



Propagation of spinors on a noncommutative spacetime: equivalence of the formal and the effective approach

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Abstract Some noncommutative (NC) theories possess a certain type of dualities that are implicitly built within their structure. In this paper we establish still another example of this kind, and we do this perturbatively in the first order of the Seiberg–Witten expansion. More precisely, we show that a particular model of noncommutative $U(1)_*$ gauge field coupled to a NC scalar field and to a classical geometry of the Reissner–Nordström (RN) type is to a first order in deformation completely equivalent at the level of equations of motion to the commutative $U(1)$ gauge theory coupled to a commutative scalar field and to a classical geometry background, different from the starting RN background. The new (effective) metric is obtained from the RN metric by switching on an additional nonvanishing $r - \phi$ component. Using this first order duality between two theories and physical systems they describe, we formulate an effective approach to studying a dynamics of spin $\frac{1}{2}$ fields on the curved background of RN type with an abiding noncommutative structure. As opposed to that, we also investigate in a more formal way a dynamics of spin $\frac{1}{2}$ fields, and we do this perturbatively, within a first order in deformation parameter, by studying a semiclassical theory which describes the NC $U(1)_*$ gauge field coupled to NC spin $\frac{1}{2}$ field and also coupled to gravity, which is however treated classically. Upon utilising the Seiberg–Witten (SW) map in order to write the NC spinor and NC gauge fields in terms of their corresponding commutative degrees of freedom, we find that the equation of motion for the fermion field obtained within the formal approach exactly coincides with the equation of motion obtained within the effective approach that utilises first order noncommutative duality. Therefore, linearized equations of motion for a spinor field in SW expansion turn out to be the same as equations

of motion in a perturbed metric. We then use these results to analyze the problem of stability of solutions of the equations of motion and the associated issue of superradiance, as related to fermions in RN spacetime with an all-pervasive noncommutative structure.

1 Introduction

Many distinct approaches to a unification of quantum mechanics with gravity in the ultraviolet sector point toward the existence of absolute minimal length scale, whose very presence puts a lower bound to the minimal possible resolution of space as its intrinsic property. As a consequence, one of the cornerstones of quantum mechanics, the Heisenberg uncertainty relations, start to call for a revision, resulting in a generalized uncertainty principle [1,2], which is also suggested by perturbative string theory [3,4], quantum gravity [5,6] and black hole physics [7]. Moreover, in loop quantum gravity a process of quantization gives rise to the area and volume operators which have discrete spectra, whose lowest possible eigenvalues are being proportional to the square and cube of the Planck length, respectively [8,9]. This, together with the generalized uncertainty principle, implies the existence of a minimum uncertainty in position [7,10–12].

One of the more known patterns to implement a minimal length scale in quantum mechanics, quantum field theory (QFT) and gravity is provided by the frame of noncommutative (NC) geometry, which is characterized by the fact that the spacetime coordinates get raised to the level of the operators and thus generally fail to commute [13–18]. The idea of noncommutativity of spacetime was first clearly articulated by Snyder [19] and expounded further in terms of geometric notions by Connes [20,21]. Correspondingly, a possibility to observe the consequences of spacetime noncommutativity

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and the existence of a minimal length scale led to the intensive study of noncommutative versions of quantum mechanics, QFT and gravity with the aim of revising the standard theories as diverse as the gauge theory, particle physics and the Quantum Hall effect (QHE), thermodynamics and the black-hole physics and cosmology, so that they may keep the track with and accommodate the eventual novel features into their framework [22–33].

It is known in general that various noncommutative models, including models of noncommutative field theory and noncommutative gravity allow for a representation in terms of classical commutative fields either in a compact and closed form or in a form of a perturbation expansion up to a certain order in a deformation parameter. The most known example of this kind is obtained by means of the Seiberg–Witten (SW) map [16], which is a field transformation that allows to rewrite a gauge theory on noncommutative space as a gauge theory on commutative space. In an attempt to map a noncommutative gauge field theory to its commutative counterpart from which it emerged as a result of deforming an associated Poisson structure, the SW map undoubtedly becomes highly important.

Related to this, it is noteworthy to recall that the SW map helped to resolve some of the issues that had appeared by the introduction of the star-product into the action, among them being the advent of the field operator ordering ambiguities, as well as the breaking of the ordinary gauge invariance and the problem with the charge quantisation. This map also ensures that by going from a set of degrees of freedom describing noncommutative gauge symmetry to a corresponding set describing local commutative gauge symmetry the number of degrees of freedom stays the same. This way a number of NC deformed QFT's could be properly defined for arbitrary gauge group representations, which has facilitated the building of the whole range of semi-realistic NC deformed particle physics [34–42] and gravity models [43–47].

Another interesting situation where the NC theory allows for an interpretation in terms of commutative degrees of freedom involves a class of noncommutative models which have been shown to exhibit a specific type of duality relations that are implicitly built-in at the level of equations of motion. This feature has been observed in some hybrid or semi-hybrid noncommutative models investigated within a so called realization framework [48–54], which utilizes the representation of NC coordinates and NC field operators in terms of formal power series of generators of the undeformed Heisenberg algebra. The Lagrangian density in these models, which is expressed partially or completely in terms of NC degrees of freedom (that's why these models are being labelled as hybrid or semi-hybrid), allowed not only for a reinterpretation of the Lagrangian density in terms of commutative degrees of freedom, but also allowed for a radical refashioning of the initial semi-hybrid NC model so that it could take on a form

of an effective commutative model realized within a similar or even the same physical setting, but with modified system parameters. Such reinterpretation was then able to give noncommutativity a definite physical meaning. For example, in [55–59] it has been found that the semi-hybrid model of NC massless scalar field coupled to a classical nonrotational BTZ geometry is dual to the model of massive commutative scalar field probing the geometry of a rotating BTZ black hole. In this way, the noncommutativity took on the role of an agent medium that has put a black hole into a state of rotation. Besides, the noncommutativity was shown to be responsible for a mass generating mechanism, as applied to a scalar probe, and for inducing certain back-reaction effects.

A similar situation has been encountered in [60], where the analogy between NC version of the Schwarzschild black hole and the commutative Reissner–Nordström black hole with a stretched horizon was drawn. Likewise, in [61] the authors have shown that the minimal $U(1)$ NCQED based on a reversible Seiberg–Witten (SW) map is equivalent to the Moyal NCQED without SW map, as manifested at the level of tree-level scattering amplitudes [61]. In this case the equivalence between two models comes as a result of a mutual cancellation between terms induced by the reversible SW map, which might also be viewed as being due to a presence of the specific duality relations that are inherent to the noncommutative model being considered. Similarly, the features of this kind can be found in supersymmetric noncommutative field theories related by the theta-exact Seiberg–Witten map [62]. In most of these cases (certainly for the case studied in [55–59]) the notion of duality refers to an exact mathematical correspondence that may be drawn between two different physical systems having different system parameters, though governed by the same Lagrangian density and the associated equation of motion. A duality understood that way gives an example of the equivalence that can be established between noncommutative and commutative model, where each of these models separately describes its own respective physical system. As these two systems that commutative and noncommutative models refer to are actually being governed by the same equations of motion and the same Lagrangian density, they may too be characterized as being equivalent with each other. Therefore, referring in the current context to a duality itself that exists between two different physical systems, it shouldn't come as a surprise that it gets manifested through a set of exact mathematical transformations that connect the parameters of these two physical systems, thus making them dual with each other.

In this paper we set out to find another example of this kind within the semi-hybrid NC model studied in [63–65]. In particular, here we show that noncommutative $U(1)_*$ gauge theory coupled to NC scalar field and to a classical geometry of the Reissner–Nordström type is equivalent within a first order of deformation to a commutative $U(1)$ gauge the-

ory coupled to a commutative scalar field and to a classical geometry background, which however does not coincide with the initial RN metric, but instead represents an effective metric which encodes the impacts of the spacetime deformation. In other words the former model can be recast into a latter by redefining the components of the initial RN metric. In this way we end up with an effective, but equivalent description in which the redefined metric has an important role, forming one of the crucial building blocks that characterize the dual system and the duality transformation itself. From now on we refer to this redefined metric as the effective metric characterizing the first order dual picture, or the first order effective dual metric in short. It can be viewed as having ensued from a specific deformation of the RN metric which brings in a nonvanishing $r - \phi$ component. This whole scheme thus establishes a first order duality relation between two models, one (semiclassical hybrid) noncommutative and the other fully commutative, and two physical systems they describe.

As a further step, we study a dynamics of fermions on a curved background with a deformed spacetime structure, where as an exemplar for the curved geometry we use the RN black hole background. This study has been carried out within two different approaches: effective and formal. The effective approach uses standard notions of commutative differential geometry, with parallel implementation of the noncommutative-born effects through the utilisation of the type of duality just explained, where the effective dual metric mimics the impacts of noncommutativity. The formal approach is a level beyond more rigorous in a sense that it attempts to stay in line and be compatible with the requirements that the NC $U(1)_*$ Dirac action on a curved background remains invariant under NC gauge transformations, as well as under the undeformed local $SO(1, 3)$. With that in mind, we put forth a proposal for the NC $U(1)_*$ Dirac action that could meet these requirements. As this NC $U(1)_*$ Dirac action that we propose is being expressed in terms of the NC spin $\frac{1}{2}$ field and the NC $U(1)_*$ gauge field that are both coupled to gravitational degrees of freedom, it is important to stress that it will remain invariant under the NC $U(1)_*$ gauge transformations as long as we assume that the gravity is unaffected by them, i.e. $\delta_* g_{\mu\nu} = 0$ (or alternatively $\delta_* e^a{}_\mu = \delta_* \omega_\mu{}^{ab} = 0$). In other words, our formal approach assumes that the NC spin $\frac{1}{2}$ field, and the NC $U(1)_*$ gauge field are the only degrees of freedom that get affected by the NC gauge transformations. In addition, we will assume that the local $SO(1, 3)$ symmetry is unaffected by NC deformation. The NC $U(1)_*$ Dirac action that we propose is in a line with the proposals made in [43, 44].

The main result of this paper is to show that these two different approaches surprisingly give rise to the same equation of motion describing a dynamics of fermions on a curved

space in presence of a NC structure of spacetime, therefore explicitly demonstrating the equivalence between the effective and formal, more rigorous approach. Namely, what has been shown in the present paper is that the equation of motion obtained by using the SW map for spin $\frac{1}{2}$ and gauge fields, and by varying the NC $U(1)_*$ Dirac action that is invariant under NC gauge transformations and $SO(1, 3)_*$ group at the end of the day appears to be the same as the equation of motion obtained by simply writing the Dirac equation on a geometric background described by the effective dual metric.

Dirac equation in the context of noncommutative spaces is important from many reasons and was studied intensively in the literature. The range of topics where it finds application is vast, going from the high energy physics, all through the gravitational physics and cosmology and all the way down to the problems in condensed matter. In particular, the problems related to high energy physics involve a study of the hydrogen atom spectrum on Moyal [66] and kappa-Minkowski space [67], the impact of quantum deformation on the spin- $\frac{1}{2}$ Aharonov–Bohm effect [68], the problem of Yukawa couplings and seesaw neutrino masses in noncommutative gauge theory [40], photon-neutrino interaction in noncommutative field theory [69], renormalizability and dispersion of chiral fermions in NCQED [70] and the impact on neutrino oscillations due to noncommutativity of spacetime [71] to name just a few. Besides canonical and kappa-Minkowski type of noncommutativity (for a recent review see [72]), other types of spacetime noncommutativity that have been frequently studied in the past include that of Snyder type (for the review see [73, 74]), as well as that which is usually referred to as the “spin noncommutativity”, first introduced in [75, 76], and which could be theoretically understood as a non-relativistic analog of the original Snyder’s model [19]. In [77], the spin noncommutativity was obtained by means of a consistent deformation of the Berezin–Marinov pseudoclassical model for the spinning particle [78]. Like the Snyder model, spin noncommutativity exhibits preservation of the Lorentz symmetry. Within this framework a modification of the Dirac equation was proposed and a dynamics of a Dirac fermion in the presence of spin noncommutativity was studied in [77, 79].

In condensed matter, the topic of special interest is the integer and in particular the fractional quantum Hall effect and a related attempt to procure the explanation for the latter in terms of the Dirac oscillator [80] and especially in terms of the Dirac oscillator on NC space [81–89]. Due to the same reasons the study of relativistic Landau levels or Dirac-Landau levels, including the breaking of their degeneracy, becomes increasingly more important and even more so as these results were being applied to the graphene and the related nanostructures [90, 91].

Another intriguing field where the Dirac equation under conditions of discretized spacetime was being analyzed

involves deformed relativistic wave equations, namely the Klein–Gordon and Dirac equations in a Doubly Special Relativity (DSR) scenario [92–97]. Besides the algebraic approach to this problem, which originally started by considering the standard real form of the quantum anti-de Sitter algebra, $SO_q(3, 2)$ and then by consistently modifying the related coproduct [98], there recently appeared a geometric approach [99–102] to the same problem, which is based on the geometry of a curved momentum space [103–109]. While Dirac equation obtained in [98] is invariant under the spin-half representation of the κ -Poincaré algebra, it doesn't yield the Casimir after squaring. Instead, its square gives rise to the κ -deformed Pauli–Lubanski vector. Contrary to that, Dirac equation obtained in [102] gives rise to the κ -Poincaré Casimir upon squaring, along with having the required symmetry properties. This geometric approach should be seen as complementary to the more spread algebraic one [110–116]. Finally, it is worthy to mention that Dirac equation has an important role in studying wide range of physical processes that occur near the black hole horizon, such as the scattering and absorption processes for Dirac particles [117], the spectral power emission of Dirac fermions [118, 119], including the study of superradiance [120] and the quasinormal mode (QNM) spectrum for the fermionic perturbations. In the latter case, of special interest is the study of the impact of NC spacetime deformation on QNM spectrum of the fermionic perturbations of black holes [58].

The paper is organized as follows. After a very brief review of NC deformation that we analyze in this paper, in Sect. 3 we study semiclassical NC $U(1)_*$ gauge theory coupled with NC spin $\frac{1}{2}$ field and NC gravitational degrees of freedom, whose action is invariant under NC gauge transformations and the undeformed local $SO(1, 3)$ group. Upon utilising the Seiberg–Witten map in order to write NC spinor and NC gauge fields in terms of their corresponding commutative degrees of freedom, the equation of motion for the fermion field is obtained by varying the action over $\bar{\Psi}$. The effective dual metric (derived in Appendix A) is then used in Sect. 4 to write a noncommutative version of the equation of motion for the fermions in a curved background of RN type, thus putting forth an effective approach to the same problem that is treated in a formal, more rigorous way in Sect. 3. Here we find that surprisingly, both of these two approaches, formal and effective, yield the same final result. In Sect. 5 we show that the resulting equation of motion is separable, yielding two pairs of equations, one for the angular part and the other for the radial part. Noncommutative deformation appears to affect only the radial part, with the angular part being solved by the same spin $\frac{1}{2}$ spherical harmonics as in the case of Dirac equation for the hydrogen atom in flat or Schwarzschild case. At the end, we utilise the general properties of the fermionic solutions deduced here to investigate the problem of their stability. In particular, the issue of superradiance is considered

as related to the solutions of the Dirac equation in RN spacetime, and especially the impact of noncommutative deformation on the effect of superradiance is addressed. We end up with two Appendices. In Appendix A we demonstrate the equivalence between semiclassical NC $U(1)_*$ gauge theory with NC scalar field on a classical RN background and commutative $U(1)$ gauge theory with ordinary scalar field on the background with the effective dual metric. Furthermore, we find the explicit form of the effective dual metric in a first order of deformation. In Appendix B we briefly discuss possible generalizations of our model to include settings with more nontrivial geometric backgrounds, as well as types of deformation.

2 Preliminary settings

A solution to Einstein equations representing a charged non-rotating black hole with mass M and charge Q is given by the Reissner–Nordström (RN) metric

$$ds^2 = \left(1 - \frac{2MG}{r} + \frac{Q^2G}{r^2}\right)dt^2 - \frac{dr^2}{1 - \frac{2MG}{r} + \frac{Q^2G}{r^2}} - r^2(d\theta^2 + \sin^2\theta d\phi^2). \quad (1)$$

Being static and spherically symmetric, the spacetime of RN black hole has four Killing vectors, among which ∂_t and ∂_ϕ are included, and t and ϕ are the time and polar variables of the spherical coordinate system $x^\mu = (t, r, \theta, \phi)$.

In the previous paper [63] we have introduced a semiclassical model describing a charged NC scalar field $\hat{\Phi}$ and NC $U(1)$ gauge field \hat{A} on a classical gravitational background of RN type. By semiclassical we mean that while the gravitational field in this model was assumed to be a classical degree of freedom (i.e. not deformed by noncommutativity), the scalar and gauge field propagating in that classical gravitational background were assumed to be affected by noncommutative nature of spacetime. In a sense, we are therefore dealing with a situation where the scalar and gauge field are quantized and gravitational field is not. It is however important to stress that the gauge and scalar field are not quantized in a sense of quantum field theory.

The model was built by using deformation quantization techniques based on Drinfeld twist operator and the explicit twist operator that was used in the construction was the so called angular twist operator [63, 64]

$$\begin{aligned} \mathcal{F} &= e^{-\frac{i}{2}\theta^{\alpha\beta}\partial_\alpha\otimes\partial_\beta} \\ &= e^{-\frac{i}{2}(\partial_t\otimes\partial_\phi - \partial_\phi\otimes\partial_t)}, \end{aligned} \quad (2)$$

with $\alpha, \beta = t, r, \theta, \phi$ and $\theta^{t\phi} = -\theta^{\phi t} = a$ as the only non-zero components of the deformation tensor $\theta^{\alpha\beta}$. The small constant parameter a is the deformation parameter that sets

up the NC scale, commonly related to the Planck length. This twist operator is a Killing twist, since it is built from the vector fields that are actually Killing vectors for the metric (1). In this way it is ensured that the geometry (1) stays unaffected by the deformation as the twist (2) does not act on the RN metric.

The star product, the wedge star product between forms, the coproduct and other structural maps of the related symmetry algebra can all be obtained from the twist operator (2). In particular, the star product between functions is given by

$$f \star g = \mu \circ \mathcal{F}^{-1}(f \otimes g) = \mu \{ e^{\frac{ia}{2}(\partial_t \otimes \partial_\phi - \partial_\phi \otimes \partial_t)} f \otimes g \} \\ = fg + \frac{ia}{2}(\partial_t f(\partial_\phi g) - \partial_t g(\partial_\phi f)) + \mathcal{O}(a^2), \tag{3}$$

where the map μ represents the usual pointwise multiplication. The remaining ingredients of the differential calculus are described in [63].

3 Spinor field on the noncommutative RN background: formal analysis

A massive and charged spinor field Ψ on a fixed gravitational background can be described by the following action

$$S = \int d^4x |e| \bar{\Psi} \left(i \gamma^\mu D_\mu \Psi - m \Psi \right). \tag{4}$$

The mass and the charge of the spinor field Ψ are respectively m and q . The determinant of the vierbein e^a_μ we label with $|e| = \sqrt{-g}$ and the covariant derivative D_μ includes both the spin connection ω_μ and the $U(1)$ gauge field A_μ

$$D_\mu \Psi = \partial_\mu \Psi - \frac{i}{2} \omega_\mu^{ab} \Sigma_{ab} \Psi - iq A_\mu \Psi. \tag{5}$$

Matrices Σ^{ab} are the (hermitian) generators of the local Lorentz transformations and they close the Lorentz algebra $[\Sigma^{ab}, \Sigma^{cd}] = i(\eta^{ad} \Sigma^{bc} + \eta^{bc} \Sigma^{ad} - \eta^{ac} \Sigma^{bd} - \eta^{bd} \Sigma^{ac})$.

The spin connection is not an independent field, but a function of e^a_μ , calculated from the torsion free condition

$$T^a_{\mu\nu} = \nabla_\mu e^a_\nu - \nabla_\nu e^a_\mu = 0,$$

with $\nabla_\mu e^a_\nu = \partial_\mu e^a_\nu + \omega_\mu^a_b e^b_\nu$. For more details on the spin connection, vierbeins and the notation we use, we refer to the beginning of Sect. 4.

The action (4) is invariant under the local $U(1)$ transformations

$$\delta_\alpha \Psi = i\alpha(x)\Psi, \quad \delta_\alpha \bar{\Psi} = -i\bar{\Psi}\alpha(x), \quad \delta_\alpha A_\mu = \frac{1}{q}\partial_\mu\alpha(x). \tag{6}$$

Note that these transformations do not act on the gravitational background, that is on e^a_μ and ω_μ . The action (4) is

also invariant under the general coordinate transformations and the local $SO(1, 3)$ symmetry. In this paper we use the semiclassical analysis and promote only the $U(1)$ gauge symmetry to the noncommutative $U(1)_\star$ gauge symmetry. In our future work we will lift this approximation and allow for the noncommutative local $SO(1, 3)_\star$ symmetry. The equation of motion for the spinor field Ψ is obtained by varying the action (4) with respect to $\bar{\Psi}$ and it is given by

$$i\gamma^\mu \left(\partial_\mu \Psi - i\omega_\mu \Psi - iq A_\mu \Psi \right) - m\Psi = 0. \tag{7}$$

Following the steps from [63], we now introduce an action functional that describes the NC $U(1)_\star$ gauge theory of a charged spinor field on the RN background¹ (1)

$$S_\star = \int d^4x |e| \star \bar{\hat{\Psi}} \star \left(i\gamma^\mu (\partial_\mu \hat{\Psi} - i\omega_\mu \star \hat{\Psi} - iq \hat{A}_\mu \star \hat{\Psi}) - m \hat{\Psi} \right). \tag{8}$$

Noncommutative fields are now labeled with a $\hat{}$ and the \star -product is given by (3). One can show that this action is invariant under the following infinitesimal $U(1)_\star$ gauge transformations:

$$\delta_\star \hat{\Psi} = i\hat{\Lambda} \star \hat{\Psi}, \\ \delta_\star \hat{A}_\mu = \partial_\mu \hat{\Lambda} + i(\hat{\Lambda} \star \hat{A}_\mu - \hat{A}_\mu \star \hat{\Lambda}), \\ \delta_\star \omega_\mu = \delta_\star e^a_\mu = 0, \tag{9}$$

where $\hat{\Lambda}$ is the NC gauge parameter. In particular, note that

$$\delta_\star D_\mu \hat{\Psi} = i\hat{\Lambda} \star D_\mu \hat{\Psi}$$

since the twist (2) does not act on the gravitational field and therefore $\omega_\mu \star \Lambda = \omega_\mu \cdot \Lambda = \Lambda \star \omega_\mu$. Note that the action (8) can be written in a more geometric way [121] such that the general coordinate transformation invariance is manifest

$$S \sim \int \left((\overline{D\hat{\Psi}})_B \star \hat{\Psi}_A - \bar{\hat{\Psi}}_B \star (D\hat{\Psi})_A \right) \wedge_\star (e \wedge_\star e \wedge_\star e \gamma_5)_{BA}, \tag{10}$$

with spinor indices A, B explicitly written and the vierbein one form e . The covariant derivative one-form is given by $D\hat{\Psi} = d\hat{\Psi} - i\omega \star \hat{\Psi} - i\hat{A} \star \hat{\Psi}$, with the gauge potential one-form \hat{A} and the spin connection one-form ω . When the Killing twist is used and the action (10) is expanded in a chosen coordinate basis, the action (8) is obtained.

Similarly to [63], we can add the action for the NC $U(1)_\star$ gauge field \hat{A}_μ to (8), promoting the gauge field \hat{A}_μ into a dynamical field. However, since later on we will be interested in propagation of NC spinor field on the fixed RN background, we do not write the action for the NC $U(1)_\star$ gauge field \hat{A}_μ explicitly here.

¹ Actually, it can be any background with Killing vectors ∂_t and ∂_ϕ .

To simplify the calculation, from now on we redefine $A_\mu = qA_\mu$. Then we use the Seiberg–Witten (SW)-map [16,27] in order to express NC fields $\hat{\Psi}$ and \hat{A}_μ as functions of the corresponding commutative fields and the deformation parameter a . The SW-map assumes an expansion in orders of the deformation parameter and this expansion is known to all orders for an arbitrary Abelian twist deformation [27,44,45] of which the twist (2) is only one example. For the twist operator (2), SW-map gives rise to the following expansions for the fields:

$$\hat{\Psi} = \Psi - \frac{1}{2}\theta^{\rho\sigma}A_\rho(\partial_\sigma\Psi), \tag{11}$$

$$\hat{A}_\mu = A_\mu - \frac{1}{2}\theta^{\rho\sigma}A_\rho(\partial_\sigma A_\mu + F_{\sigma\mu}). \tag{12}$$

The expanded action up to first order in a is given by

$$S_\star = \int d^4x |e|\bar{\Psi}\left(i\gamma^\mu D_\mu\Psi - m\Psi\right) + \frac{1}{2}\theta^{\alpha\beta}\left(-iF_{\mu\alpha}\bar{\Psi}\gamma^\mu D_\beta^{U(1)}\Psi - \frac{i}{2}\bar{\Psi}\gamma^\mu\omega_\mu F_{\alpha\beta}\Psi - \frac{1}{2}F_{\alpha\beta}\bar{\Psi}(i\gamma^\mu D_\mu^{U(1)}\Psi - m\Psi)\right).$$

Remembering that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ and choosing the electromagnetic potential to be that of the RN black hole, the only non-zero component of $F_{\alpha\beta}$ is $F_{rt} = \frac{qQ}{r^2}$. This leads to a simplified NC action

$$S_\star = \int d^4x |e|\bar{\Psi}\left(i\gamma^\mu D_\mu\Psi - m\Psi\right) - \frac{i}{2}\theta^{\alpha\beta}\bar{\Psi}F_{\mu\alpha}\gamma^\mu D_\beta^{U(1)}\Psi \tag{13}$$

and the corresponding equation of motion for the spinor Ψ

$$i\gamma^\mu\left(\partial_\mu\Psi - i\omega_\mu\Psi - iA_\mu\Psi\right) - m\Psi - \frac{ia}{2}F_{rt}\gamma^r\partial_\phi\Psi = 0. \tag{14}$$

Inserting the explicit expressions for F_{rt} and $\gamma^r = e_a{}^r\gamma^a$, this equation reduces to

$$i\gamma^\mu\left(\partial_\mu\Psi - i\omega_\mu\Psi - iA_\mu\Psi\right) - m\Psi - \frac{ia}{2}\frac{qQ}{r^2}\sqrt{f}\gamma^1\partial_\phi\Psi = 0. \tag{15}$$

For a later comparison with the result that will be obtained in an effective approach, it is instructive to write this equation in terms of two-component spinors $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$. Note that the spin connection part ω_μ and vielbeins refer to RN background.²

² How they look like in RN background may be inferred from relations (28), (22) and (23), by taking into account that $\omega_\mu \equiv -\frac{1}{2}\omega_\mu{}^{cd}\Sigma_{cd}$ and by setting a to zero.

$$i\frac{1}{\sqrt{f}}i\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\partial_r\Psi + i\sqrt{f}i\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\partial_r\Psi + i\frac{1}{r}i\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}\partial_\theta\Psi + i\frac{1}{r\sin\theta}i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\partial_\phi\Psi + (e_0{}^t\gamma^0\omega_t + e_2{}^\theta\gamma^2\omega_\theta + e_3{}^\phi\gamma^3\omega_\phi)\Psi + e_0{}^t\gamma^0A_t\Psi - m\Psi + \frac{a}{2}\frac{qQ}{r^2}\sqrt{f}\begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\partial_\phi\Psi = 0. \tag{16}$$

The particular model of NC gauge theory applied to NC spinor field, that we consider here, involves NC spinor field which is minimally coupled to both, NC $U(1)$ gauge field and classical gravitational field of the RN background. While the gauge field itself is fixed to be the Coulomb field, but only until after rejecting all except the first order terms in the SW expansion and varying the action to get the equation of motion, the gravitational field is fixed from the very beginning to be that of the RN background. In practice this means that the only propagating degrees of freedom in this model are those of the matter fields. We point out that the working setting just described is completely analogous (even identical) to the one that we used in our previous work [63] when studying a particular model of NC gauge theory, as applied to NC scalar field. The main assumption of this setting is that a gravitational field is being considered as a classical (commutative) object, and matter fields along with a gauge field are being considered as noncommutative objects. From this reason we term this kind of working framework as *semi-classical*, bearing on the fact that it does not correspond to a full NC gauge theory, but only to a description of NC matter field in a particular setup (static charge, black hole geometry). The immediate consequence of the gravitational degrees of freedom (either components of the metric or vielbeins) being classical is that they do not change/transform under infinitesimal NC gauge transformations. Therefore, an immutability of the gravitational degrees of freedom is an assumption and a starting point of this particular NC framework, and not its consequence.

Mathematically, the semiclassical approximation manifests itself in (13) in the following way: the covariant derivative $D_\mu\Psi = \partial_\mu\Psi - iA_\mu\Psi - i\omega_\mu\Psi$ includes both the electromagnetic ($U(1)$) and the gravitational part, while the covariant derivative $D_\beta^{U(1)}\Psi = \partial_\beta\Psi - iA_\beta\Psi$ has only the electromagnetic part. In the NC correction only the $U(1)$ part appears.

4 Spinor fields on the noncommutative RN background: effective approach

In our previous paper [63] we have analyzed a propagation of the NC scalar field in the RN background. The equation

of motion governing the evolution of the scalar field is given by

$$\left(\frac{1}{f}\partial_t^2 - \Delta + (1-f)\partial_r^2 + \frac{2MG}{r^2}\partial_r + 2iqQ\frac{1}{rf}\partial_t - \frac{q^2Q^2}{r^2f} - \mu^2\right)\Phi + \frac{aqQ}{r^3}\left(\left(\frac{MG}{r} - \frac{GQ^2}{r^2}\right)\partial_\phi + rf\partial_r\partial_\phi\right)\Phi = 0, \tag{17}$$

with $f = 1 - \frac{2MG}{r} + \frac{Q^2G}{r^2}$. In [56,57], within first order of deformation, an equivalence between the NC scalar field on the non-rotating BTZ background and a commutative scalar field on the rotating BTZ background was established. We will follow that idea here and try to understand if there is an effective description of the NC scalar field on the RN background.

It can be shown that the equation of motion (17) may be rewritten as the equation of motion governing a charged commutative scalar field with the same charge q as its NC counterpart, and propagating in some effective metric. That this process of finding a metric from the given equation of motion can indeed be carried out³ within a first order of deformation is shown in Appendix A. There the first order effective dual metric has been derived and shown to pick up the form of a modified RN geometry

$$ds^2 = \left(1 - \frac{2MG}{r} + \frac{Q^2G}{r^2}\right)dt^2 - \frac{dr^2}{1 - \frac{2MG}{r} + \frac{Q^2G}{r^2}} - aqQ \sin^2\theta drd\phi - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{18}$$

It appears that new, first order effective dual metric (18) acquires an additional off-diagonal term which is induced purely by noncommutative nature of spacetime. This feature comes into play only in the presence of charged matter. Unlike in the case of scalar field in the (NC) BTZ background, in this case the effective metric cannot be interpreted as a metric of a background rotating geometry (of either RN or any other type).

Having established the effective metric (18), we now investigate the propagation of a charged massive spinor field Ψ in this geometry. In particular, it is interesting to see if this effective approach agrees with the more rigorous approach from Sect. 2. The Dirac equation in a curved background given by the effective noncommutative (NC) metric

$$(i\gamma^a\nabla_a - m)\Psi = 0, \tag{19}$$

where the Latin indices such as a , ($a = 0, 1, 2, 3$) refer to intrinsic coordinates and γ^a are the standard flat space Dirac

gamma matrices, $\{\gamma_a, \gamma_b\} = 2\eta_{ab}$, where

$$\eta_{ab} = \eta^{ab} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{20}$$

If in addition, the spinor field is charged, this gives rise to a Dirac equation in which the gauge potential A_μ is minimally coupled to a Dirac operator on a curved background

$$\left(i\gamma^a(\nabla_a - iA_a) - m\right)\Psi = \left(i\gamma^a e_a^\mu(\nabla_\mu - iA_\mu) - m\right)\Psi = 0. \tag{21}$$

The gravitational covariant derivative ∇_μ is defined as $\nabla_\mu\Psi = \partial_\mu\Psi - \frac{i}{2}\omega_\mu^{ab}\Sigma_{ab}\Psi$. The Dirac operator $\gamma^a\nabla_a$ on a curved space is introduced in terms of tetrads (vierbeins) e^a_μ and their inverse e_a^μ , satisfying $e^a_\mu e_a^\nu = \delta_\mu^\nu$ and $e^a_\mu e_b^\mu = \delta^a_b$. Tetrads written in components are $e^a_\mu = (e^a_t, e^a_r, e^a_\theta, e^a_\phi)$ and $e_a^\mu = (e_0^\mu, e_1^\mu, e_2^\mu, e_3^\mu)$. They also satisfy $g_{\mu\nu} = e^a_\mu e_b^\nu \eta_{ab}$ and $g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}$. In what follows we use the setting defined in [122] with the vierbein frame chosen to be

$$e^a_\mu = \begin{pmatrix} \sqrt{f} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{f}} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & \frac{aqQ}{2r} \sin\theta & 0 & r \sin\theta \end{pmatrix} \tag{22}$$

with the corresponding inverse matrix

$$e_a^\mu = \begin{pmatrix} \frac{1}{\sqrt{f}} & 0 & 0 & 0 \\ 0 & \sqrt{f} & 0 & -\frac{aqQ}{2r^2}\sqrt{f} \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & 0 & 0 & \frac{1}{r \sin\theta} \end{pmatrix}. \tag{23}$$

The representation of gamma matrices is

$$\begin{aligned} \gamma^0 &= i\tilde{\gamma}^0 = i \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, & \gamma^1 &= i\tilde{\gamma}^3 = i \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}, \\ \gamma^2 &= i\tilde{\gamma}^1 = i \begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix}, & \gamma^3 &= i\tilde{\gamma}^2 = i \begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}, \end{aligned} \tag{24}$$

where $\tilde{\gamma}^0, \tilde{\gamma}^1, \tilde{\gamma}^2$ and $\tilde{\gamma}^3$ are gamma matrices in chiral (Weyl) representation, while σ_i , ($i = 1, 2, 3$) are the usual Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

³ It can be carried out at least for the type of geometry and deformation considered in this paper, i.e. a deformation with the Killing twist operator. For more general examples of deformation, see Appendix B.

By writing out a detailed structure of the covariant derivative ∇_a , the Dirac equation (21) takes the form⁴

$$\left[i\gamma^a e_a{}^\mu \left(\partial_\mu + \frac{i}{2} \omega_\mu{}^{cd} \Sigma_{cd} - iA_\mu \right) - m \right] \Psi = 0. \tag{26}$$

Here $\Sigma_{cd} = \frac{i}{4}[\gamma_c, \gamma_d]$ and the coefficients of the spin connection $\omega_\mu{}^{ab}$ are given by

$$\begin{aligned} \omega_\mu{}^{ab} &= e^a{}_\nu \eta^{bc} \partial_\mu e_c{}^\nu + e^a{}_\nu \eta^{bc} e_c{}^\lambda \Gamma^\nu{}_{\mu\lambda} \\ &= \frac{1}{2} e^{a\nu} \left(\partial_\mu e^b{}_\nu - \partial_\nu e^b{}_\mu \right) - \frac{1}{2} e^{b\nu} \left(\partial_\mu e^a{}_\nu - \partial_\nu e^a{}_\mu \right) \\ &\quad - \frac{1}{2} e^{a\rho} e^{b\sigma} \left(\partial_\rho e_{c\sigma} - \partial_\sigma e_{c\rho} \right) e^c{}_\mu, \end{aligned}$$

where $\Gamma^\nu{}_{\mu\lambda} = \frac{1}{2} g^{\nu\delta} \left(\partial_\mu g_{\delta\lambda} + \partial_\lambda g_{\mu\delta} - \partial_\delta g_{\mu\lambda} \right)$ are the coefficients of the affine connection. Note that $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$.

With the tetrads given in (22), one gets that the only non zero components of the spin connection are

$$\begin{aligned} \omega_t{}^{01} &= -\omega_t{}^{10} = -\frac{Mr-Q^2}{r^3}, \quad \omega_\theta{}^{12} = -\omega_\theta{}^{21} = \sqrt{f}, \\ \omega_\phi{}^{13} &= -\omega_\phi{}^{31} = \sqrt{f} \sin \theta, \quad \omega_\phi{}^{23} = -\omega_\phi{}^{32} = \cos \theta, \tag{27} \\ \omega_r{}^{23} &= -\omega_r{}^{32} = \frac{aqQ}{2r^2} \cos \theta, \quad \omega_r{}^{13} = -\omega_r{}^{31} = \frac{aqQ\sqrt{f}}{2r^2} \sin \theta. \end{aligned}$$

In subsequent analysis we will also use the sums $\omega_\mu{}^{cd} \Sigma_{cd}$:

$$\begin{aligned} \omega_t{}^{cd} \Sigma_{cd} &= 2\omega_t{}^{01} \Sigma_{01} = -2 \frac{Mr-Q^2}{r^3} \frac{i}{4} [\gamma_0, \gamma_1] \\ &= -i \frac{Mr-Q^2}{r^3} \begin{pmatrix} -\sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \\ \omega_r{}^{cd} \Sigma_{cd} &= 2\omega_r{}^{23} \Sigma_{23} + 2\omega_r{}^{13} \Sigma_{13} \\ &= -\frac{aqQ}{2r^2} \cos \theta \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} + \frac{aqQ\sqrt{f}}{2r^2} \sin \theta \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \\ \omega_\theta{}^{cd} \Sigma_{cd} &= 2\omega_\theta{}^{12} \Sigma_{12} = -\sqrt{f} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \\ \omega_\phi{}^{cd} \Sigma_{cd} &= 2\omega_\phi{}^{13} \Sigma_{13} + 2\omega_\phi{}^{23} \Sigma_{23} = \sqrt{f} \sin \theta \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \\ &\quad - \cos \theta \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}. \tag{28} \end{aligned}$$

Inserting these into (26) leads to the Dirac equation

$$\begin{aligned} &\left[i\gamma^0 e_0{}^t \left(\partial_t - iA_t + \frac{i}{2} \omega_t{}^{cd} \Sigma_{cd} \right) \right. \\ &\quad \left. + i\gamma^1 e_1{}^r \left(\partial_r - iA_r + \frac{i}{2} \omega_r{}^{cd} \Sigma_{cd} \right) \right. \end{aligned}$$

⁴ Since implementation of (24) as our representation of γ -matrices involves a flip in their hermiticity properties (hermitian turns into antihermitian), the covariant derivative gets changed, $\nabla_\mu = \partial_\mu - \frac{i}{2} \omega_\mu{}^{cd} \frac{i}{4} [\gamma_c, \gamma_d] \rightarrow \nabla_\mu = \partial_\mu - \frac{1}{2} \omega_\mu{}^{cd} \frac{1}{4} [\gamma_c, \gamma_d]$, which amounts to changing the sign in the covariant derivative in front of the spin part, $\nabla_\mu = \partial_\mu + \frac{i}{2} \omega_\mu{}^{cd} \Sigma_{cd} = \partial_\mu + \frac{1}{2} \omega_\mu{}^{cd} \frac{i}{4} [\gamma_c, \gamma_d]$.

$$\begin{aligned} &+ i\gamma^2 e_2{}^\theta \left(\partial_\theta - iA_\theta + \frac{i}{2} \omega_\theta{}^{cd} \Sigma_{cd} \right) \\ &+ i\gamma^1 e_1{}^\phi \left(\partial_\phi - iA_\phi + \frac{i}{2} \omega_\phi{}^{cd} \Sigma_{cd} \right) \\ &\left. + i\gamma^3 e_3{}^\phi \left(\partial_\phi - iA_\phi + \frac{i}{2} \omega_\phi{}^{cd} \Sigma_{cd} \right) - m \right] \Psi = 0. \tag{29} \end{aligned}$$

With the spinor field Ψ written in terms of two two-component spinors Ψ_1 and Ψ_2 , namely $\Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix}$ and the gauge potential $A_\mu = (A_t, \mathbf{A}) = \left(-\frac{qQ}{r}, \mathbf{0} \right)$, the Eq. (29) splits into two two-component equations

$$\begin{aligned} &\left[-\frac{1}{\sqrt{f}} \mathbb{1} \partial_t - \sqrt{f} \sigma_3 \partial_r - \frac{1}{2} \frac{Mr-Q^2}{r^3} \frac{1}{\sqrt{f}} \sigma_3 - \frac{\sqrt{f}}{r} \sigma_3 \right. \\ &\quad \left. - \frac{1}{r} \sigma_1 \partial_\theta + \frac{aqQ}{2r^2} \sqrt{f} \sigma_3 \partial_\phi - \frac{1}{r \sin \theta} \sigma_2 \partial_\phi \right. \\ &\quad \left. - \frac{1}{2r} \cot \theta \sigma_1 - \frac{iqQ}{r\sqrt{f}} \mathbb{1} \right] \Psi_2 - m \mathbb{1} \Psi_1 = 0, \\ &\left[-\frac{1}{\sqrt{f}} \mathbb{1} \partial_t + \frac{1}{2} \frac{Mr-Q^2}{r^3} \frac{1}{\sqrt{f}} \sigma_3 + \sqrt{f} \sigma_3 \partial_r + \frac{\sqrt{f}}{r} \sigma_3 \right. \\ &\quad \left. + \frac{1}{r} \sigma_1 \partial_\theta - \frac{aqQ}{2r^2} \sqrt{f} \sigma_3 \partial_\phi + \frac{1}{r \sin \theta} \sigma_2 \partial_\phi \right. \\ &\quad \left. + \frac{1}{2r} \cot \theta \sigma_1 - \frac{iqQ}{r\sqrt{f}} \mathbb{1} \right] \Psi_1 - m \mathbb{1} \Psi_2 = 0. \tag{30} \end{aligned}$$

We see that these equations have the same form as the Eq. (15). The only NC correction is of the form $\frac{aqQ}{2r^2} \sqrt{f} \gamma^1 \partial_\phi \Psi$. Therefore we can conclude that the rigorous approach of the NC gauge theory and the SW expansion described in Sect. 3 and the effective approach described here lead to the same result. However, two comments are in order. Firstly, our results are valid up to first order in the deformation parameter a , implying that the linearized equations of motion for a spinor field in SW expansion turn out to be the same as equations of motion in a perturbed (first order effective dual) metric. Secondly, the result in Sect. 3 was deduced using the semiclassical approximation. In our future work we plan to investigate if this duality holds more generally.

5 Discussion and outlook

Before we go on to analyze the set of equations (30), note that the main result of our paper may be restated in a slightly different way, by using interpretation in terms of a process of reversed engineering described in Sect. 4 and Appendix B, which has led to the first order effective dual metric (18). In this respect, it is important to emphasize that the process of reversed engineering as applied to the equations of motion resulting from two different NC gauge theory models, one for NC scalar field, and the other for NC spin one-half field, may not necessarily lead to the same first order effective dual

metric. Quite opposite, it would be highly unlikely for this to happen. However, within our semiclassical model of NC gauge theory, we have manifestly demonstrated that this is indeed the case, and this constitutes the main result of our paper. More precisely, two first order effective dual metrics, obtained by two different and independent back engineering processes, one applied on the particular NC scalar field model and the other on the related NC spin one-half model, are the same! We have shown this by establishing the first order equivalence between the formal and the effective approach to the semiclassical $U(1)$ gauge theory applied on NC spinor field. This way we have gone around and bypassed the process of reversed engineering on the equation of motion for the NC spinor field. To be more precise, we took one of the first order effective dual metrics, the one obtained by back engineering the NC scalar field equation of motion and then we used this metric to write down the equation of motion for the ordinary commutative spinor field. Interestingly enough, it turned out that this equation is the same as the equation of motion obtained in a more formal approach to the semiclassical model of NC gauge theory, as applied to NC spin one-half field.

Let us now analyze the Eq. (30) in more detail. This equation can be used to study various effects, such as NC spinor bound states or quasinormal modes in the RN background.

In order to solve (30) for the wavefunction $\Psi \equiv \Psi(t, r, \theta, \phi)$, we follow [122] and take the ansatz

$$\begin{aligned} \Psi &= \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} = \begin{pmatrix} \Psi_1^{(1)} \\ \Psi_1^{(2)} \\ \Psi_2^{(1)} \\ \Psi_2^{(2)} \end{pmatrix} = \frac{1}{r} f^{-1/4} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = (r^4 f)^{-1/4} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\ &= \left(r^4 - 2Mr^3 + Q^2r^2 \right)^{-1/4} \begin{pmatrix} \psi_1^{(1)} \\ \psi_1^{(2)} \\ \psi_2^{(1)} \\ \psi_2^{(2)} \end{pmatrix}. \end{aligned} \tag{31}$$

After plugging this ansatz into (30) and performing some simplifications, the set of Eq. (30) reduces to

$$\begin{aligned} &\left[-\frac{r}{\sqrt{f}} \mathbb{1} \partial_t - r\sqrt{f} \sigma_3 \partial_r - \sigma_1 \partial_\theta - \frac{1}{\sin \theta} \sigma_2 \partial_\phi \right. \\ &\quad \left. - \frac{1}{2} \cot \theta \sigma_1 + \frac{aqQ}{2r} \sqrt{f} \sigma_3 \partial_\phi - \frac{iqQ}{\sqrt{f}} \mathbb{1} \right] \psi_2 - mr \mathbb{1} \psi_1 = 0, \\ &\left[-\frac{r}{\sqrt{f}} \mathbb{1} \partial_t + r\sqrt{f} \sigma_3 \partial_r + \sigma_1 \partial_\theta + \frac{1}{\sin \theta} \sigma_2 \partial_\phi \right. \\ &\quad \left. + \frac{1}{2} \cot \theta \sigma_1 - \frac{aqQ}{2r} \sqrt{f} \sigma_3 \partial_\phi - \frac{iqQ}{\sqrt{f}} \mathbb{1} \right] \psi_1 - mr \mathbb{1} \psi_2 = 0. \end{aligned} \tag{32}$$

We further make a factorization of the spinor wavefunctions ψ_1 and ψ_2 according to

$$\begin{aligned} \psi_1 &\equiv \psi_1(t, r, \theta, \phi) = e^{i(v\phi - \omega t)} \begin{pmatrix} \psi_1^{(1)}(r, \theta) \\ \psi_1^{(2)}(r, \theta) \end{pmatrix} \\ &= e^{i(v\phi - \omega t)} \begin{pmatrix} -R_2(r) S_1(\theta) \\ -R_1(r) S_2(\theta) \end{pmatrix}, \\ \psi_2 &\equiv \psi_2(t, r, \theta, \phi) = e^{i(v\phi - \omega t)} \begin{pmatrix} \psi_2^{(1)}(r, \theta) \\ \psi_2^{(2)}(r, \theta) \end{pmatrix} \\ &= e^{i(v\phi - \omega t)} \begin{pmatrix} R_1(r) S_1(\theta) \\ R_2(r) S_2(\theta) \end{pmatrix}, \end{aligned} \tag{33}$$

where ω and ν are respectively energy and projection of the angular momentum of the spin 1/2 particle. Note that this factorization is not arbitrary, but is singled out by a demand of having separable equation of motion. Indeed, it gives rise to a straightforward separation of the equation of motion into radial and angular parts, as we show below.

The first step in utilizing the factorization (33), which includes a separation of the azimuthal and time variables, gives rise to the set of two 2-component equations

$$\begin{aligned} &\left[\frac{i\omega r}{\sqrt{f}} \mathbb{1} - r\sqrt{f} \sigma_3 \partial_r - \sigma_1 \partial_\theta - \frac{iv}{\sin \theta} \sigma_2 \right. \\ &\quad \left. - \frac{1}{2} \cot \theta \sigma_1 + i\nu \frac{aqQ}{2r} \sqrt{f} \sigma_3 - \frac{iqQ}{\sqrt{f}} \mathbb{1} \right] \psi_2(r, \theta) \\ &\quad - mr \mathbb{1} \psi_1(r, \theta) = 0, \\ &\left[\frac{i\omega r}{\sqrt{f}} \mathbb{1} + r\sqrt{f} \sigma_3 \partial_r + \sigma_1 \partial_\theta + \frac{iv}{\sin \theta} \sigma_2 \right. \\ &\quad \left. + \frac{1}{2} \cot \theta \sigma_1 - i\nu \frac{aqQ}{2r} \sqrt{f} \sigma_3 - \frac{iqQ}{\sqrt{f}} \mathbb{1} \right] \psi_1(r, \theta) \\ &\quad - mr \mathbb{1} \psi_2(r, \theta) = 0. \end{aligned} \tag{34}$$

The second step, which involves a separation of the radial and polar angle variables, leads to the set of four coupled partial differential equations

$$\begin{aligned} &\frac{i\omega r}{\sqrt{f}} R_1 S_1 - r\sqrt{f} (\partial_r R_1) S_1 - R_2 \partial_\theta S_2 - \frac{iv}{\sin \theta} (-i) R_2 S_2 \\ &\quad - \frac{1}{2} \cot \theta R_2 S_2 + i\nu \frac{aqQ}{2r} \sqrt{f} R_1 S_1 + mr R_2 S_1 \\ &\quad - \frac{iqQ}{\sqrt{f}} R_1 S_1 = 0, \\ &\frac{i\omega r}{\sqrt{f}} R_2 S_2 + r\sqrt{f} (\partial_r R_2) S_2 - R_1 \partial_\theta S_1 - \frac{iv}{\sin \theta} i R_1 S_1 \\ &\quad - \frac{1}{2} \cot \theta R_1 S_1 - i\nu \frac{aqQ}{2r} \sqrt{f} R_2 S_2 + mr R_1 S_2 \\ &\quad - \frac{iqQ}{\sqrt{f}} R_2 S_2 = 0, \\ &-\frac{i\omega r}{\sqrt{f}} R_2 S_1 - r\sqrt{f} (\partial_r R_2) S_1 - R_1 \partial_\theta S_2 - \frac{iv}{\sin \theta} (-i) R_1 S_2 \\ &\quad - \frac{1}{2} \cot \theta R_1 S_2 + i\nu \frac{aqQ}{2r} \sqrt{f} R_2 S_1 - mr R_1 S_1 \end{aligned}$$

$$\begin{aligned}
 & + \frac{iqQ}{\sqrt{f}} R_2 S_1 = 0, \\
 & - \frac{i\omega r}{\sqrt{f}} R_1 S_2 + r\sqrt{f}(\partial_r R_1)S_2 - R_2 \partial_\theta S_1 \\
 & - \frac{iv}{\sin\theta} iR_2 S_1 - \frac{1}{2} \cot\theta R_2 S_1 - iv \frac{aqQ}{2r} \sqrt{f} R_1 S_2 \\
 & - mr R_2 S_2 + \frac{iqQ}{\sqrt{f}} R_1 S_2 = 0.
 \end{aligned} \tag{35}$$

After dividing above equations respectively with $R_2 S_1, R_1 S_2, R_1 S_1, R_2 S_2$ one finds that this system of equations is completely separable,

$$\begin{aligned}
 & \frac{i\omega r}{\sqrt{f}} \frac{R_1}{R_2} - r\sqrt{f} \frac{\partial_r R_1}{R_2} + iv \frac{aqQ}{2r} \sqrt{f} \frac{R_1}{R_2} - \frac{iqQ}{\sqrt{f}} \frac{R_1}{R_2} \\
 & + mr = \frac{\partial_\theta S_2}{S_1} + \frac{\nu}{\sin\theta} \frac{S_2}{S_1} + \frac{1}{2} \cot\theta \frac{S_2}{S_1} \equiv \lambda, \\
 & \frac{i\omega r}{\sqrt{f}} \frac{R_2}{R_1} + r\sqrt{f} \frac{\partial_r R_2}{R_1} - iv \frac{aqQ}{2r} \sqrt{f} \frac{R_2}{R_1} - \frac{iqQ}{\sqrt{f}} \frac{R_2}{R_1} \\
 & + mr = \frac{\partial_\theta S_1}{S_2} - \frac{\nu}{\sin\theta} \frac{S_1}{S_2} + \frac{1}{2} \cot\theta \frac{S_1}{S_2} \equiv \lambda_1, \\
 & - \frac{i\omega r}{\sqrt{f}} \frac{R_2}{R_1} - r\sqrt{f} \frac{\partial_r R_2}{R_1} + iv \frac{aqQ}{2r} \sqrt{f} \frac{R_2}{R_1} + \frac{iqQ}{\sqrt{f}} \frac{R_2}{R_1} \\
 & - mr = \frac{\partial_\theta S_2}{S_1} + \frac{\nu}{\sin\theta} \frac{S_2}{S_1} + \frac{1}{2} \cot\theta \frac{S_2}{S_1} = \lambda, \\
 & - \frac{i\omega r}{\sqrt{f}} \frac{R_1}{R_2} + r\sqrt{f} \frac{\partial_r R_1}{R_2} - iv \frac{aqQ}{2r} \sqrt{f} \frac{R_1}{R_2} + \frac{iqQ}{\sqrt{f}} \frac{R_1}{R_2} \\
 & - mr = \frac{\partial_\theta S_1}{S_2} - \frac{\nu}{\sin\theta} \frac{S_1}{S_2} + \frac{1}{2} \cot\theta \frac{S_1}{S_2} = \lambda_1.
 \end{aligned} \tag{36}$$

Moreover, it is easily seen that two separation constants λ and λ_1 , which have appeared in a process of separation are not mutually independent, but subject to the requirement $\lambda = -\lambda_1$. In effect, the system of equations (36) gives rise to two angular equations

$$\begin{aligned}
 & \partial_\theta S_2 + \frac{\nu}{\sin\theta} S_2 + \frac{1}{2} \cot\theta S_2 = \lambda S_1, \\
 & \partial_\theta S_1 - \frac{\nu}{\sin\theta} S_1 + \frac{1}{2} \cot\theta S_1 = -\lambda S_2,
 \end{aligned} \tag{37}$$

and two radial equations

$$\begin{aligned}
 & \frac{i\omega r}{\sqrt{f}} R_1 - r\sqrt{f} \partial_r R_1 + iv \frac{aqQ}{2r} \sqrt{f} R_1 \\
 & - \frac{iqQ}{\sqrt{f}} R_1 = \left(\lambda - mr\right) R_2, \\
 & \frac{i\omega r}{\sqrt{f}} R_2 + r\sqrt{f} \partial_r R_2 - iv \frac{aqQ}{2r} \sqrt{f} R_2 \\
 & - \frac{iqQ}{\sqrt{f}} R_2 = -\left(\lambda + mr\right) R_1.
 \end{aligned} \tag{38}$$

This system of radial equations can be used to study the behaviour of spinor quasinormal modes in the RN background.

This discussion we close with the analysis of the stability of chargeless, but massive fermionic modes. For that purpose we recall that for the bosonic fields on Kerr spacetime there exists a regime in which bosonic modes become unstable, due to superradiant growth [120, 123–128]. Contrary to that, in the same regime where the bosonic modes manifest instability, the fermionic fields on Kerr spacetime under condition of extra slow rotation do not,⁵ resulting in them being stable and subject to a decay only [129–133]. In other words, unlike the equations of motion governing the bosonic fields, the single-particle Dirac equation is not subject to superradiance, and thus all modes decay in that particular regime, which includes a setting where $mM \lesssim (l + \frac{1}{2})$, as well as the limit of slow rotation, $\Omega/M \ll 1$.⁶ Interestingly, fermionic fields on Schwarzschild or Reissner–Nordström spacetime display somewhat different characteristics, which makes them more susceptible of exhibiting the effect of superradiance and thus not remaining stable. These observations may be drawn by inspecting the bosonic and fermionic modes in question, either by inspecting the imaginary part of their bound state frequencies or by investigating the properties of the flux passing into the horizon and the corresponding conservation law.

Here we set to examine a possibility that a noncommutative deformation of RN spacetime, realized in a form of the effective metric (A7), introduces certain changes to the above statements. To start with, let us recall the form of the wave function that solves the Dirac equation

$$\begin{aligned}
 \Psi &= e^{i(v\phi - \omega t)} (r^4 f)^{-1/4} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \\
 &= e^{i(v\phi - \omega t)} \left(r^4 - 2Mr^3 + Q^2 r^2 \right)^{-1/4} \begin{pmatrix} -R_2(r)S_1(\theta) \\ -R_1(r)S_2(\theta) \\ R_1(r)S_1(\theta) \\ R_2(r)S_2(\theta) \end{pmatrix}.
 \end{aligned} \tag{39}$$

Its corresponding hermitian conjugate is defined as

$$\bar{\Psi} = -\Psi^\dagger \gamma^0 = -i\Psi^\dagger \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \tag{40}$$

and the covariant derivatives that include the spin connection part are given by

$$\begin{aligned}
 \nabla_\mu \Psi &= \partial_\mu \Psi - \Gamma_\mu \Psi = \partial_\mu \Psi - \frac{1}{4} \omega_\mu^{bc} \gamma_b \gamma_c \Psi, \\
 \nabla_\mu \bar{\Psi} &= \partial_\mu \bar{\Psi} + \bar{\Psi} \Gamma_\mu = \partial_\mu \bar{\Psi} + \frac{1}{4} \omega_\mu^{bc} \bar{\Psi} \gamma_b \gamma_c.
 \end{aligned}$$

⁵ The absence of superradiance for Dirac field in a Kerr or Kerr–Newman background was proved for a first time in [120].

⁶ Note that m is the mass of the perturbing field, M is the mass of the black hole, l is the orbital angular momentum number and ΩM is the angular momentum of a black hole.

With these quantities at hand, one may define the stress–energy tensor as

$$T_{\mu\nu} = \frac{i}{4} \left[\bar{\Psi} \gamma_\mu \nabla_\nu \Psi + \bar{\Psi} \gamma_\nu \nabla_\mu \Psi - (\nabla_\mu \bar{\Psi}) \gamma_\nu \Psi - (\nabla_\nu \bar{\Psi}) \gamma_\mu \Psi \right]. \tag{41}$$

For the fermionic field on the Kerr spacetime the form of the solution for the wave function essentially (up to a different prefactor) has the same general form as (39). The radial Dirac current and the corresponding conservation law in this case give rise to the condition

$$\frac{dN}{dt} = \left(|R_1|^2 - |R_2|^2 \right)_{r=r_h} \leq 0, \tag{42}$$

where N is the number density and r_h is the outer horizon radius, signalling the absence of superradiance [122]. For the case considered in this paper, i.e. deformed RN metric (A7), the radial component of the Dirac current $J^\mu = \bar{\Psi} \gamma^\mu \Psi = \bar{\Psi} e_a^\mu \gamma^a \Psi$ may be shown to have the form

$$\begin{aligned} J^r &= \bar{\Psi} \gamma^r \Psi = \bar{\Psi} e_a^r \gamma^a \Psi \\ &= -i \left(r^4 - 2Mr^3 + Q^2 r^2 \right)^{-1/2} \\ &\quad \times \sqrt{f} \left(R_1^* S_1^* R_2^* S_2^* - R_2^* S_1^* - R_1^* S_2^* \right) \\ &\quad \times \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix} \begin{pmatrix} -R_2 S_1 \\ -R_1 S_2 \\ R_1 S_1 \\ R_2 S_2 \end{pmatrix} \\ &= \frac{1}{r^2} \left(|R_1|^2 - |R_2|^2 \right) \left(|S_1|^2 + |S_2|^2 \right). \end{aligned} \tag{43}$$

At first glance, with the result (43) at hand, one might be led to think that the fermionic field on deformed RN spacetime might also exhibit the absence of superradiance. This is due to the fact that the integral of the radial current evaluated at the lower bound, i.e. at the outer horizon $r = r_h$, has a natural interpretation as the flux passing into the horizon and the expression on the rhs of (43) not being positive-definite.

However, a deeper inspection seems not to confirm this conclusion. In fact, a superradiance may be seen as a direct consequence of the second law of black hole thermodynamics. On the other hand, a crucial assumption that underlies the second law of black hole thermodynamics is that the condition $T_{\mu\nu} t^\mu t^\nu \geq 0$ must hold for any time-like vector field, $t_\mu t^\mu > 0$, with $T_{\mu\nu}$ being the stress–energy tensor. This condition is known as the weak energy condition. Therefore, for the purpose of demonstrating that a superradiance is absent in a given system, or at least that it doesn’t show up for a particular choice of the system parameters, we would first have to make evident that there exists a sector in the parameter space of that system where the weak energy condition is violated. In order to examine if the weak energy

condition might possibly be violated within the context of fermionic field on NC deformed RN background, opening in this way a window for a possible violation of the second law of thermodynamics and a consequent loss of the effect of superradiance within the same context, we take the time-like vector $t^\mu \equiv e_0^\mu = (e_0^t, e_0^r, e_0^\theta, e_0^\phi) = (\frac{1}{\sqrt{f}}, 0, 0, 0)$, $t_\mu t^\mu > 0$, and first evaluate and then analyse the bilinear form $T_{\mu\nu} t^\mu t^\nu = T_{\mu\nu} e_0^\mu e_0^\nu = T_{tt} e_0^t e_0^t$. This gives

$$\begin{aligned} T_{\mu\nu} t^\mu t^\nu &= \frac{i}{4} \left[\bar{\Psi} \gamma_t \nabla_t \Psi + \bar{\Psi} \gamma_t \nabla_t \Psi - (\nabla_t \bar{\Psi}) \gamma_t \Psi - (\nabla_t \bar{\Psi}) \gamma_t \Psi \right] e_0^t e_0^t \\ &= 2 \left[\frac{i}{4} \bar{\Psi} \gamma_t \nabla_t \Psi e_0^t e_0^t - \frac{i}{4} (\nabla_t \bar{\Psi}) \gamma_t \Psi e_0^t e_0^t \right]. \end{aligned} \tag{44}$$

Inserting (22), (24) and (39) into this equation leads to

$$\begin{aligned} T_{\mu\nu} t^\mu t^\nu &= 2 \frac{\left(r^4 - 2Mr^3 + Q^2 r^2 \right)^{-1/2}}{4\sqrt{f}} \\ &\quad \times \left[\omega \left(|R_1|^2 + |R_2|^2 \right) \left(|S_1|^2 + |S_2|^2 \right) \right. \\ &\quad \left. + i \frac{Mr - Q^2}{2r^3} \left(|R_1|^2 - |R_2|^2 \right) \left(|S_1|^2 + |S_2|^2 \right) \right] \\ &\quad - 2 \frac{\left(r^4 - 2Mr^3 + Q^2 r^2 \right)^{-1/2}}{4\sqrt{f}} \\ &\quad \times \left[-\omega \left(|R_1|^2 + |R_2|^2 \right) \left(|S_1|^2 + |S_2|^2 \right) \right. \\ &\quad \left. + i \frac{Mr - Q^2}{2r^3} \left(|R_1|^2 - |R_2|^2 \right) \left(|S_1|^2 + |S_2|^2 \right) \right] \\ &= \frac{\left(r^4 - 2Mr^3 + Q^2 r^2 \right)^{-1/2}}{2\sqrt{f}} 2\omega \left(|R_1|^2 + |R_2|^2 \right) \\ &\quad \times \left(|S_1|^2 + |S_2|^2 \right). \end{aligned} \tag{45}$$

It is clear that outside the outer horizon the expression under the square root is greater than zero. Moreover, the expression (45) as a whole is strictly positive-definite, that is $T_{\mu\nu} t^\mu t^\nu \geq 0$, implying that the noncommutative deformation of the Reissner–Nordström spacetime does not violate the weak energy condition for the fermionic field. This in turn implies that the weak energy condition is not violated for the Dirac particle in a Reissner–Nordström spacetime subject to a noncommutative deformation. Since the key assumption for the second law of black hole thermodynamics is not violated, the law continues to hold and the superradiance is expected to occur for the fermionic field in the spacetime described by the effective (deformed RN) metric, contrary to a first naive impression obtained by considering the radial component of the Dirac current. This result is different from the case of the

fermionic field on Kerr spacetime in the near horizon region $f(r) \rightarrow 0$ and in the superradiant⁷ regime $\omega < \nu \frac{\Omega}{2Mr_h}$, where ω is the frequency of the mode and ν is its azimuthal number, $\frac{\Omega}{2Mr_h}$ is the angular frequency of the horizon and r_h is the horizon radius. In the latter case the weak-energy condition is violated for the Dirac field on Kerr spacetime and consequently the effect of superradiance is absent. In our future work we plan to use the results obtained in this paper in order to study the massless as well as the massive fermionic perturbations of RN black hole in the presence of spacetime noncommutativity.

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Appendix A: Effective metric from first order noncommutative duality-calculation of the first order effective dual metric

Here we show that the Eq. (17) can be reversely engineered to yield the first order (in the deformation parameter a) effective metric (18).

Equation (17) can be symbolically written in terms of an extended Klein–Gordon operator, extended to include a coupling to a gauge field

⁷ The term superradiant is here used because in this regime bosonic fields rapidly grow in time, thus exhibiting a superradiance. Dirac fields though remain stable in this regime, as they are not superradiant there.

$$(\square_{g'} + \mathcal{O}(a))\Phi \equiv \left(g'^{\mu\nu} (\nabla'_\mu - iA_\mu) (\nabla'_\nu - iA_\nu) + \mathcal{O}(a) \right) \Phi = 0. \tag{A1}$$

Corrections are included in $\mathcal{O}(a)$ that is a generic expression and it designates symbolically a whole set of correction terms in the Eq. (17) that are induced by the noncommutativity and are therefore linear in NC parameter a . Likewise, ∇'_μ is a covariant derivative with respect to the metric $g'^{\mu\nu}$ ⁸ (1) and $\square_{g'}$ is the Klein–Gordon operator for the metric $g'^{\mu\nu}$. Note that by switching off a noncommutativity by letting $a \rightarrow 0$, all corrections that scale with a disappear, and the KG equation reduces to

$$\square_{g'}\Phi \equiv g'^{\mu\nu} (\nabla'_\mu - iA_\mu) (\nabla'_\nu - iA_\nu) \Phi = \frac{1}{\sqrt{-g'}} (\partial_\mu - iA_\mu) \left(\sqrt{-g'} g'^{\mu\nu} (\partial_\nu - iA_\nu) \right) \Phi = 0. \tag{A2}$$

At this stage one is naturally led to ponder over a possibility that the terms in (17) which scale linearly with the NC parameter a can actually be soaked up by the already present KG operator $\square_{g'}$ to yield a KG operator \square_g with a redefined metric that has managed to absorb within itself noncommutative features of the original problem. More concisely, the question to be posed is if there exists a metric which is able to meet the requirement

$$(\square_{g'} + \mathcal{O}(a))\Phi = \left(g'^{\mu\nu} (\nabla'_\mu - iA_\mu) (\nabla'_\nu - iA_\nu) + \mathcal{O}(a) \right) \Phi \equiv \square_g \Phi = g^{\mu\nu} (\nabla_\mu - iA_\mu) (\nabla_\nu - iA_\nu) \Phi = \frac{1}{\sqrt{-g}} (\partial_\mu - iA_\mu) \left(\sqrt{-g} g^{\mu\nu} (\partial_\nu - iA_\nu) \right) \Phi = 0, \tag{A3}$$

where ∇_μ is a covariant derivative with respect to the new, effective metric $g_{\mu\nu}$. We point out that the gauge potential did not change upon switching to a new setting and rewriting dynamics of the system in terms of the effective metric.

In order to find the metric tensor which satisfies the requirement (A3), one may try with the following ansatz

$$g_{\mu\nu} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & -\frac{1}{f} & 0 & g_{r\phi} \\ 0 & 0 & -r^2 & 0 \\ 0 & g_{r\phi} & 0 & -r^2 \sin^2 \theta \end{pmatrix}. \tag{A4}$$

The novel nonvanishing entry $g_{r\phi}$ is assumed to depend only on variables r and θ , since we expect that ∂_t and ∂_ϕ are Killing vectors for the effective metric as well. Moreover, it is assumed to be at least linear in NC parameter a , $g_{r\phi} \sim \mathcal{O}(a)$ since the effective metric $g_{\mu\nu}$ has to reduce to the original

⁸ $\nabla'_\mu A^\nu = \partial_\mu A^\nu + \Gamma'^{\nu}_{\lambda\mu} A^\lambda$ and $\nabla'_\mu A_\nu = \partial_\mu A_\nu - \Gamma'^{\lambda}_{\mu\nu} A_\lambda$.

RN metric $g'_{\mu\nu}$ in the limiting case $a \rightarrow 0$. The inverse of the metric tensor (A4) has nonvanishing entries at the same places

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{f} & 0 & 0 & 0 \\ 0 & -f + \frac{f^2 g_r^2}{f g_r^2 - r^2 \sin^2 \theta} & 0 & \frac{f g_r \phi}{f g_r^2 - r^2 \sin^2 \theta} \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & \frac{f g_r \phi}{f g_r^2 - r^2 \sin^2 \theta} & 0 & \frac{1}{f g_r^2 - r^2 \sin^2 \theta} \end{pmatrix}. \tag{A5}$$

It can be seen that while off-diagonal elements have a leading correction term that is linear in a , the diagonal elements g^{rr} and $g^{\phi\phi}$ have a leading correction term that is quadratic in a . Likewise, the determinant and the square-root of the determinant of the effective metric (A4) have a leading correction term that is quadratic in the NC parameter a , $\sqrt{-g} = r^2 \sin \theta + \mathcal{O}(a^2)$. These observations will have a crucial role in the subsequent analysis, whose aim is to deduce the metric $g_{\mu\nu}$, satisfying the requirement (A3).

The form of the metric (A4) dictates which terms are going to survive after the Eq. (A3) is written out explicitly

$$\begin{aligned} \square_g \Phi &= \frac{1}{\sqrt{-g}} (\partial_\mu - i A_\mu) \left(\sqrt{-g} g^{\mu\nu} (\partial_\nu - i A_\nu) \right) \Phi \\ &= \frac{1}{\sqrt{-g}} \left[(\partial_t - i A_t) \left(\sqrt{-g} g^{tt} (\partial_t - i A_t) \right) \right. \\ &\quad + (\partial_r - i A_r) \left(\sqrt{-g} g^{rr} (\partial_r - i A_r) \right) \\ &\quad + (\partial_r - i A_r) \left(\sqrt{-g} g^{r\phi} (\partial_\phi - i A_\phi) \right) \\ &\quad + (\partial_\theta - i A_\theta) \left(\sqrt{-g} g^{\theta\theta} (\partial_\theta - i A_\theta) \right) \\ &\quad + (\partial_\phi - i A_\phi) \left(\sqrt{-g} g^{\phi r} (\partial_r - i A_r) \right) \\ &\quad \left. + (\partial_\phi - i A_\phi) \left(\sqrt{-g} g^{\phi\phi} (\partial_\phi - i A_\phi) \right) \right] \Phi. \end{aligned}$$

Taking into account the fact that the gauge potential has only time component, one finds that the equation of motion (A3) further boils down to

$$\begin{aligned} &\frac{1}{f} \left[\partial_t^2 \Phi - 2i A_t \partial_t \Phi - A_t^2 \Phi \right] + \frac{1}{\sqrt{-g}} \left[\partial_r (\sqrt{-g} g^{rr}) \right] \partial_r \Phi \\ &\quad + g^{rr} \partial_r^2 \Phi + \frac{1}{\sqrt{-g}} \left[\partial_r (\sqrt{-g} g^{r\phi}) \right] \partial_\phi \Phi \\ &\quad + 2g^{r\phi} \partial_r \partial_\phi \Phi + \frac{1}{\sqrt{-g}} \left[\partial_\theta (\sqrt{-g} g^{\theta\theta}) \right] \partial_\theta \Phi \\ &\quad + g^{\theta\theta} \partial_\theta^2 \Phi + g^{\phi\phi} \partial_\phi^2 \Phi = 0. \end{aligned}$$

Focusing only on terms in the above equation that are at most linear in a , and stacking it up against the Eq. (17) leads

to the following two relations:

$$\begin{aligned} \frac{aqQ}{r^3} \left(\frac{MG}{r} - \frac{GQ^2}{r^2} \right) \partial_\phi \Phi &= \frac{1}{\sqrt{-g}} \left[\partial_r (\sqrt{-g} g^{r\phi}) \right] \partial_\phi \Phi, \\ \frac{aqQ}{r^2} f \partial_r \partial_\phi \Phi &= 2g^{r\phi} \partial_r \partial_\phi \Phi. \end{aligned} \tag{A6}$$

The solution to this set of relations, which is consistent with the requirement $g^{r\phi} = \frac{f g_r \phi}{f g_r^2 - r^2 \sin^2 \theta}$, finally gives for the dual effective metric

$$g_{\mu\nu} = \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & -\frac{1}{f} & 0 & -\frac{aqQ}{2} \sin^2 \theta \\ 0 & 0 & -r^2 & 0 \\ 0 & -\frac{aqQ}{2} \sin^2 \theta & 0 & -r^2 \sin^2 \theta \end{pmatrix} \tag{A7}$$

and for its inverse metric

$$g^{\mu\nu} = \begin{pmatrix} \frac{1}{f} & 0 & 0 & 0 \\ 0 & -f & 0 & \frac{aqQ}{2r^2} f \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & \frac{aqQ}{2r^2} f & 0 & -\frac{1}{r^2 \sin^2 \theta} \end{pmatrix}. \tag{A8}$$

Note that we demand $g_{\mu\nu} g^{\nu\rho} = \delta_\mu^\rho + \mathcal{O}(a^2)$.

We have thus shown that the equation of motion for a charged NC scalar field in a classical RN background, coupled to NC $U(1)$ gauge field may be rewritten in terms of the equation of motion governing behaviour of a charged commutative scalar field (having the same charge q as its NC counterpart), propagating in a modified RN geometry

$$\begin{aligned} ds^2 &= \left(1 - \frac{2MG}{r} + \frac{Q^2 G}{r^2} \right) dt^2 - \frac{dr^2}{1 - \frac{2MG}{r} + \frac{Q^2 G}{r^2}} \\ &\quad - aqQ \sin^2 \theta dr d\phi - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \end{aligned} \tag{A9}$$

Appendix B: More general choices of the twist

In this Appendix we briefly discuss more general forms of the twist operator, leading to more general NC deformations. This discussion is far from being complete, the detailed analysis we postpone for our future research.

1. For a deformation with a Killing twist operator, an arbitrary static, spherically symmetric metric

$$ds^2 = f(r) dt^2 - \frac{dr^2}{f(r)} - r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \tag{B1}$$

and an arbitrary static, spherically symmetric electromagnetic potential $A_\mu dx^\mu = A(r) dt$, with arbitrary functions $f(r)$ and $A(r)$, our results continue to be valid. More precisely, from equations (A6) it follows that the component $g^{r\phi}$ is related to the function $f(r)$, while the radial derivative of $g^{r\phi}$ is related to the radial derivative of $f(r)$. Note

that in the RN case, the term $\frac{MG}{r} - \frac{GQ^2}{r^2}$ is nothing else but $\frac{r}{2} \partial_r f$. Therefore, the component $g^{r\varphi}$ will have the same form as in (A8) with an arbitrary function $f(r)$ from (B1). On the other hand, in this more general case the only non-vanishing component of the field strength tensor is $F_{tr} = -\partial_r A(r)$. We conclude that the effective metric will retain the same general form (A7) with $g_{r\phi} = a \frac{r^2}{2} F_{tr} \sin^2 \theta$, and its inverse metric will have a different $g^{r\varphi}$ component given by $g^{r\varphi} = -\frac{a}{2} F_{tr} f(r) = \frac{a}{2} f(r) \partial_r A(r)$.

2. For a definition with a semi-Killing twist operator, such as

$$\mathcal{F} = e^{-\frac{i\alpha}{2}(\partial_t \otimes \partial_r - \partial_r \otimes \partial_t)}$$

for the RN metric, the Seiberg–Witten expanded (up to first order) action for the NC scalar field is given by

$$S = \int d^4x \left(\sqrt{-g} (g^{\mu\nu} D_\mu \phi^+ D_\nu \phi - \mu^2 \phi^+ \phi + \frac{\mu^2}{2} \theta^{\alpha\beta} F_{\alpha\beta} \phi^+ \phi + \frac{1}{2} \theta^{\alpha\beta} g^{\mu\nu} (-\frac{1}{2} D_\mu \phi^+ F_{\alpha\beta} D_\nu \phi + (D_\mu \phi^+) F_{\alpha\nu} D_\beta \phi + (D_\beta \phi^+) F_{\alpha\mu} D_\nu \phi) - \frac{i}{2} \theta^{\alpha\beta} \partial_\alpha (\sqrt{-g} g^{\mu\nu}) D_\mu \phi^+ D_\beta D_\nu \phi \right).$$

where $D_\nu \phi = (\partial_\nu - i A_\nu) \phi$. We notice that an additional (compared to (3.34) in [63]) term $-\frac{i}{2} \theta^{\alpha\beta} \partial_\alpha (\sqrt{-g} g^{\mu\nu}) D_\mu \phi^+ D_\beta D_\nu \phi$ will lead to an equation of motion for the field ϕ that is third order in derivatives. This immediately signals that equation cannot be reduced to an equation of motion for a commutative scalar field in an effective metric, since that equation is necessarily a 2nd order differential equation.

3. For a completely arbitrary (within a physical reason) well defined twist there is no guaranty that the first order duality will hold. Moreover, based on the results for the semi-Killing twist deformation it is very likely that there will be no duality between the propagation of the NC scalar field on a fixed background and the propagation of commutative scalar field in an effective background.

From this brief analysis we can conclude that the deformation by a Killing twist is a special one and it is the only one that corresponds to the semi-classical approximation we use in our work.

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