

On the extension of a physical body in classical motion. An analogy between a pseudo-velocity concept and Wiener's process in (ideal) polymer solutions

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ABSTRACT

A pseudo-velocity concept, based on the extension of a linear body, is defined by a special relativity experiment. It suggests an analogy with the covariance properties of Wiener's process, ultimately implying that the scaling behavior of (Gaussian) polymer solutions can be derived from a Brownian Relativity theory, as it was formerly put forward. An ad-hoc statistical interpretation of the resulting spacetime transforms may be given by the central limit theorem.

Introduction

Brownian relativity (BwR) is the name given to a framework that combines the conceptual structures of relativity and Brownian motion [1]. It propounds that time and space in a Brownian system can be envisaged similar to the spacetime of Einstein's relativity. To avoid misinterpretations, it is worth perhaps clarifying from the outset that BwR does not concern or want to contribute to long-standing problems of relativistic Brownian motion and heat transport, which deal with issues as blue violations of the principles of covariance and causality by the propagator of non-relativistic diffusion, the (non-)existence of Markov's processes and random space-like curves that are covariant upon the (inhomogeneous) Lorentz group [2–4]. BwR was rather restricted to studying (universal limits in) polymer solutions, building up a language and analogies that would only conflict with the previous points if BwR were extended or applied outside its formal and phenomenological domains.

Accordingly, average size and characteristic time of a polymer fluctuating in a liquid were derived similarly to a Lorentz–FitzGerald length contraction and a time dilation rule, if the system is short-range correlated (or uncorrelated), and by means of an equivalence relation for geometry and statistics when correlations are long-ranged. BwR predicts new universal laws for the relevant scaling exponents in polymer solutions, i.e. for chain size (ν), diffusion coefficient (δ), characteristic time (σ) and viscosity (ϵ), which are fulfilled by theoretical and experimental values in both unentangled and entangled systems (see e.g. reptation [5] and renormalization [6] theories, athermal, good and θ -solvents [7]), e.g. [1,8]:

$$\delta + \nu = 3\nu - \sigma = -\frac{\epsilon}{2} \quad (1)$$

Eq. (1) is holding in different chain unit number ($N \gg 1$) and volume fraction (ϕ) regimes, each of the above quantities behaving proportionally to N^a or $\phi^{a'}$, with $a = \delta, \nu, \sigma, \epsilon$ and $a' = \delta', \nu', \sigma', \epsilon'$. Such an agreement, based on the link that BwR settles between molecular and macromolecular scales [9,10], can be of utmost importance in disciplines such as turbulence in liquids, where the idea to probe the statistical properties of a turbulent flow by large molecules (whether they are called fibers, polymers, or material lines) has attracted attention quite recently [11–13], or in life sciences, in which BwR may be promising e.g. to the systems biologist's work of modeling cell pathways at any relevant spatiotemporal scale [14], in the less developed field of intercellular processes or when diffusion is slower than biochemical reactions [15]. Diffusing molecules must get to specific absorbing sub-domains of cell membranes to trigger a reaction, this being normally described by diffusion–reaction models and their master equations [16,17].

Purpose of this letter thus is clarifying and delving into some of the main aspects of the special version of BwR. To this end, the next section describes a special relativity experiment representative of BwR, where the pseudo-velocity = (linear body extension)/time is introduced in analogy to the role played in BwR by a polymer molecule.

Observer's velocity from a pseudo-velocity

In BwR the perturbation to the local diffusion coefficient induced by a polymer chain in a liquid may be formulated to give rise to a transformation conceptually similar to the Lorentz–Poincaré type. The

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light speed (c) in Special Relativity (SR) is formally replaced here by the diffusion coefficient of liquid molecules (D_1), taken to be a constant and homogeneous diffusivity unit throughout the solution. That, clearly, was only an analogy, as the following SR example is going to expound. It had nothing to do with either the impassible limit introduced by the second SR postulate, nor with the related causality prescriptions.

A pointwise observable and a rectilinear line of length ℓ_0 move uniformly between points A and B , with $\overline{AB} = \Delta l$. Let an extreme of the line to lie on A , while the other falls within \overline{AB} . From a frame K_0 at rest with A and B , the point and the line reach the extreme B at equal times (Δt), with velocities $v = \Delta l / \Delta t$ and $v' = (\Delta l - \ell_0) / \Delta t$. From another frame K_u , moving with velocity u relative to K_0 , Lorentz–Poincaré’s transform [18] returns two time intervals, for the point and line, equal respectively to:

$$\Delta t' = \frac{\gamma_u}{v} (1 - \beta_u \beta_v) \Delta l \quad (2)$$

$$\overline{\Delta t'} = \frac{\gamma_u}{v'} (1 - \beta_u \beta_{v'}) (\Delta l - \ell_0) \quad (3)$$

where $\gamma_u = (1 - \beta_u^2)^{-\frac{1}{2}}$ is the Lorentz factor evaluated at the velocity u , while $\beta_u = u/c$, $\beta_v = v/c$ and $\beta_{v'} = v'/c$. By imposing that K_u agrees with the simultaneous arrival in B of the point and line ($\overline{\Delta t'} = \Delta t'$), and defining the pseudo-velocity $\gamma_0 \equiv \ell_0 / \Delta t$ (in the same direction and orientation of v), it turns out:

$$v' = \frac{v - \gamma_0}{1 - \beta_u \beta_0} \quad (4)$$

still with $\beta_0 = \gamma_0/c$. The velocity v' belongs to an observer moving in the space reduced by ℓ_0 , between the extreme of the line closest to B and point B . The quantity γ_0 resembles instead the speed of a movement on the spot, traveling *within* the rectilinear rod. Let K_v be the frame that gets linked to the pointwise observer ($u \rightarrow v$), Eq. (4) returns the velocity composition law of SR, i.e.:

$$v' = v \ominus \gamma_0 \quad (5)$$

However, while γ_0 has the dimensions of a velocity, from a conceptual point of view it is *not*. Its limiting behavior for $\beta_0 \rightarrow 0^+$ should comply with time dilation length and contraction rules, $\gamma_0 \rightarrow (1 - \beta_0^2)^{-\frac{1}{2}} \gamma_0$. More generally, $\forall \beta_0 > 0$, Eqs. (4) and (5) suggest an extended transformation law given by:

$$\gamma'_0 = \frac{1 - \beta_v^2}{1 - \beta_v \beta_0} \gamma_0 \quad (6)$$

One may form now a new quantity, sum of measurements in Eqs. (5) and (6), and observe that it points out an identity relation of the Galilean type, holding $\forall v \leq c$:

$$v - \gamma'_0 = v \ominus \gamma_0 \quad (7)$$

or, identically:

$$(v \ominus \gamma_0) + \gamma'_0 = v \quad (8)$$

the light speed being the only value of velocity at which $\gamma'_0 = 0$.

In conclusion, the observer’s v is regained in Eq. (8) by summing a pseudo-velocity contribution, of a motion *in place* or *internal to the body*, to the velocity measurement (see sketch in Fig. 1).

Wiener’s process and Brownian relativity

The quantity γ_0 is reminiscent of a self-correlation term for the slower movement, pointing out the sought BwR analogy to SR. In what follows, neither Itô’s calculus [19] nor a phase space representation with momentum coordinates is necessary. It does suffice to recall a fundamental feature of Wiener–Levy’s process ($W(t)$), representing a Brownian motion with time increments much larger than Rayleigh’s

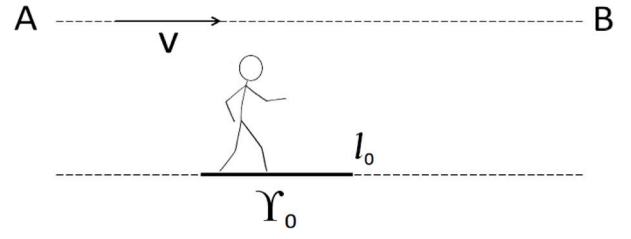


Fig. 1. Scheme of the experiment bringing to Eq. (8).

damping times [20]. Without losing generalities, let a one-dimensional motion and two time instants, $t > t' \geq 0$. Since [21]:

$$\overline{W(t)W(t')} = \min\{t, t'\} = \min\{\overline{W^2(t)}, \overline{W^2(t')}\} \quad (9)$$

it turns out a relation having the structure of Eq. (8):

$$\overline{(W(t) - W(t'))^2} + \overline{W(t)W(t')} = \max\{t, t'\} = t \quad (10)$$

with the first term accounting for the relative Brownian motion and the second for the self-correlation. It can be applied to a molecule long n units ($W_n(t')$) and a liquid molecule ($W_1(t)$) sharing the same paths:

$$\overline{W_1^2(t) + W_n^2(t')} - \overline{W_1(t)W_n(t')} = D_1 t \quad (11)$$

because, let $p_i(s, x, y) = e^{-\frac{(x-y)^2}{2D_i s}} / \sqrt{2\pi D_i s}$ be the Gaussian transition density, one obtains [22]:

$$\int_{\mathbb{R}^2} xy p_n(t', 0, x) p_1(t - t', x, y) dx dy = D_n t' \quad (t > t') \quad (12)$$

Eqs. (10) and (11) express how in BwR the time for a liquid molecule to run across two (radial) positions is regarded as invariant, regardless of chain paths in between. Eq. (10) may be also translated in random walk terms by the total step number N , step size l and Kuhn’s length l' , e.g.:

$$\frac{(N - n) l^2}{D_1} + \frac{l'^2}{D_n} = \frac{N l^2}{D_1} \quad (13)$$

that reduces to an identity law for a Rouse ideal coil with $l' = l$, $D_n = D_1/n$ accounting for the motion of its gravity center. These natural assumptions will be kept below, upon the hydrodynamic limit of large t and the (real-valued) continuation $(N, n) \rightarrow (t, t')$ of polymer physics [5].

Sum of second and third addenda on the left side of Eq. (11) is the zero mean of two uncorrelated increments, i.e. $(W_n(t') - W_1(0))(W_n(t') - W_1(t))$. One can perturb it by a random disturbance, and the two sampling frames of reference K_1, K_n will measure:

$$K_1(\tau, W^*) : D_1(t - \Delta^\alpha \tau) + \overline{W_1 \Delta^\alpha W^*} \quad (14)$$

$$K_n(\tau', W'^*) : D_1(t - \Delta^\alpha \tau') + \overline{W_n \Delta^\alpha W'^*} \quad (15)$$

the perturbation order being still unspecified, to be discussed later. Eqs. (14) and (15) cannot be equated, as $\Delta^\alpha \tau \neq \Delta^\alpha \tau'$ and Eq. (11) prescribe t to be fixed. This is possible in the limit of large time intervals, when the displacement between dynamic and static variances in K_n^* :

$$\Delta^\alpha \sigma^2(\tau', W'^*; n) = D_1 \Delta^\alpha \tau' - \overline{W_n \Delta^\alpha W'^*} \quad (16)$$

tends to the value taken on in K_1^* :

$$\Delta^\alpha \sigma^2(\tau, W^*; 1) = \Delta^\alpha \sigma'^2(\tau', W'^*; n) \quad (17)$$

$K_{n,1}^*$ indicate the two frames at the hydrodynamic limit, at which the polymer scaling laws were formerly derived from BwR. They comprise

a time, linked to dynamic fluctuations of a Brownian particle, and a measure of space which stems from shape fluctuations (see Eq. (9)):

$$\overline{W_1 \Delta^\alpha W^*} = \overline{(\Delta^\alpha W^*)^2} \equiv \Delta^\alpha \delta_1^2 \quad (18)$$

$$\overline{W_n \Delta^\alpha W^*} = \overline{(\Delta^\alpha W^*)^2} \equiv \Delta^\alpha \delta_n^2 \quad (19)$$

The relative diffusivity is pointed out as usual by the first average on the left side of Eq. (10), here evaluated at equal times and quantifying how much the diffusive limit set by D_1 is displaced by the slower chain molecule. To dismiss negative values, a time-ordering operation may be carried out by introducing, for any W_k , the time $T_{\epsilon k} = T + \epsilon/k$. Then we can benefit from the continuity of Wiener's process and take:

$$D_n^1 = \frac{1}{T} \lim_{\epsilon \rightarrow 0^+} \lim_{\substack{t' \rightarrow T_{\epsilon n} \\ t \rightarrow T_{\epsilon 1}}} \overline{(W_1(t) - W_n(t'))^2} = D_1 - D_n \quad (20)$$

This term conceptually replaces v^2 in Lorentz–Poincaré's transforms. Eq. (18), however, denotes a spatial perturbation to the root mean square traced by the liquid molecule, and generally is different from zero.

To proceed, note that line elements in Eq. (17) should be formally second-order's and, physically, still dimensioned to a length squared. Let the increment order be defined by:

$$\Delta^\alpha x \equiv (\Delta \sqrt{x})^\alpha \quad \alpha \in \mathbb{N}, x \geq 0 \quad (21)$$

one may therefore set $\alpha = 2$:

$$(\Delta \theta)^2 - (\Delta \rho_1)^2 = (\Delta \theta')^2 - (\Delta \rho_n)^2 \quad (22)$$

being $\theta^2 = D_1 \tau$, $\rho_m^2 = \overline{\delta_m^2}$. The case with $\alpha = 1$ anyway returns consistent results [23] and reduces to $\alpha = 2$ upon $\Delta \tau \sim \tau$ (unless of a proportionality constant, $\frac{1}{4}$, on both sides of Eq. (22)) [24]. On this basis, the BwR transforms taking the place of Lorentz–Poincaré's in SR read:

$$\begin{aligned} \Delta \rho_n &= \frac{1}{\sqrt{1 - d_n^1}} (\Delta \rho_1 - \sqrt{d_n^1} \Delta \theta) \\ \Delta \theta' &= \frac{1}{\sqrt{1 - d_n^1}} (\Delta \theta - \sqrt{d_n^1} \Delta \rho_1) \end{aligned} \quad (23)$$

having set the diffusion displacement in units of D_1 :

$$d_n^1 = \frac{D_n^1}{D_1} = 1 - \frac{1}{n} \quad (24)$$

Eqs. (23) specify a unitary matrix connecting different domains, i.e.:

$$\mathbf{L}_1(\Delta \rho_1, \Delta \theta) = \mathbf{L}_n^{-1}(\Delta \rho_n, \Delta \theta'). \quad (25)$$

As expected, they are not defined for diffusion coefficients $> D_1$, $d_n^1 \rightarrow -d_n^1$, upon which they anyway admit the trivial solution $\rho_m = \rho_m$ and $\theta' = \theta$. When $d_n^1 \rightarrow 1^-$, two identity relations follow, expressing Einstein's law of Brownian movement in both direct and inverse transforms, i.e. $\sqrt{d_n^1} \rightarrow -\sqrt{d_n^1}$:

$$\overline{\delta_n^2} / \tau' \sim \overline{\delta_1^2} / \tau \sim D_1. \quad (26)$$

Fig. 2 illustrates the view settled by Eq. (17), building up a framework that formally resembles SR. Note that Eqs. (23) indicate a relationship between statistical domains. Obviously, they do not form an ordinary Lorentz–Poincaré-covariant transformation of coordinates. Another point to remark is that the relative motion in BwR is the Brownian motion of the displacement from D_1 (Eq. (20)). That is why, in the former work, the new terminology 'diffusive horizon' was adopted.



Fig. 2. Schematic BwR representation (3000 steps). A Brownian liquid molecule fluctuates with diffusion coefficient D_1 until time t (gray path). A chain molecule (blackened snapshot) substitutes a portion of this path ($0 < t' < t$) at a constant diffusivity, similarly to the way the body in Fig. 1 extends within the trajectory, taking up a part of distance to be traveled.

Universal scaling and Gaussian chain

A proof of consistency of the basic spacetime scaling in the polymer picture stems naturally from the time dilation and length contraction rules entailed by Eqs. (23). The limit $n \gg 1$ equals to annul the additive constants (i.e. $\tau \sim \Delta \tau$), and the results are:

$$\begin{aligned} \rho_n &= \sqrt{\frac{D_1}{D_n}} \rho_1 \\ \tau &= \sqrt{\frac{D_n}{D_1}} \tau' \end{aligned} \quad (27)$$

whereas, for a system with molecularity $m = 1$, one gets again the trivial solution (i.e. $d_m^m \rightarrow 0^+$). Eqs. (27) can be applied to n molecules of a liquid sub-ensemble, Kuhn's step size being associated to a bare molecular time scale τ_0 . Let $\rho_1 = n l$ and $\theta = n \tau_0$, one gets:

$$\begin{aligned} \rho_n &= n l^2 \\ \tau &= n^2 \tau_0 \end{aligned} \quad (28)$$

that is, the random walk end-to-end size and Rouse's longest relaxation time of a chain molecule in solution [25]. These laws here come from the insertion of polymer extension into the spacetime structure. As in SR a length contracts and time expands as a consequence of motion, conformation and characteristic time of a macromolecule undergo in BwR an similar transform caused by a diffusive displacement from the host liquid. This will have more profound implications in a general BwR approach, when correlations are explicit, suggesting further scaling laws and inquiries on the significance of universality [1]. Eqs. (1) come themselves from a general version of BwR.

Finally, stepping back from BwR to SR, we put forward that Lorentz–Poincaré's transformations of SR admit a statistical interpretation in light of BwR and the previous experiment bringing to Eq. (4). We

assume that proper lengths and times are expressible as sums of independent aleatory variables, and compare the results of two observers, one (K') that moves with the rectilinear rod, the other (K) traveling at velocity equal e.g. to $-\gamma_0$, relative to K' . The length ℓ_0 gives a natural unit of measure for the time elapsed in this measurement, and the spatial Lorentz–Poincaré transform becomes:

$$\mathbf{X}'_n = \gamma_0 (\mathbf{X}_n - \lambda n) \quad (29)$$

where $\gamma_0 = 1/\sqrt{1-\beta_0^2}$ is Lorentz' factor with $\ell_0 \equiv n\lambda$. Letting correspondingly $\mathbf{x}'_i = \lambda \mathbf{z}'_i$, the quantity:

$$\frac{1}{\lambda} \mathbf{X}'_n = \sum_{i=1}^n \mathbf{z}'_i \quad (30)$$

is a sum of independent random variables, characterized by:

$$\overline{\mathbf{z}'_i} = -\gamma_0, \quad \text{Var}(\mathbf{z}'_i) = 1 \quad (31)$$

A natural application of the central limit theorem [26] now consists of partitioning time as space ($t = n\theta$) and forming the aleatory variable:

$$\mathbf{L}_n = \sqrt{\frac{\theta}{t}} \mathbf{X}_n \quad (32)$$

which converges in law to a Gaussian distribution upon $n \rightarrow \infty$:

$$\mathbf{L}_n \xrightarrow{\mathcal{L}} \mathbf{L} \sim \mathcal{N}(0, \gamma_0^{-2}) \quad (33)$$

As the unitary variance reduces to $1/\gamma_0^2$, one is left with Lorentz–FitzGerald's length contraction, reinterpreted in a statistical fashion. Clearly, the inverse transform i.e.:

$$\overline{\mathbf{z}'_i} = \gamma_0, \quad \text{Var}(\mathbf{z}'_i) = 1 \quad (34)$$

leaves Eq. (33) invariant, as it had to be.

A similar argument applies to the temporal domain. As usual in deriving the time dilation rule, a rest clock is placed in K to measure a distribution of (here, non-dimensional) times:

$$\frac{1}{\theta} \mathbf{J}_n = \sum_{i=1}^n \mathbf{y}_i \quad (35)$$

still with unitary variance, and mean values:

$$\overline{\mathbf{y}_i} = \pm \beta_0^2, \quad \text{Var}(\mathbf{y}_i) = 1 \quad (36)$$

In this dual case, the central limit theorem is applied to the random variable:

$$\mathbf{T}'_n = \sqrt{\frac{\lambda}{x}} \mathbf{J}'_n \quad (37)$$

and, since the unit variance now dilates by γ_0^2 , it returns the time dilation rule:

$$\mathbf{T}'_n \xrightarrow{\mathcal{L}} \mathbf{T} \sim \mathcal{N}(0, \gamma_0^2) \quad (38)$$

Observe that the Fourier transforms in non-dimensional time and space values (\mathcal{F}) of the normal laws in Eq. (33) (p_L) and Eq. (38) (p_T) are related, e.g. symbolically, by:

$$\gamma_0 \mathcal{F}^{-1} \{ p_T \} = p_L, \quad \mathcal{F} \{ p_L \} = \gamma_0 p_T \quad (39)$$

and, interestingly, a tradeoff relationship takes place:

$$\text{Var}(\mathbf{L}) \text{Var}(\mathbf{T}) = 1 \quad (40)$$

We thus see that BwR, connecting the theoretical structures of Brownian motion and relativity, may lead to the emergence of probabilistic laws for time and space, as Eq. (40) provides with an example.

Conclusions

It has been examined more in depth how a BwR approach can model scaling and universality in polymer solutions. A pseudo-velocity concept, for the length extension of a rectilinear body, is introduced

by a thought experiment, bringing back to a composition law of the Galilean type. The covariance behavior of Wiener's process then is able to hint an analogy with the second SR postulate. A statistical reading of spacetime transforms is finally set by the central limit theorem. Inclusion of the extension of a body into kinematics may lead to a new picture with an extended notion of movement for material shapes, with given statistical mechanics or microscopic properties of their constituent matter.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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