Article
From Hopf Algebra to Braided L_∞-Algebra

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Abstract: We show that an L_∞-algebra can be extended to a graded Hopf algebra with a codifferential. Then, we twist this extended L_∞-algebra with a Drinfel'd twist, simultaneously twisting its modules. Taking the L_∞-algebra as its own (Hopf) module, we obtain the recently proposed braided L_∞-algebra. The Hopf algebra morphisms are identified with the strict L_∞-morphisms, whereas the braided L_∞-morphisms define a more general L_∞-action of twisted L_∞-algebras.

Keywords: L_∞-algebra; Hopf algebra; Drinfel'd twist

1. Introduction

L_∞-algebras or homotopy Lie algebras are generalizations of Lie algebras with infinitely many higher brackets, related to each other by higher homotopy versions of the Jacobi identity [1–3]. From a physical point of view, these higher algebras represent the geometrical structure providing a deeper understanding of the quantization of field theory and gravity. In particular, there exists a correspondence between the BV formalism used in the quantization of gauge theories and L_∞-algebras, as noted by Zwiebach in his seminal work on closed string field theory [2]. Recently, the BV L_∞-algebra of the Yang–Mills theory was used in relating the squares of gluon amplitudes to those of ($\mathcal{N} = 0$ super) gravity amplitudes and toall orders of perturbative quantum field theory [4]. On the other hand, in the framework of deformation quantization [5], Kontsevich's famous formality theorem on the existence and classification of star products on Poisson manifolds has been proven using the concept of L_∞-quasi-isomorphisms [6]. Subsequently, Cattaneo and Felder [7] provided an interpretation of the Kontsevich quantization formula in terms of the perturbative expansion of the path integral for the Poisson sigma model [8,9].

More recently, attempts to improve the understanding of consistent non-commutative deformations of field theory using an L_∞-framework resulted in two interesting proposals. In [10], the homotopy relations defining an L_∞-algebra were used to bootstrap a consistent star-gauge-invariant theory starting from the star-deformed commutator of the symmetry algebra. However, there is another consistent way of introducing non-commutative deformations based on Drinfel'd twists of the symmetric Hopf algebra [11]. The analysis of twisted gauge symmetries was initiated in [12–15], but only recently was it shown that these twisted symmetries can be understood in the framework of braided L_∞-algebras [16,17]. The construction of braided L_∞-algebras in [16] was based on a reinterpretation of all the defining relations of an L_∞-algebra in terms of morphisms in a suitable category. By twisting the enveloping Hopf algebra of vector fields on a manifold M to a non-cocommutative Hopf algebra and simultaneously deforming its category of modules, the L_∞-algebra was deformed into a braided L_∞-algebra. Here we show that an L_∞-algebra itself can be extended to a graded Hopf algebra with a compatible codifferential, and using this observation we construct the braided L_∞-algebra using a Drinfel'd twist of its underlying Hopf algebra structure.

In the next section we recall the coalgebraic structure of an L_∞-algebra and show that it can be extended to a graded Hopf algebra with a codifferential. In Section 3 we

apply Drinfel'd twists to the Hopf algebras underlying L_∞-algebras and obtain twisted $L_\infty^{\mathcal{F}}$-algebras. In the spirit of deformation quantization, after twisting the algebra, one can further twist its modules. Taking the Hopf algebra as its own module, we obtain another Hopf algebra, $L_\infty^\star$, which is exactly the same as the braided L_∞-algebra defined in [16]. Furthermore, we reinterpret the Hopf algebra (iso)morphism in terms of strict L_∞-(iso)morphisms, and define a more general class of braided L_∞-morphism. In the conclusion, we briefly discuss the relevance of our results for braided gauge field theory.

2. L_∞ as Hopf Algebra

An L_∞-algebra can be defined in several different ways, depending on the context. In the standard approach one defines an L_∞-algebra on a graded vector space X as a generalization of a Lie algebra with possibly infinitely many higher brackets, related to each other by higher-homotopy versions of the Jacobi identity [1–3] Alternatively, one can describe an L_∞-structure as a degree 1 coderivation on the coalgebra generated by the suspension[1] of X, as shown in [3,19]. In order to identify the Hopf algebra structure underlying an L_∞-algebra, this coalgebraic picture is more appropriate and we shall review it here[2].

Let us start with a graded symmetric tensor algebra:

$$\mathbf{S}(X) := \bigoplus_{n=0}^{\infty} S^n X \,,$$

where X is a $\mathbb{Z}$-graded vector space $X = \bigoplus_{d \in \mathbb{Z}} X_d$ over the field $K = S^0 X$ and the degree of a homogeneous element $x_i \in X$ is denoted as $|x_i|$. The tensor product $\mu : \mathbf{S}(X) \otimes \mathbf{S}(X) \to \mathbf{S}(X)$ is graded-symmetric,

$$\mu(x_1 \otimes x_2) = x_1 \vee x_2 = (-1)^{|x_1||x_2|} x_2 \vee x_1 = (-1)^{|x_1||x_2|} \mu(x_2 \otimes x_1), \qquad x_1, x_2 \in X \,,$$

and we use $\vee$ to denote the product in $\mathbf{S}(X)$. The algebra structure can be endowed with a unit map $\eta : K \to \mathbf{S}(X)$, where $\eta(1) = 1$.

The coalgebra structure on $\mathbf{S}(X)$ is given by the coproduct:

$$\Delta(x_1 \vee \cdots \vee x_m) = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p, m-p)} \epsilon(\sigma; x)(x_{\sigma(1)} \vee \cdots \vee x_{\sigma(p)}) \otimes (x_{\sigma(p+1)} \vee \cdots \vee x_{\sigma(m)}) \,, \quad (1)$$

where $\epsilon(\sigma; x)$ is the Koszul sign,

$$x_1 \vee \cdots \vee x_k = \epsilon(\sigma; x) x_{\sigma(1)} \vee \cdots \vee x_{\sigma(k)}, \qquad x_i \in X \,,$$

and $\mathrm{Sh}(p, m-p) \in S_m$ denotes those permutations, ordered as $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(m)$. We use the conventions that $\mathrm{Sh}(n,0) = \mathrm{Sh}(0,n)$ equals ID$\in S_n$ and that an empty slot in the product equals the unit $1 \in K$. Thus, we have

$$\Delta(1) = 1 \otimes 1 \,,$$
$$\Delta(x) = 1 \otimes x + x \otimes 1 \,,$$
$$\Delta(x_1 \vee x_2) = 1 \otimes (x_1 \vee x_2) + (-1)^{|x_1||x_2|} x_2 \otimes x_1 + x_1 \otimes x_2 + (x_1 \vee x_2) \otimes 1 \,,$$
$$\cdots$$

As a map $\Delta : \mathbf{S}(X) \to \mathbf{S}(X) \otimes \mathbf{S}(X)$, this reads:

$$\Delta \circ \mathrm{id}^{\vee m} = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p, m-p)} (\mathrm{id}^{\vee p} \otimes \mathrm{id}^{\vee (m-p)}) \circ \tau^{\sigma} \,, \qquad p, m \geq 0 \,, \quad (2)$$

where τ^σ denotes the action of permutations [16], e.g., the non-identity permutation of two elements is:

$$\tau^\sigma(x_1 \vee x_2) = (-1)^{|x_1||x_2|} x_2 \vee x_1 \, ,$$

and this includes the Koszul sign. Furthermore, the coalgebraic structure of $\mathbf{S}(X)$ includes counit $\varepsilon : \mathbf{S}(X) \to K$, where $\varepsilon(1) = 1$ and $\varepsilon(x) = 0$, $x \in X$.

Next, we introduce a coderivation D that squares to zero and thus generates the appropriate homotopy relations. The coderivation is a map $D : \mathbf{S}(X) \to \mathbf{S}(X)$ of degree 1, such that the co-Leibniz property is satisfied:

$$\Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta \, . \tag{3}$$

This coderivation is given as [3]:

$$D = \sum_{i=0}^{\infty} b_i \, , \tag{4}$$

where the graded symmetric multilinear maps b_i are of degree 1. When b_0 is non-vanishing, one talks about curved L_∞-algebras [20,21], whereas for $b_0 = 0$ we have flat L_∞-algebras[3]. The b_i maps act on the full tensor algebra as a coderivation:

$$b_i : S^j X \to S^{j-i+1} X \, , \tag{5}$$

$$b_i(x_1 \vee \ldots \vee x_j) = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x) b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \ldots \vee x_{\sigma(j)}, \qquad j \geq i \, ,$$

and can be written using the permutation map τ^σ as:

$$b_i \circ \mathrm{id}^{\vee j} = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} (b_i \vee \mathrm{id}^{\vee(j-i)}) \circ \tau^\sigma \, , \qquad j \geq i \, . \tag{6}$$

Note that $b_0(1) = b_0$ is a degree 1 element of $S^1 X = X$. Now one can define an L_∞-algebra as a $\mathbb{Z}$-graded vector space with multilinear graded symmetric maps $b_i : X^{\otimes i} \to X$ of degree 1 such that the coderivation $D = \sum_{i=0}^{\infty} b_i$ is nilpotent [19]. As an example, we calculate the first few homotopy relations:

$$D^2(x_1 \vee x_2) = \sum_{i=0}^{\infty} b_i \sum_{j=0}^{2} b_j(x_1 \vee x_2)$$

$$= \sum_{i=0}^{3} b_i(b_0 \vee x_1 \vee x_2 + b_1(x_1) \vee x_2 + (-1)^{|x_1||x_2|} b_1(x_2) \vee x_1 + b_2(x_1, x_2))$$

$$= b_1(b_0) \vee x_1 \vee x_2 +$$
$$+ b_1^2(x_1) \vee x_2 + (-1)^{|x_1||x_2|} b_1^2(x_2) \vee x_1 + b_2(b_0, x_1) \vee x_2 + (-1)^{|x_1||x_2|} b_2(b_0, x_2) \vee x_1 +$$
$$+ b_1(b_2(x_1, x_2)) + b_2(b_1(x_1), x_2) + (-1)^{|x_1||x_2|} b_2(b_1(x_2), x_1) + b_3(b_0, x_1, x_2) \, .$$

The vanishing of the above expression is equivalent to the following three identities:

$$b_1 b_0 = 0 \, ,$$
$$b_2 b_0 + b_1^2 = 0 \, , \tag{7}$$
$$b_3 b_0 + b_2 b_1 + b_1 b_2 = 0 \, .$$

Additionally, we can use the fact that b_0^2 is trivially zero due to the odd degree of b_0, and is therefore not a constraint. The homotopy relations defining an L_∞-algebra can be written in the closed form [18]:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{i} b_{i-j+1}(b_j \circ \mathrm{id}^{\vee i}) = 0 \, . \tag{8}$$

Moreover, we have $\varepsilon \circ D = 0$ as $\varepsilon(b_0(1)) = 0$.

So far we have identified an L_∞-structure with a counital coalgebra over a graded vector space with compatible coderivation that squares to zero. Now we wish to compare this structure with that of a Hopf algebra. In short, a Hopf algebra is a bialgebra that admits an antipode map with certain compatibility properties. Although the formal definition is given in Appendix B, we discuss here the prototypical example—a tensor algebra.

A tensor algebra $T(V) = \bigoplus_{n=0}^{\infty} T^n V$, where V is a vector space over the field K, can be seen as a Hopf algebra $(T(V), \cdot, \Delta, \varepsilon, S)$. The coproduct Δ, counit ε and antipode S are defined on $v \in V$ as:

$$\begin{aligned}
\Delta(v) &= v \otimes 1 + 1 \otimes v \, , & \Delta(1) &= 1 \otimes 1 \, , \\
\varepsilon(v) &= 0 \, , & \varepsilon(1) &= 1 \, , \\
S(v) &= -v \, , & S(1) &= 1 \, .
\end{aligned}$$

Since the coproduct and counit are algebra homomorphisms and the antipode is an algebra (and coalgebra) anti-homomorphism, we can extend the definition from the basis elements to the full tensor algebra:

$$S(v_1 \cdot \ldots \cdot v_m) = (-1)^m v_m \cdot \ldots \cdot v_1 \, ,$$

$$\Delta(v_1 \cdot \ldots \cdot v_m) = \sum_{p=0}^{m} \sum_{\sigma \in \mathrm{Sh}(p, m-p)} \left(v_{\sigma(1)} \cdot \ldots \cdot v_{\sigma(p)} \right) \otimes \left(v_{\sigma(p+1)} \cdot \ldots \cdot v_{\sigma(m)} \right) \, ,$$

where we use $\cdot$ for the product in the tensor algebra. This example can be trivially extended to the symmetric-graded tensor algebra used in the construction of an L_∞-algebra, described above. In particular, the coproduct will be of the form (1), and the antipode will be extended to the graded antipode:

$$S(x_1 \vee \cdots \vee x_m) = (-1)^m (-1)^{\sum_{i=2}^{m} \sum_{j=1}^{i-1} |x_i||x_j|} x_m \vee \cdots \vee x_1. \tag{9}$$

Using the axioms of a Hopf algebra, given in Appendix B, one can easily verify that the symmetric-graded tensor algebra is indeed a Hopf algebra. Thus, we arrive at the following theorem.

Theorem 1. *An extended L_∞-algebra is a bialgebra $(\mathbf{S}(X), \mu, \eta, \Delta, \varepsilon)$ with coderivation D: $\mathbf{S}(X) \to \mathbf{S}(X)$ of degree 1 s.t. the co-Leibniz property is satisfied*

$$\Delta \circ D = (1 \otimes D + D \otimes 1) \circ \Delta \, ,$$

and $D^2 = 0$. This naturally inherits the structure of a Hopf algebra from the graded symmetric tensor algebra, with:

$$S \circ D = \widetilde{D} \circ S, \quad \varepsilon \circ D = 0,$$

where the codifferential $\widetilde{D}$

$$\widetilde{D} = \sum_{i=0}^{\infty} \tilde{b}_i = \sum_{i=0}^{\infty} (-1)^{1-i} b_i \, ,$$

induces the same homotopy relations as D.

Note that the unit map $\eta : K \to \mathbf{S}(X)$ is in general a morphism of graded coalgebras, and only for a flat L_∞-algebra, i.e., when $b_0 = 0$, does it become a morphism of differential graded coalgebras with $D \circ \eta = 0$.

Proof. We need to show that the Hopf algebra structure of the symmetric-graded tensor algebra is compatible with the L_∞-algebra structure encoded in the nilpotent coderivation. The compatibility of the unit and counit with coderivation was already discussed; therefore, we need to check only the antipode compatibility relation. We apply the coderivations b_i and $\tilde{b}_i$ on an element of $S^j X$ using (5). The left-hand side is

$$S(b_i(x_1 \vee \ldots \vee x_j)) = \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x) S(b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \ldots \vee x_{\sigma(j)})$$

$$= \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x)(-1)^{\mathcal{P}} S(x_{\sigma(j)}) \vee \ldots \vee S(x_{\sigma(i+1)}) \vee S(b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}))$$

$$= \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x)(-1)^{j-i+1}(-1)^{\mathcal{P}} x_{\sigma(j)} \vee \ldots \vee x_{\sigma(i+1)} \vee b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)})$$

$$= \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x)(-1)^{j-i+1} b_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \ldots \vee x_{\sigma(j)} \,,$$

where, in the second line, we introduced the sign

$$\mathcal{P} = \left(\sum_{m=1}^{i} |x_{\sigma(m)}| + 1 \right) \sum_{n=i+1}^{j} |x_{\sigma(n)}| + \sum_{m=i+2}^{j} \sum_{n=i+1}^{m-1} |x_{\sigma(n)}||x_{\sigma(m)}| \,,$$

induced by the graded action of antipode (9) and in the third line we use $S(x) = -x$, $\forall x \in X$. Similarly, the right-hand side gives

$$\tilde{b}_i(S(x_1 \vee \ldots \vee x_j)) = \tilde{b}_i((-1)^{\widetilde{\mathcal{P}}} S(x_j) \vee \ldots \vee S(x_1)) = (-1)^j \tilde{b}_i(x_1 \vee \ldots \vee x_j)$$

$$= \sum_{\sigma \in \mathrm{Sh}(i,j-i)} \epsilon(\sigma; x)(-1)^j \tilde{b}_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}) \vee x_{\sigma(i+1)} \vee \ldots \vee x_{\sigma(j)} \,,$$

where $\widetilde{\mathcal{P}} = \sum_{m=2}^{j} \sum_{n=1}^{m-1} |x_n||x_m|$. Equating the two sides gives $\tilde{b}_i = (-1)^{1-i} b_i$. Inspecting the homotopy relations (7), it is easy to see that the relations induced by $\widetilde{D}^2 = 0$ are the same. $\square$

A coderivation of a graded Hopf algebra with similar properties was previously introduced in [22] in the context of the BRST formulation of quantum group gauge theory.

3. Braided L_∞-Algebra from a Drinfel'd Twist

The Hopf algebra underlying an L_∞-algebra that we have discussed so far is cocommutative and coassociative, as it is based on a (graded) symmetric tensor algebra. A systematic way to introduce a non-(co)commutative deformation is by applying the Drinfel'd twist approach [11]. We twist a Hopf algebra H using a twist element $\mathcal{F} \in H \otimes H$, which is invertible and satisfies:

$$(\mathcal{F} \otimes 1)(\Delta \otimes \mathrm{id})\mathcal{F} = (1 \otimes \mathcal{F})(\mathrm{id} \otimes \Delta)\mathcal{F} \,, \tag{10}$$

$$(\varepsilon \otimes \mathrm{id})\mathcal{F} = 1 \otimes 1 = (\mathrm{id} \otimes \varepsilon)\mathcal{F} \,. \tag{11}$$

Relation (10) is known as the 2-cocycle condition, whereas condition (11) is known as (normalized) counitality. The 2-cocyle condition ensures that the deformed algebra remains coassociative. Using Sweedler's summation notation we can write the twist element and its inverse as:

$$\mathcal{F} = f^\alpha \otimes f_\alpha, \qquad \mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha \,. \tag{12}$$

It was shown in [23,24] that a twist $\mathcal{F}$ of a Hopf algebra H results in a new Hopf algebra, $H^{\mathcal{F}}$, which is given by $(H, \mu, \Delta^{\mathcal{F}}, \varepsilon, S^{\mathcal{F}})$. On the level of vector spaces $H^{\mathcal{F}} = H$, the product μ and counit ε are unchanged, whereas the coproduct transforms as follows:

$$\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \qquad h \in H. \tag{13}$$

In the case of an Abelian twist[4], which we assume in the following, the antipode is not deformed, $S^{\mathcal{F}} = S$. Thus, using the Drinfel'd twist, we obtain a twisted L_∞-algebra, namely, $(L_\infty^{\mathcal{F}}, \vee, \Delta^{\mathcal{F}}, \varepsilon, S)$, where $L_\infty^{\mathcal{F}}$ and L_∞ are the same as vector spaces.

Drinfel'd twist deformation quantization consists of twisting the Hopf algebra as above, while simultaneously twisting all of its modules [25]. Taking the Hopf algebra L_∞ as a module itself, one obtains another Hopf algebra $(L_\infty^\star, \vee_\star, \Delta_\star, \varepsilon, S_\star)$ with the corresponding vector space once again being the same as before, namely, L_∞, with the following product:

$$x_1 \vee_\star x_2 = \bar{f}^\alpha(x_1) \vee \bar{f}_\alpha(x_2). \tag{14}$$

This algebra is a Hopf algebra with:

$$\Delta_\star(x) = x \otimes 1 + \bar{R}^\alpha \otimes \bar{R}_\alpha(x), \tag{15}$$
$$S_\star(x) = -\bar{R}^\alpha(x)\bar{R}_\alpha. \tag{16}$$

The $\mathcal{R}$-matrix $\mathcal{R} \in \mathbf{S}(X) \otimes \mathbf{S}(X)$ is an invertible matrix induced by the twist,

$$\mathcal{R} = \mathcal{F}_{21}\mathcal{F}^{-1} =: R^\alpha \otimes R_\alpha, \ \mathcal{R}^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha \tag{17}$$

where $\mathcal{F}_{21} = f_\alpha \otimes f^\alpha$. In the case of an Abelian twist, $\mathcal{R}$ is triangular $R_\alpha \otimes R^\alpha = \bar{R}^\alpha \otimes \bar{R}_\alpha$, and $\mathcal{R} = \mathcal{F}^{-2}$. The inverse $\mathcal{R}$-matrix controls the non-commutativity of the $\vee_\star$-product and provides a representation of the permutation group [25] and, in particular, the action of a non-identity permutation of two elements is:

$$\tau_R^\sigma(x_1 \vee_\star x_2) = (-1)^{|x_1||x_2|}\bar{R}^\alpha(x_2) \vee_\star \bar{R}_\alpha(x_1).$$

As $\mathcal{R}$ is triangular, τ_R^σ squares to the identity. Now we can extend the coproduct (15) to the whole tensor algebra:

$$\Delta_\star \circ \mathrm{id}^{\vee_\star m} = \sum_{\sigma \in \mathrm{Sh}(p, m-p)} (\mathrm{id}^{\vee_\star p} \otimes \mathrm{id}^{\vee_\star(m-p)}) \circ \tau_R^\sigma, \qquad p, m \geq 0. \tag{18}$$

The coderivation $D_\star = \sum_{i=0}^\infty b_i^\star$ is defined in terms of braided graded-symmetric maps $b_i^\star$:

$$b_i^\star \circ \mathrm{id}^{\vee_\star j} = \sum_{\sigma \in \mathrm{Sh}(i, j-i)} (b_i^\star \vee_\star \mathrm{id}^{\vee_\star(j-i)}) \circ \tau_R^\sigma, \qquad j \geq i, \tag{19}$$

$$b_i^\star(x_1, \ldots, x_m, x_{m+1}, \ldots, x_i) = (-1)^{|x_m||x_{m+1}|} b_i^\star(x_1, \ldots, \bar{R}^\alpha(x_{m+1}), \bar{R}_\alpha(x_m), \ldots, x_i),$$

with the condition $D_\star^2 = 0$ reproducing the deformed homotopy relations. In particular, we have:

$$D_\star^2(x_1 \vee_\star x_2) = \sum_{i=0}^\infty b_i^\star \sum_{j=0}^2 b_j^\star(x_1 \vee_\star x_2)$$

$$= \sum_{i=0}^3 b_i^\star(b_0^\star \vee_\star x_1 \vee_\star x_2 + b_1^\star(x_1) \vee_\star x_2 + (-1)^{|x_1||x_2|} b_1^\star(\bar{R}^\alpha(x_2)) \vee_\star \bar{R}_\alpha(x_1) + b_2^\star(x_1, x_2))$$

$$= \sum_{i=0}^3 b_i^\star(b_0^\star \vee_\star x_1 \vee_\star x_2 + b_1^\star(x_1) \vee_\star x_2 + (-1)^{|x_1|} x_1 \vee_\star b_1^\star(x_2) + b_2^\star(x_1, x_2)).$$

In passing to the last line we assume the equivariance of the maps $b_i^\star$, i.e., we assume that they commute with the action of the twist generators. Thus, one can show that the homotopy relations are the same[5] as (7). The braided coproduct (18) and the compatible coderivation (19), equivariant under the action of the degree-zero twist element, reproduce, in the coalgebra picture, the braided L_∞-algebra constructed in [16], c.p., Definition 4.73 in [17]. Formally, the homotopy relations have the same form as (8):

$$\sum_{i=0}^{\infty}\sum_{j=0}^{i} b_{i-j+1}^\star (b_j^\star \circ \text{id}^{\vee\star i}) = 0 \,.$$

Moreover, we have $\varepsilon \circ D = 0$ as $\varepsilon(b_0(1)) = 0$.

The Hopf algebras $L_\infty^\star$ and $L_\infty^{\mathcal{F}}$ are isomorphic and there exists an invertible map φ between the underlying vector spaces [26]

$$\varphi(1) = 1, \ \varphi(x) = \bar{f}^\alpha(x)\bar{f}_\alpha \,, \tag{20}$$

such that:

$$\varphi(x_1 \vee_\star x_2) = \varphi(x_1) \vee \varphi(x_2) \,, \tag{21}$$

$$\Delta_\star = (\varphi^{-1} \otimes \varphi^{-1}) \circ \Delta^{\mathcal{F}} \circ \varphi \,, \tag{22}$$

$$\varepsilon_\star = \varepsilon \circ \varphi \,, \tag{23}$$

$$S_\star = \varphi^{-1} \circ S \circ \varphi \,. \tag{24}$$

On the other hand, there exist maps between L_∞-algebras: an L_∞-morphism is a collection of graded symmetric maps $\phi = \{\phi_i : S^i X \to X', \ i \geq 0\}$ of degree zero from $\mathbf{S}(X)$ to $\mathbf{S}(X')$, such that they define a coalgebra morphism, i.e., they satisfy

$$\Delta' \circ \phi = (\phi \otimes \phi) \circ \Delta \,, \tag{25}$$

and such that ϕ is compatible with the coderivations:

$$D' \circ \phi = \phi \circ D \,. \tag{26}$$

The first few components are:

$$\phi(1) = 1 + \phi_0 + \tfrac{1}{2!}\phi_0 \vee' \phi_0 + \cdots \,,$$
$$\phi(x) = \phi_1(x) + \phi_0 \vee' \phi_1(x) + \tfrac{1}{2!}\phi_0 \vee' \phi_0 \vee' \phi_1(x) + \cdots \,,$$
$$\phi(x_1 \vee x_2) = \phi_1(x_1) \vee' \phi_1(x_2) + \phi_0 \vee' \phi_1(x_1) \vee' \phi_1(x_2) + \cdots +$$
$$+ \phi_2(x_1, x_2) + \phi_0 \vee' \phi_2(x_1, x_2) + \cdots \,. \tag{27}$$

When $\phi_0 \neq 0$, we talk about curved L_∞-morphisms. From the compatibility of coderivations (26) one obtains the explicit relation between coderivation maps b_i and b_i' [27]

$$\sum_{\sigma \in \text{Sh}(l,n-l)} \phi_{1+l} \circ (b_{(n-l)} \otimes \text{id}^{\otimes l}) \circ \tau^\sigma = \sum_{j=0}^{\infty}\sum_{k_1+\cdots+k_j=n}\sum_{\sigma \in \text{Sh}(k_1,\ldots,k_j)} \tfrac{1}{j!} b_j'(\phi_{k_1} \vee \cdots \vee \phi_{k_j}) \circ \tau^\sigma \,. \tag{28}$$

When the map ϕ_1 is invertible, we have an L_∞-isomorphism. Applying the L_∞-morphism to our case of interest, namely, finding a map $\phi^\star : L_\infty^\star \to L_\infty^{\mathcal{F}}$, we need to define the component maps $\phi_i^\star$ which are braided graded symmetric, i.e.,

$$\phi_i^\star(x_1,\ldots,x_m,x_{m+1},\ldots,x_i) = (-1)^{|x_m||x_{m+1}|}\phi_i^\star(x_1,\ldots,\bar{R}^\alpha(x_{m+1}),\bar{R}_\alpha(x_m),\ldots,x_i) \,, \tag{29}$$

and equivariant with respect to the action of twist generators. The expression for the morphism $\phi^\star$ is then obtained from (28) by exchanging b_i with $b_i^\star$ and the action of the permutation τ^σ with τ_R^σ.

However, things are much simpler here; the Hopf algebra morphism is both an algebra morphism (21) and a coalgebra morphism (22) and (25); so we obtain that the only non-vanishing component of the morphism $\phi^\star$ is $\phi_1^\star$:

$$\phi_1^\star(x) = \varphi(x) = \bar{f}^\alpha(x)\bar{f}_\alpha \ . \tag{30}$$

The relation we established between Hopf and L_∞-algebras implies that the morphism φ between Hopf algebras (21)–(24) can be extended to a strict L_∞-morphism by demanding compatibility of the morphism with the coderivation (26)

$$\begin{aligned}
\varphi \circ D_\star &= D_\mathcal{F} \circ \varphi \ , \\
\varphi(b_n^\star(x_1, \ldots, x_n)) &= b_n(\varphi(x_1), \ldots, \varphi(x_n)) \ .
\end{aligned} \tag{31}$$

Note that in [28,29] the authors discussed the special example of the twisting of an L_∞-algebra, where the L_∞-morphisms of twisted algebras went beyond strict L_∞-morphisms. The difference comes from the difference between Hopf algebra modules which we discuss here, and more general L_∞-algebra modules; see [19,30].

Finally, in complete analogy with Thm 1., we can relate the braided L_∞-algebra $(L_\infty^\star, D_\star)$ with the Hopf algebra $(L_\infty^\star, \vee_\star, \Delta_\star, \epsilon, S_\star)$. The compatibility relation between the antipode $S_\star$ and the codifferential $D_\star$ follows from the equivariance of the coderivation maps $b_i^\star$ and the fact that the antipode $S_\star$ is a graded algebra anti-homomorphism.

4. Concluding Remarks

In this paper we have identified the cocommutative and coassociative Hopf algebra structure underlying L_∞-algebras. Thus, we were able to introduce a non-(co)commutative deformation by applying the Drinfel'd twist approach [11] and obtaining the braided L_∞-algebra of [16] as a module of the twisted one. In [16], the braided L_∞-algebra was used in the construction of a non-commutative deformation of the Chern–Simons and Einstein–Cartan–Palatini actions with a braided gauge symmetry. However, the physical interpretation of braided gauge symmetries encountered in these models was not well understood. One way to improve this situation is to construct an appropriate generalization of the BV formalism [17,31] that could help in identifying equivalent physical configurations. This is particularly natural in the coalgebra formulation, in which one can interpret the dual of the codifferential as locally being a cohomological vector field Q of degree 1 on a manifold M, i.e., $Q = D^*$ or:

$$Q = \sum_{i=0}^\infty \frac{1}{i!} C_{\alpha_1 \ldots \alpha_i}^\beta z^{\alpha_1} \cdots z^{\alpha_i} \frac{\partial}{\partial z^\beta} \ .$$

Here, the structure constants of the L_∞ algebra are the components of the coderivation D on a basis $\{\tau_\alpha\}$ of X:

$$b_i(\tau_{\alpha_1}, \ldots, \tau_{\alpha_i}) = C_{\alpha_1 \ldots \alpha_i}^\beta \tau_\beta \ ,$$

and $\{z^\alpha\}$ represents a basis of the dual[6] vector space $X^\star$. In the BV formalism, Q becomes the BRST operator and z^α denotes the physical fields.

Furthermore, in the L_∞-framework there exists a well-defined notion, at least for flat L_∞-algebras, of an L_∞-quasi-isomorphism that relates physically (gauge) equivalent configurations. Namely, when the 0-bracket vanishes, the 1-bracket is a differential (see (7)) and there is a cochain complex underlying the L_∞-algebra. In that case, one defines the L_∞-quasi-isomorphisms by the requirement that the linear morphism component ϕ_1 induces an isomorphism of cohomologies of the respective L_∞-algebras; see detailed discussion in [18]. For the case of a non-vanishing 0-bracket, a natural setting would be that of σ-models and

L_∞-spaces, introduced by Costello [33]. An L_∞-space includes target manifold data and a 0-bracket can be identified with the curvature of a connection on the target. Said differently, the connection $x \in X$ is a degree-zero solution of Maurer–Cartan equation

$$\sum_{i=0}^{\infty} \tfrac{1}{i!} b_i(\underbrace{x, \ldots, x}_{i \text{ times}}) = 0 \, .$$

Using this solution, one can define new L_∞-algebra on the same vector space, but with vanishing curvature [34,35].

Appendix A. On L_∞-Algebras

Definition A1 (L_∞-**algebra** [3]). *An L_∞-algebra (X, μ_i) is a graded vector space X equipped with a collection of multilinear maps that are graded totally antisymmetric:*

$$\mu_i : X^{\otimes i} \to X \, ,$$

of degree $2 - i$, where $i \in \mathbb{N}_0$ and satisfy the homotopy Jacobi identities:

$$\sum_{j+k=n} \sum_{\sigma} \chi(\sigma; x)(-1)^k \mu_{k+1}(\mu_j(x_{\sigma(1)}, \ldots, x_{\sigma(j)}), x_{\sigma(j+1)}, \ldots, x_{\sigma(n)}) = 0 \, ;$$

for all $x_i \in X$, $n \in \mathbb{N}_0$. Here, $\chi(\sigma; l)$ indicates the graded Koszul sign, including the sign from the parity of the permutation of $\{1, \ldots, n\}$ that is ordered as: $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(n)$.

We use the convention that totally graded antisymmetric means:

$$\mu_i(\ldots, x_r, x_s, \ldots) = -(-1)^{|x_r||x_s|} \mu_i(\ldots, x_s, x_r, \ldots) \, ,$$

with $|x_r|$ indicatingthe degree of homogeneous element $x_r \in X$. When $\mu_0 \neq 0$, this algebra is called a curved L_∞-algebra, whereas the name flat L_∞-algebra refers to the case $\mu_0 = 0$.

The homotopy Jacobi identities defining the L_∞ structure exists for any given level n, and there can be, in principle, an infinite number of these. The first few homotopy relations are:

$$n = 0 : \quad \mu_1 \mu_0 = 0 \, ,$$
$$n = 1 : \quad \mu_1^2(x) = \mu_2(\mu_0, x) \, ,$$
$$n = 2 : \quad \mu_1(\mu_2(x_1, x_2)) - \mu_2(\mu_1(x_1), x_2) - (-1)^{1 + xl_1||x_2|} \mu_2(\mu_1(x_2), x_1) = -\mu_3(\mu_0, x_1, x_2) \, .$$

These homotopy relations can be related to (7) using a degree -1 map s between the algebra and coalgebra pictures, called a suspension or shift isomorphism:

$$s : X \to X[1] \quad \text{s.t.} \quad (X[1])_d = X_{d+1} \, ,$$

which induces an isomorphism of the graded tensor algebras,

$$s^{\otimes i} : x_1 \wedge \cdots \wedge x_i \to (-1)^{\sum_{j=1}^{i-1}(i-j)} s x_1 \vee \cdots \vee s x_i \,,$$

and décalage isomorphism of the brackets:

$$\mu_i = (-1)^{\frac{1}{2}i(i-1)+1} s^{-1} \circ b_i \circ s^{\otimes i} \,.$$

Appendix B. On Hopf Algebras

Definition A2 (**Bialgebra**). *A bialgebra* $(A, \mu, \eta, \Delta, \varepsilon)$ *over K is a vector space which is both an algebra and a coalgebra in a compatible way:*

$$\Delta(hg) = \Delta(h)\Delta(g), \quad \Delta(1) = 1 \otimes 1, \quad \varepsilon(hg) = \varepsilon(h)\varepsilon(g), \quad \varepsilon(1) = 1, \quad \forall h, g \in A. \quad \text{(A1)}$$

The comultiplication $\Delta : A \to A \otimes A$ *and counit map* $\varepsilon : A \to K$ *are both algebra homomorphisms, whereas the multiplication* $\mu : A \otimes A \to A$ *and unit map* $\eta : K \to A$ *are coalgebra homomorphisms.*

Definition A3 (**Hopf algebra**). *A Hopf algebra* $(H, \mu, \Delta, \varepsilon, S)$ *over K is a bialgebra over K equipped with an antipode map* $S : H \to H$ *satisfying the following:*

$$\mu \circ (id \otimes S) \circ \Delta = \mu \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon \,. \quad \text{(A2)}$$

Remark A1. *If an antipode exists, it is unique [23].*

The existence of an inverse antipode map S^{-1} is not assumed, but if $S^2 = \text{id}$, the inverse is equivalent to the antipode map itself. A consequence of the antipode's uniqueness is that it obeys the following relations $\forall h, g \in H$:

$$S(hg) = S(g)S(h) \,, \qquad S(1) = 1 \,, \quad \text{(A3)}$$
$$(S \otimes S) \circ \Delta(h) = \Delta \circ S(h) \,, \qquad \varepsilon S(h) = \varepsilon(h) \,. \quad \text{(A4)}$$

The first two relations state that the antipode is an antialgebra map, whereas the second two state that it is an anticoalgebra map.

Notes

1.   The suspension map is also called a shift isomorphism [18]; see Appendix A for more details.
2.   In the rest of the paper, we work in the coalgebraic picture and denote with X the underlying graded vector space in order to simplify the notation.
3.   We shall use the term L_∞-algebra for both cases when the distinction is not relevant.
4.   The twist generators commute in the case of an Abelian twist.
5.   Only when acting on explicit elements of the tensor algebra does one have to take into account the braided transposition map. In that case, the first difference with respect to the untwisted algebra appears when acting on three or more elements, as shown in [16].
6.   In the infinite-dimensional case, one either restricts $X^\star$ to the space spanned by $\{z^\alpha\}$, or considers continuous duals in infinite-dimensional topological vector spaces; see discussion in [32].

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