where $g_{\mu\mu}/g_{ee} = 1$ (see the dashed line in the right panel in Fig. 6). Going beyond this simplified scenario, we can find the simultaneous solution of Δa_e and Δa_μ for $g_{\mu\mu}^V > g_{ee}^V$. A particular solution for $g_{\mu\mu}/g_{ee} = 40$, properly taking into account the limits from the LFV processes (see Fig. 2), is presented in the right panel of Fig. 6. We note that this solution is pretty hierarchical, requiring separation of two couplings of the similar nature in more than order of magnitude, which looks unnatural. Another possibility avoiding this kind of hierarchy would be to extent the field content of the model, amending it with the dark sector (pseudo)scalars providing additional contributions to $(g-2)_l$. The study of this possibility is beyond the scope of this work.

VII. CONCLUSIONS

We constructed a phenomenological Lagrangian approach which combines SM and DM sectors based on the Stueckelberg mechanism for the generation mass of the dark $U_D(1)$ gauge boson (or dark photon). The DM sector contains dark photon, dark scalar, and generic dark fermion fields. Note that the dark scalar generates the mass of the dark photon and plays a role of Goldstone boson in our gauge-invariant formalism. The Stueckelberg portal opens new possibilities for the study of phenomenology of BSM physics and can be important for running and planning experiments at worldwide facilities (e.g., for the NA64 Experiment at SPS CERN [2,3]).

We derived the limits on the effective couplings of our Lagrangian using data on lepton AMMs, LFV lepton decays $l_i \to \gamma l_k$, and $\mu - e$ conversion. It is known that the latter are very useful because they give more stringent limits on the couplings of effective Lagrangian. We also found that the $(g-2)$ anomaly cannot be preferably solved within the Stueckelberg portal scenario by the light dark photon in the framework of the conservative scenario with taking into account lepton universality. However, the simultaneous explanation of these both anomalies becomes possible once we allow approximately one–order of magnitude hierarchy between the flavor diagonal couplings of A' to electron g_{ee} and to muon $g_{\mu\mu}$, which can be treated as moderately unnatural. We mentioned the possible ways for relaxing this tension with the naturalness. We leave a detailed study of these aspects of our model for the future publications.

In the future, we plan to study a possible role of the Stueckelberg portal in different LFV processes including semileptonic decays. We plan include scalar and pseudoscalar dark bosons into the Stueckelberg portal of DM. We also intend to extended our ideas on non-Abelian scenario for the dark sector.

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APPENDIX A: CONTRIBUTION OF DARK PHOTON TO LEPTON ANOMALOUS MAGNETIC MOMENT IN R_ξ GAUGES

The most simple calculation of the contribution of the dark photon to the lepton anomalous magnetic moment in R_ξ gauges can be performed in the unitary gauge specified by the choice $\xi = \infty$. In particular, in this case, the contribution of Goldstone bosons explicitly vanishes. It is convenient to perform all calculations in dimensional regularization with $D = 4 - 2\epsilon$ in order to explicitly show that all potentially divergent terms cancel each other, leading to finite results. In particular, the sum of these terms is given by the integral

$$I = \int_0^1 dx \int \frac{d^D k}{i\pi^{D/2}} \frac{k^2}{[\Delta(x, y_A, y_l) - k^2]^3} \times \frac{2}{D}[4 - (D+2)(1-x)], \tag{A1}$$

where x is the Feynman parameter of integration.

Performing integration of the loop integral in D dimensions using master integral

$$\begin{aligned} I &= \int_0^1 dx \int \frac{d^D k}{i\pi^{D/2}} \frac{(k^2)^s}{[\Delta - k^2]^n} \\ &= \int_0^1 dx (-1)^s \frac{\Gamma(s + D/2)\Gamma(n - s - D/2)}{\Gamma(D/2)\Gamma(n)(\Delta)^{n-s-D/2}}, \end{aligned} \tag{A2}$$

we get

$$I = -\int_0^1 dx (1-x)(4 - (D+2)(1-x))\Delta^{D/2-2}(y_A, y_l). \tag{A3}$$

Next, using $D = 4 - 2\epsilon$ and performing ϵ expansion

$$\Delta^{-\epsilon} = 1 - \epsilon \log[\Delta], \tag{A4}$$

we verify that the integral is finite and after straightforward simplifications at the limit $\epsilon \to 0$ is given by

$$I = -\int_0^1 dx \frac{(1-x)^2}{\Delta(x, y_A, y_l)}[1 - y_l^2 x^2]. \tag{A5}$$

Then, summing all finite terms, we arrive at the final results, which are in full agreement with results of Ref. [37]. For convenience, we display partial contributions to the integrand over Feynman parameter x, e.g., in case of the dark photon connecting vector Dirac matrices γ^μ and γ^ν. As was stressed in Ref. [37] and was pointed out before in the present manuscript, the axial case is simply obtained from the vector case upon inverting a sign in front of the mass of external lepton m_f:

(i) contribution induced by transverse part of the dark photon propagator, i.e., by the $g^{\mu\nu}$ part:

$$\frac{x(1-x)}{\Delta(x, y_A, y_l)}[2 - y_l(1+x)]; \tag{A6}$$

(ii) contribution induced by longitudinal part of the dark photon propagator, i.e., by the $p^\mu p^\nu / m_{A'}^2$ part:

$$\frac{(1-x)^2}{\Delta(x, y_A, y_l)} \frac{(1-y_l)^2}{2y_A^2}(1 + y_l x). \tag{A7}$$

APPENDIX B: GORDON IDENTITIES

The Gordon identities for the matrix elements describing the coupling of the external gauge field with fermions having different masses read

$$
\begin{aligned}
i\sigma_{\mu\nu} q^\nu &= -P_\mu + (m_i + m_j)\gamma_\mu, \\
i\sigma_{\mu\nu} P^\nu &= -q_\mu + (m_j - m_i)\gamma_\mu, \qquad \text{(B1)}
\end{aligned}
$$

$$
\begin{aligned}
i\sigma_{\mu\nu} q^\nu \gamma_5 &= -P_\mu\gamma_5 + (m_j - m_i)\gamma_\mu\gamma_5, \\
i\sigma_{\mu\nu} P^\nu \gamma_5 &= -q_\mu\gamma_5 + (m_i + m_j)\gamma_\mu\gamma_5, \qquad \text{(B2)}
\end{aligned}
$$

where $P = p_1 + p_2$ and $q = p_1 - p_2$

APPENDIX C: DIAGRAMS IN FIGS. 4

In this Appendix, we explicitly demonstrate that the set of three diagrams in Figs. 4(a)–4(c) is gauge invariant and finite. To simplify the proof of gauge invariance, we split the contribution of each diagram M from the set 4 into a part which is manifestly gauge invariant $M^{\rm GI}$ and one which is not $M^{\rm rest}$ as $M = M^{\rm GI} + M^{\rm rest}$. This separation can be achieved in the following manner. For the γ matrix contacting with photon field, we use the substitution $\gamma^\mu = \gamma^\mu_\perp + q^\mu \not{q}/q^2$ where $\gamma^\mu_\perp = \gamma^\mu - q^\mu \not{q}/q^2$ obeys the transversity condition $\gamma^\mu_\perp \cdot q_\mu = 0$. Expressions for diagrams containing only $\perp$ values are gauge invariant separately. It is easy to show that the remaining terms, which are not gauge invariant, cancel each other in total (see the detailed discussion of this technique, e.g., in Ref. [73]). Therefore, it is enough to consider only the sum of the gauge-invariant contributions from all diagrams in Fig. 4(a)–4(c). The proof of cancellation of the remaining terms is straightforward. Below, we list these terms for each diagram using dimensional regularization and for general case of exchange by the boson particle (with spin 0 or 1) (i.e., we do not restrict to the exchange of the dark photon):

$$
M^{\rm rest}_{4a} = \int \frac{d^D k}{(2\pi)^D i} \Gamma_1 \frac{1}{\not{p}' + \not{k} - m_f} \frac{q^\mu \not{q}}{q^2} \frac{1}{\not{p} + \not{k} - m_j} \Gamma_2 \frac{d^{\Gamma_1\Gamma_2}(k)}{k^2 - m^2}, \qquad \text{(C1)}
$$

$$
M^{\rm rest}_{4b} = \frac{q^\mu \not{q}}{q^2} \frac{1}{\not{p} - m_k} \int \frac{d^D k}{(2\pi)^D i} \Gamma_1 \frac{1}{\not{p} + \not{k} - m_j} \Gamma_2 \frac{d^{\Gamma_1\Gamma_2}(k)}{k^2 - m^2}, \qquad \text{(C2)}
$$

$$
M^{\rm rest}_{4c} = \int \frac{d^D k}{(2\pi)^D i} \Gamma_1 \frac{1}{\not{p}' + \not{k} - m_j} \Gamma_2 \frac{d^{\Gamma_1\Gamma_2}(k)}{k^2 - m^2} \frac{1}{\not{p}' - m_i} \frac{q^\mu \not{q}}{q^2}. \qquad \text{(C3)}
$$

Here, Γ_1 and Γ_2 are the corresponding Dirac matrices; $d^{\Gamma_1\Gamma_2}(k) = 1$ for exchange by scalar/pseudoscalar particles with $\Gamma_1 = I, \gamma^5$ and $\Gamma_2 = I, \gamma^5$, and $d^{\mu\nu}(k) = -g^{\mu\nu} + k^\mu k^\nu/m^2$ for exchange by vector/axial particles with $\Gamma_1 = \gamma^\mu, \gamma^\mu\gamma^5$ and $\Gamma_2 = \gamma^\nu, \gamma^\nu\gamma^5$.

Next, using the Ward identity for inverse fermion propagators $\not{q} = (\not{p} - m_\ell) - (\not{p}' - m_\ell)$ and free Dirac equations of motion for initial and final leptons, we simplify expressions for the individual rest matrix elements as

$$
M^{\rm rest}_{4a} = \int \frac{d^D k}{(2\pi)^D i} \Gamma_1 \left[\frac{1}{\not{p}' + \not{k} - m_j} - \frac{1}{\not{p} + \not{k} - m_j} \right] \Gamma_2 \frac{d^{\Gamma_1\Gamma_2}(k)}{k^2 - m^2}, \qquad \text{(C4)}
$$

$$
M^{\rm rest}_{4b} = \int \frac{d^D k}{(2\pi)^D i} \Gamma_1 \frac{1}{\not{p} + \not{k} - m_j} \Gamma_2 \frac{d^{\Gamma_1\Gamma_2}(k)}{k^2 - m^2}, \qquad \text{(C5)}
$$

$$
M^{\rm rest}_{4c} = -\int \frac{d^D k}{(2\pi)^D i} \Gamma_1 \frac{1}{\not{p}' + \not{k} - m_j} \Gamma_2 \frac{d^{\Gamma_1\Gamma_2}(k)}{k^2 - m^2}. \qquad \text{(C6)}
$$

Finally, summing Eqs. (C4)–(C6), we get 0, therefore, proving gauge invariance of the set 4(a)–4(c).

Now, we turn to the discussion of finiteness of the sum of the set of diagrams 4(a)–4(c). The logarithmically divergent term in Fig. 4(a) is generated by the part of numerator containing two loop momenta: $\not{k}\gamma^\mu\not{k}$. Applying dimension regularization with $D = 4 - 2\epsilon$, it gives the following divergent result in case of exchange by a scalar S or pseudoscalar P particle with $\Gamma_1 = \Gamma_2 = I$ or $i\gamma^5$:

$$
M^{UV;S/P}_{4a} = -\frac{\gamma^\mu}{2\epsilon}. \qquad \text{(C7)}
$$

The diagrams in Figs. 4(d) and 4(e) induce the following logarithmic divergencies:

$$
M^{UV;S/P}_{4b} = \frac{\gamma^\mu}{2\epsilon} \frac{m_i \pm 2m_j}{m_i - m_k}. \qquad \text{(C8)}
$$

and

$$
M^{UV;S/P}_{4c} = \frac{\gamma^\mu}{2\epsilon} \frac{m_k \pm 2m_j}{m_k - m_i}, \qquad \text{(C9)}
$$

respectively. Here and below, $\pm$ corresponds to exchange of a scalar or pseudoscalar particle.

Summing up divergent contributions of three diagrams, we get exact cancellation of the latter:

$$
\begin{aligned}
&M^{UV;S/P}_{4a} + M^{UV;S/P}_{4b} + M^{UV;S/P}_{4c} \\
&= \frac{\gamma^\mu}{2\epsilon} \left[-1 + \frac{m_i \pm 2m_j}{m_i - m_k} + \frac{m_k \pm 2m_j}{m_k - m_i} \right] = 0. \qquad \text{(C10)}
\end{aligned}
$$

In case of the diagrams induced by vector particle exchange, the logarithmic divergences induced by individual diagrams read (we explicitly show the contribution of transverse and longitudinal part of the vector V or axial A propagator, which are supplied by the subscript T and L, respectively)

$$M_{4a}^{UV;V/A} = M_{4a;T}^{UV;V/A} + M_{4a;L}^{UV;V/A},$$
$$M_{4a;T}^{UV;V/A} = -\frac{\gamma^\mu}{\epsilon},$$
$$M_{4a;L}^{UV;V/A} = \frac{\gamma^\mu}{\epsilon}\left[1 + \frac{3}{2}\frac{m_j^2}{m^2} - \frac{m_i^2 + m_k^2 + m_i m_k}{2m^2}\right], \quad \text{(C11)}$$

$$M_{4b}^{UV;V/A} = M_{4b;T}^{UV;V/A} + M_{4b;L}^{UV;V/A},$$
$$M_{4b;T}^{UV;V/A} = \frac{\gamma^\mu}{\epsilon}\frac{m_i \mp 2m_j}{m_i - m_k},$$
$$M_{4b;L}^{UV;V/A} = \frac{\gamma^\mu}{\epsilon}\frac{m_k \mp m_j}{m_i - m_k}\left[1 + \frac{m_j^2}{m^2} - \frac{m_k(m_k \pm m_j)}{2m^2}\right], \quad \text{(C12)}$$

$$M_{4c}^{UV;V/A} = M_{4c;T}^{UV;V/A} + M_{4c;L}^{UV;V/A},$$
$$M_{4c;T}^{UV;V/A} = \frac{\gamma^\mu}{\epsilon}\frac{m_k \mp 2m_j}{m_k - m_i},$$
$$M_{4c;L}^{UV;V/A} = \frac{\gamma^\mu}{\epsilon}\frac{m_i \mp m_j}{m_k - m_i}\left[1 + \frac{m_j^2}{m^2} - \frac{m_i(m_i \pm m_j)}{2m^2}\right]. \quad \text{(C13)}$$

Here, $\pm$ corresponds to exchange of a vector or axial particle.

It is easy to show that in case of vector and particle exchange we also get exact cancellation of the divergences and it occurs separately for transverse and longitudinal contribution of the propagator of the exchange particle:

$$M_{4a}^{UV;V/A} + M_{4b}^{UV;V/A} + M_{4c}^{UV;V/A} = 0,$$
$$M_{4a;T}^{UV;V/A} + M_{4b;T}^{UV;V/A} + M_{4c;T}^{UV;V/A} = 0,$$
$$M_{4a;L}^{UV;V/A} + M_{4b;L}^{UV;V/A} + M_{4c;L}^{UV;V/A} = 0. \quad \text{(C14)}$$

APPENDIX D: LOOP FUNCTIONS $h_1^V(x)$

In this Appendix, we present the analytical expressions of the loop integrals occurring in the amplitude of the LFV decays $l_i \to \gamma l_k$ for different channels and leptons propagating in the loop in the approximation $m_e \ll m_\mu \ll m_\tau$:

$$h_1^V(x) = -\frac{(4x^3 - 3x^2 - 6x^2\ \ln(x) - 1)}{x(1-x)^3}, \quad \text{(D1)}$$

$$h_2^V(x) = 2\Bigg(2\mathrm{Li}_2(1-x) - 2\mathrm{Li}_2\left(\frac{2}{-x + \sqrt{(x-4)x} + 2}\right) + 2\mathrm{Li}_2\left(\frac{2}{x + \sqrt{(x-4)x}}\right) - 2x + \log^2\left(\frac{x + \sqrt{(x-4)x}}{2x}\right)$$
$$+ \frac{(x+1)((x-4)x+2)\log(x)}{x-1} - 2x\sqrt{(x-4)x}\log\left(\frac{\sqrt{x} + \sqrt{(x-4)}}{2}\right) + 1\Bigg), \quad \text{(D2)}$$

$$h_3^V(x) = -4x + 4(x-1)^2 \ln\left(\frac{x}{x-1}\right) + 6. \quad \text{(D3)}$$

All results have been numerically and analytically cross-checked using the *Mathematica* Package-X [74] and packages FeynHelpers [75] and FeynCalc [76].

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