

PERMUTATION INVARIANT STATISTICS, DUALITY AND SIMPLE INTERPOLATIONS

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Abstract

General permutation invariant statistics in the second quantized approach are considered. Simple interpolations between dual statistics are constructed. Particularly, we present a new minimal interpolation between parabosons and parafermions of any order. The connection with a simple mixing between bosons and fermions is established. The construction is extended to anyonic-like statistics.

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In the last few years there has been increasing interest in generalized statistics. The main reason is their possible application to the theory of the fractional quantum Hall effect [1] and to the theory of anyon superconductivity [2] based on the two-dimensional concept of anyons. Haldane fractional statistics [3], generalizing the Pauli exclusion principle to any spatial dimension, has also attracted much interest. A large class of generalizations is based on permutation group invariance, for example parastatistics [4], infinite quon statistics [5,6], a simple interpolation between bosons and fermions introduced by Wu *et al.* [7] and Scipioni [8]. Similarly, braid group invariance leads to anyonic-like statistics. Recently, permutation invariant statistics has been studied in the first quantized approach [9],[10].

In this letter we follow the second quantized approach and present a unified view on all types of statistics invariant under the permutation group. For a given type of generalized statistics we introduce the notion of its dual statistics and construct a simple interpolation between these two. Particularly, we analyse the minimal interpolation between Bose and Fermi statistics and some of its physical consequences, as well as the minimal interpolation between para-Bose and para-Fermi statistics. We establish a connection with the statistics of Wu *et al.* [7] and Scipioni [8]. Finally, we briefly discuss the extension of our simple interpolation to the anyonic-like statistics which are not permutation invariant.

Fock space and generalized statistics

Let us consider a system of multi-mode oscillators described by M pairs of creation and annihilation operators a_i^\dagger, a_i ($i = 1, 2, \dots, M$) hermitian conjugated to each other. We consider operator algebras with relations defined by a normally ordered expansion Γ [11],

$$a_i a_j^\dagger = \Gamma_{ij}(a^\dagger, a) \tag{1}$$

and which possess the well-defined number operators $[N_i, a_j^\dagger] = a_i^\dagger \delta_{ij}$,

$[N_i, a_j] = -a_i \delta_{ij}$ and $[N_i, N_j] = 0$, $i, j = 1, 2, \dots, M$. In the associated Fock-like representation, let $|0\rangle$ denote the vacuum vector.

The scalar product is uniquely defined by $\langle 0|0\rangle = 1$, the vacuum condition $a_i|0\rangle = 0$, $a_i a_i^\dagger|0\rangle \neq 0$ and eq.(1). A general N -particle state is a linear combination of monomial state vectors $a_{i_1}^\dagger \cdots a_{i_N}^\dagger|0\rangle$, $i_1, \dots, i_N = 1, 2, \dots, M$.

We consider only relations (1) that may allow the norm zero vectors, but do not allow the state vectors of negative norm in the Fock space. The norm zero vectors imply relations between the creation (annihilation) operators. These relations are consequences of eq.(1) and need not be postulated independently.

For a given N -particle monomial state $a_{i_1}^\dagger \cdots a_{i_N}^\dagger|0\rangle$ we write its type as $1^{n_1} 2^{n_2} \dots M^{n_M}$, where n_1, n_2, \dots, n_M are multiplicities satisfying $n_i \geq 0$ and $\sum_{i=1}^M n_i = N$. There are in principle $N!/n_1! \cdots n_M!$ different states of the type $1^{n_1} 2^{n_2} \dots M^{n_M}$ and we define the corresponding matrix $A(n_1, \dots, n_M)$ of their scalar products. The number of linearly independent states is given by $d_{n_1, \dots, n_M} = \text{rank}[A(n_1, \dots, n_M)]$. The quantities d_{n_1, \dots, n_M} completely characterize the partition function and the thermodynamic properties of the free system defined by eq.(1). Note that the partition function of the free system, i.e. the numbers d_{n_1, \dots, n_M} , in general do not uniquely determine the operator algebra, eq.(1). The free Hamiltonian is defined by $H_0 = \sum_{i=1}^M E_i N_i$, where E_i and N_i are the energy and the number operator corresponding to the i^{th} level. The partition function of the free system described by eq.(1) is given by

$$Z(x_1, \dots, x_M; \Gamma) = \sum_{N=0}^{\infty} \sum_{n_1 + \dots + n_M = N} d_{n_1, \dots, n_M} x_1^{n_1} \cdots x_M^{n_M}, \quad (2)$$

where d_{n_1, \dots, n_M} is the degeneracy of the state with the energy $E = \sum_{i=1}^M E_i$ and $x_i = e^{-\beta/E_i}$, $\beta = 1/kT$.

Permutation invariant generalized statistics

Our aim is to unify statistics [4-8] in the second quantized algebraic approach,

eq.(1) by the simplest possible unifying principle with minimal restrictions. It is permutation invariance, meaning that the matrix element $\langle 0|a_{i_{\pi(N)}} \cdots a_{i_{\pi(1)}} a_{j_{\pi(1)}}^\dagger \cdots a_{j_{\pi(N)}}^\dagger |0\rangle$ does not depend on the permutation $\pi \in S_N$. Hence we assume that the set of relations defined by Γ_{ij} in eq.(1) is invariant under the permutation group S_M . Then the coefficients in the expansion (1) do not depend on concrete indices in normal ordered monomials, but only on certain linearly independent types of permutation invariant terms, i.e.

$$a_i a_j^\dagger = \delta_{ij} + C_{1,1} a_j^\dagger a_i + \sum_{n=1}^{\infty} \sum_{\pi, \sigma \in S_{n+1}} C_{\pi, \sigma} \sum_{k_1, \dots, k_n=1}^M [\pi(j, k_1, \dots, k_n)]^\dagger [\sigma(i, k_1, \dots, k_n)], \quad (3)$$

where the operators a_i are normalized in such a way that the coefficient of the δ_{ij} term is equal to 1. The existence of the number operators N_i implies that the annihilation and creation operators appearing in a monomial in the normal ordered expansion (3) have to come in pairs, i.e. monomials are diagonal in the variables k_1, \dots, k_n (up to permutations) [11]. The symbol $[\sigma(i, k_1, \dots, k_n)]$ denotes $a_{\sigma(i)} a_{\sigma(k_1)} \cdots a_{\sigma(k_n)} \equiv \sigma(a_i a_{k_1} \cdots a_{k_n})$. Also, $C_{\pi, \sigma} = C_{\sigma, \pi}^*$, owing to the hermiticity of the operator product $a_i a_i^\dagger$. Furthermore, the S_M -invariant relations in eq.(3) acting on the corresponding Fock space imply the following relations:

$$a_i a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0\rangle = \sum_{k=1}^N \delta_{i i_k} \sum_{\sigma \in S_{N-1}} \phi_\sigma^k [\sigma(i, \dots, \hat{i}_k, \dots, i_N)]^\dagger |0\rangle, \quad (4)$$

where \hat{i}_k denotes the omission of the index i_k . The sum is running over all linearly independent monomials and ϕ_σ^k are (complex) coefficients. The identity $\phi_{id}^1 = 1$ is implied by normalization in eq. (3). The coefficients ϕ_σ^k can be uniquely determined from $C_{\pi, \sigma}$ and vice versa.

The transition number operators N_{ij} , defined by the relations $[N_{ij}, a_k^\dagger] = \delta_{jk} a_i^\dagger$ and $N_{ii} \equiv N_i$, have a similar expansion as Γ_{ij} in eq.(3), namely

$$N_{ij} = a_j^\dagger a_i + \sum_{n=1}^{\infty} \sum_{\pi, \sigma \in S_{n+1}} D_{\pi, \sigma} \sum_{k_1, \dots, k_n=1}^M [\pi(j, k_1, \dots, k_n)]^\dagger [\sigma(i, k_1, \dots, k_n)], \quad (5)$$

where $D_{\pi,\sigma}$ are independent of i, j (by permutation invariance) and $D_{\pi,\sigma} = D_{\sigma,\pi}^*$ following from $N_i^\dagger = N_i$. Hence, it follows that $N_{ij}^\dagger = N_{ji}$.

Each of the three sets of coefficients, $\{C_{\pi,\sigma}\}$, $\{\phi_\sigma^k\}$, $\{D_{\pi,\sigma}\}$, uniquely determines the two remaining sets, fixing the structure of the Fock space [11], and each of them is equivalent to the set of matrices $A(n_1, \dots, n_M)$.

The matrix $A(n_1, \dots, n_M)$ and its rank d_{n_1, \dots, n_M} depend only on the collection of multiplicities $\{n_1, \dots, n_M\}$, which, written in the descending order $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_M \geq 0$, $|\lambda| = \sum_{i=1}^M \lambda_i = N$, give rise to a partition λ of N , i.e. $d_{n_1, \dots, n_M} = d_\lambda$ and $A(n_1, \dots, n_M) = A_\lambda$ [12]. If $\lambda_1 = \lambda_2 = \dots = \lambda_N = 1$, $\lambda_{N+1} = \dots = \lambda_M = 0$, the corresponding Young tableau, denoted by 1^N , is a column of N boxes. The $N! \times N!$ generic matrix is denoted by A_{1^N} . All other matrices A_λ , ($|\lambda| = N$) for any partition λ of N are easily obtained from the matrix A_{1^N} [12,18]. The non-generic matrix A_λ , for $\lambda \neq 1^N$, $\lambda_1 \geq \lambda_2 \geq \dots \lambda_k > 0$, $|\lambda| = N$, is the matrix of the type $\frac{N!}{\lambda_1! \dots \lambda_k!} \times \frac{N!}{\lambda_1! \dots \lambda_k!}$, whose matrix elements are enumerated by orbits $\bar{\alpha}$, $\bar{\beta}$ of permutations $\alpha, \beta \in S_N$ acting on the multi-set $\{i_1 \leq i_2 \dots \leq i_N\}$ with multiplicities $\lambda_1, \lambda_2, \dots, \lambda_k$, and

$$(A_\lambda)_{\bar{\alpha}, \bar{\beta}} = \sum_{\sigma \in S_N, \sigma \bar{\beta} = \bar{\alpha}} (A_{1^N})_{\alpha, \sigma \beta}.$$

(Note that the matrix elements do not depend on i_1, \dots, i_N , only on their multiplicities.)

By the permutation symmetry of Γ , eq.(3), it follows that A_{1^N} can be written as

$$A_{1^N} = \sum_{\pi \in S_N} f(\pi) R(\pi), \quad (6)$$

where $R, R(\pi)_{\mu,\nu} = \delta_{\mu\pi,\nu}$ is the right regular representation of the permutation group S_N and $f(\pi)$ are complex numbers completely determining all matrix elements and statistics. The matrix A_{1^N} is hermitian with non-negative eigenvalues and rank $d_{1^N} \leq N!$.

The S_M -invariant partition function can be expanded into the form

$$Z_N(x_1, \dots, x_M) = \sum_{\lambda, |\lambda|=N} d_\lambda m_\lambda(x_1, \dots, x_M) = \sum_{\mu, |\mu|=N} n(\mu) s_\mu(x_1, \dots, x_M), \quad (7)$$

where $m_\lambda(x_1, \dots, x_M)$ is the monomial S_M -invariant function and $s_\mu(x_1, \dots, x_N)$ is the Schur function [13] satisfying

$$s_\mu(x_1, \dots, x_M) = \sum_{\lambda} K_{\mu\lambda} m_\lambda(x_1, \dots, x_M). \quad (8)$$

Hence, from eqs.(7) and (8) it follows that

$$d_\lambda = \sum_{\mu} n(\mu) K_{\mu\lambda}, \quad (9)$$

where $n(\mu)$, $dim(\mu) \geq n(\mu) \geq 0$, is the number of equivalent IRREP's (irreducible representation) μ of physical states contributing to the decomposition of A_{1N} , and $K_{\mu\lambda}$ are Kostka's numbers denoting the number of linearly independent states $a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0\rangle$ of type λ which fill the Young frame μ in the column strict way and $K_{\mu\lambda} \leq K_{\mu 1^N} = dim(\mu)$. The number of N -particle independent states is $D(M, N) = Z_N(1, 1, \dots, 1)$.

The numbers $n(\mu)$ completely determine the partition function of the free S_M -invariant system defined by eq.(1), but do not determine the operator algebra itself. The two free systems with the same partition function can differ in the following properties: (i) in the commutation relations of their creation (annihilation) operators, (ii) in the probabilities of finding the monomial state $a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0\rangle$ in the IRREP μ of S_N , (iii) in the probabilities of finding the particular IRREP ρ_k of $S_{n_1+n_2}$ in the decomposition of $\mu_1 \times \mu_2 = \sum_k \rho_k$, where μ_i is the IRREP of S_{n_i} , $i = 1, 2$ and (iv) in the probabilities of finding a particular subsystem characterized by the IRREP $\mu_1 \times \mu_2 \cdots$ of $S_{n_1} \times S_{n_2} \times \cdots$ in the larger system μ of S_n , $n \geq n_1 + n_2 + \cdots$. Examples of permutation invariant statistics defined by eq.(1) are parastatistics [4], interpolation between parastatistics [14] and infinite quon statistics [5,6].

Duality and simple interpolation

Let us first discuss a duality between Bose and Fermi statistics. For Bose statistics, $f(\pi) = 1$ in the expression (6), and for Fermi statistics, $f(\pi) = (-1)^{I(\pi)}$,

$\forall \pi \in S_N$, where $I(\pi)$ is the number of inversions in π . Hence, the Bose and Fermi generic matrices are hermitian of rank one with the same spectrum (generic matrices are similar). If these properties of the Bose and Fermi generic matrices were true for all partitions λ , Bose and Fermi statistics would be the same. However, for Bose statistics, all matrix elements are equal to $\lambda_1! \cdots \lambda_k!$ (the rank is one with the eigenvalue $N!$) and for Fermi statistics, all matrix elements are zero. Hence, the crucial difference between Bose and Fermi statistics is in the structure of non-generic matrices. However, a duality transformation between completely symmetric (Bose) and antisymmetric (Fermi) eigenvectors of generic matrices can be defined.

Here we generalize this duality between Bose and Fermi statistics to any permutation invariant statistics defined by a set of generic matrices A_{1^N} . The dual generic matrix $A_{1^N}^d$ is given by

$$A_{1^N}^d = D_{1^N} A_{1^N} D_{1^N}, \quad (10)$$

where D_{1^N} is the $N! \times N!$ diagonal matrix with matrix elements

$$(D_{1^N})_{\pi,\sigma} = (-)^{I(\pi)} \delta_{\pi,\sigma}, \quad (11)$$

where $\pi, \sigma \in S_N$ and $I(\pi)$ is the number of inversions of permutation π . We point out that the duality transformation has non-trivial consequences on the non-generic matrices A_λ , $\lambda \neq 1^N$.

It follows that $D_{1^N}^\dagger = D_{1^N}$, $D_{1^N}^2 = 1$ and $Tr D_{1^N} = 0$. If $A_{1^N} = \sum_\pi f(\pi) R(\pi)$, eq.(6), then $A_{1^N}^d = \sum_\pi f^d(\pi) R(\pi)$, where $f^d(\pi) = (-)^{I(\pi)} f(\pi)$ since $D_{1^N} R(\pi) D_{1^N} = (-)^{I(\pi)} R(\pi)$. Furthermore, we have the following proposition:

Proposition. If the matrix A_{1^N} is hermitian, then $A_{1^N}^d$ is also hermitian and possesses the same eigenvectors and spectrum as A_{1^N} . Hence, $A_{1^N}^d$ and A_{1^N} commute.

Proof. Let us denote the eigenvectors of A_{1^N} as $|a, b, \mu\rangle$, $a, b = 1, 2, \dots, \dim(\mu)$, where μ fixes the IRREP of S_N , b enumerates equivalent IRREP's μ and a enumerates states in the b^{th} IRREP μ , in accordance with the decomposition of a regular representation. The components of the given eigenvector $|a, b, \mu\rangle$ are

$$|a, b, \mu\rangle = \sum_{\pi \in S_N} R_{a,b}^\mu(\pi) |\pi\rangle, \quad (12)$$

where R^μ is the unitary IRREP μ of S_N , characteristic of the generic matrix A_{1N} .

The corresponding eigenvalue of A_{1N} is Λ_b^μ

$$\begin{aligned} A_{1N} |a, b, \mu\rangle &= \Lambda_b^\mu |a, b, \mu\rangle, \quad a = 1, 2, \dots, \dim(\mu), \\ \Lambda_b^\mu &= \sum_{\pi \in S_N} R_{bb}^\mu(\pi) f(\pi). \end{aligned} \quad (13)$$

Let us show that $|a, b, \mu\rangle$ are the eigenvectors of A_{1N} as well:

$$\begin{aligned} A_{1N}^d |a, b, \mu\rangle &= D_{1N} A_{1N} D_{1N} |a, b, \mu\rangle = D_{1N} A_{1N} |a', b', \mu^T\rangle \\ &= D_{1N} \Lambda_{b'}^{\mu^T} |a', b', \mu^T\rangle = \Lambda_b^{\mu^T} |a, b, \mu\rangle, \\ \Lambda_b^{\mu^T} &= \sum_{\pi \in S_N} R_{bb}^{\mu^T}(\pi) f(\pi) = \sum_{\pi \in S_N} (-)^{I(\pi)} R_{bb}^\mu(\pi) f(\pi), \end{aligned} \quad (14)$$

since $D_{1N} |a, b, \mu\rangle = |a', b', \mu^T\rangle$ and $R^{\mu^T}(\pi) = (-)^{I(\pi)} R^\mu(\pi)$. Hence, the operator D transforms the IRREP μ to its dual IRREP μ^T . The A_{1N}^d and A_{1N} have common eigenvectors $|a, b, \mu\rangle$ with eigenvalues $(\Lambda^d)_b^\mu = \Lambda_b^{\mu^T}$ and Λ_b^μ , respectively.

We point out that the complete characterization of the statistics considered is not given only by $n(\mu)$'s (determining the number of equivalent IRREP's μ of independent physical states that contribute), but also requires the eigenvalues $\Lambda_b^\mu \geq 0, \forall \mu, b$, which determine all relevant probabilities of finding the monomial state $a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0\rangle$ in equivalent IRREP's μ of S_N , i.e. $w(\mu) = K_{\mu 1N} / N! \sum_b \Lambda_b^\mu$. Hence, although the dual generic matrix A_{1N}^d has the same eigenvectors as A_{1N} and $d_{1N}^d = d_{1N}$, they basically differ since $\Lambda_b^{\mu^T} \neq \Lambda_b^\mu$. Thus, the two permutation invariant statistics related through the duality transformation, eq.(10), are qualitatively different. They are connected by conjugation of the Young tableaux.

Simple interpolation

There is a general simple construction of mixing the given permutation invariant statistics with its dual statistics. Since our generic matrices A_{1N} have non-negative

eigenvalues, their dual generic matrices have non-negative eigenvalues, too. The mixed generic matrices $A_{1^N}^q$ are defined by

$$A_{1^N}^q = \frac{1+q}{2}A_{1^N} + \frac{1-q}{2}A_{1^N}^d = \sum_{\pi \in S_N} f^q(\pi)R(\pi), \quad (15)$$

$$f^q(\pi) = \begin{cases} f(\pi) & \text{for } \pi \text{ even} \\ qf(\pi) & \text{for } \pi \text{ odd} \end{cases},$$

with $|q| \leq 1$. The matrix $A_{1^N}^q$ has the same eigenvectors as A_{1^N} , with eigenvalues $\Lambda_{1^N}^q$:

$$A_{1^N}^q |a, b, \mu\rangle = (\Lambda^q)_b^\mu |a, b, \mu\rangle,$$

$$(\Lambda^q)_b^\mu = \frac{1+q}{2}\Lambda_b^\mu + \frac{1-q}{2}\Lambda_b^{\mu^T} \geq 0, \quad |q| \leq 1. \quad (16)$$

It is obvious that $(\Lambda^q)_b^\mu \geq 0$ if $\Lambda_b^\mu \geq 0$ and $|q| \leq 1$ and that $d_{1^N} \leq d_{1^N}^q \leq 2d_{1^N}$.

Let us analyse a few examples.

Minimal interpolation between bosons and fermions

Here we construct the minimal generalized statistics with permutation group invariance, interpolating between Bose and Fermi statistics. The generic matrix A_{1^N} , eq.(6), for Bose statistics is characterized by $f(\pi) = 1$, for all $\pi \in S_N$ and for Fermi statistics by $f(\pi) = 1$, if π is an even permutation and $f(\pi) = -1$ if π is an odd permutation. We suggest a simple interpolation defined by

$$f^q(\pi) = \begin{cases} 1 & \text{for } \pi \text{ even} \\ q & \text{for } \pi \text{ odd} \end{cases}, \quad (17)$$

where $|q| \leq 1$. Note that $q = 1$ (-1) corresponds to Bose (Fermi) statistics. The non-zero non-degenerate eigenvalues of the matrix A_{1^N} are $(1+q)N!/2$ and $(1-q)N!/2$ with eigenvectors in the symmetric and antisymmetric representation, respectively. Hence, if $|q| < 1$, the rank of the matrix A_{1^N} is $d_{1^N} = 2$ and the Fock space does

not contain vectors of negative squared norm. Then it follows that the multiplicity is $n(\mu) = 1$ for $\mu = 1^N$ and $\mu = N$, and $n(\mu) = 0$ otherwise. Null vectors imply that the state vectors are divided into two classes:

$$\begin{aligned}\pi(a_1^\dagger \cdots a_N^\dagger) &\equiv a_1^\dagger a_2^\dagger \cdots a_N^\dagger |0\rangle, \quad \pi \text{ even}, \\ \pi(a_1^\dagger \cdots a_N^\dagger) &\equiv a_2^\dagger a_1^\dagger a_3^\dagger \cdots a_N^\dagger |0\rangle, \quad \pi \text{ odd},\end{aligned}\tag{18}$$

and any generic monomial state can be decomposed into the sum of symmetric and antisymmetric states:

$$a_1^\dagger \cdots a_N^\dagger |0\rangle = \frac{1}{2}(a_1^\dagger a_2^\dagger + a_2^\dagger a_1^\dagger) a_3^\dagger \cdots a_N^\dagger |0\rangle + \frac{1}{2}(a_1^\dagger a_2^\dagger - a_2^\dagger a_1^\dagger) a_3^\dagger \cdots a_N^\dagger |0\rangle.\tag{19}$$

The probability for the state $a_1^\dagger \cdots a_N^\dagger |0\rangle$ to be found in the symmetric (resp. antisymmetric) state is $w_s = (1 + q)/2$ (resp. $w_a = (1 - q)/2$).

The matrix A_λ , $\lambda \neq 1^N$, has rank $d_\lambda = 1$ and it is identical to the matrix A_λ for Bose statistics, A_λ^B , i.e completely has the Bose character,

$$A_\lambda^q = \frac{1+q}{2} A_\lambda^B.\tag{20}$$

Using the generic matrix A_{1^N} one easily finds

$$a_i a_{i_1}^\dagger a_{i_2}^\dagger |0\rangle = [\delta_{ii_1} a_{i_2}^\dagger + q \delta_{ii_2} a_{i_1}^\dagger],\tag{21}$$

and for $N \geq 3$,

$$\begin{aligned}a_i a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0\rangle &= [\delta_{ii_1}(i_2, \dots, i_N)_{\text{even}} + \delta_{ii_2}(i_1, i_3, \dots, i_N)_{\text{odd}} + \dots \\ &\quad + \delta_{ii_N}(i_1, i_2, \dots, i_{N-1})_{\text{even(odd)}}] |0\rangle.\end{aligned}\tag{22}$$

The subscript in the last term denotes even(odd) permutations for N odd(even).

If all indices are equal, eqs.(21,22) imply

$$\begin{aligned}a_i (a_i^\dagger)^2 |0\rangle &= (1 + q) a_i^\dagger |0\rangle, \\ a_i (a_i^\dagger)^n |0\rangle &= n (a_i^\dagger)^{n-1} |0\rangle, \quad n \neq 2.\end{aligned}\tag{23}$$

Then for a single oscillator [15], one can write $a^\dagger a = \varphi(n)$, $aa^\dagger = \varphi(n + 1)$ and $\langle 0 | a^n (a^\dagger)^n | 0 \rangle = [\varphi(n)]! = \frac{1}{2}(1 + q)n!$, where

$$\varphi(n) = \begin{cases} n & n \neq 2 \\ 1 + q & n = 2. \end{cases} \quad (24)$$

The expansion in eq.(3) implied by eqs.(21,22) can be written as follows:

$$\begin{aligned} a_i a_j^\dagger &= \delta_{ij} + q a_j^\dagger a_i \\ &+ \sum_{k=1}^M [x_1(jk)^\dagger(ik) + z_1(kj)^\dagger(ki) + y_1(kj)^\dagger(ik) + y_1(jk)^\dagger(ki)] \\ &+ \sum_{n=2}^{\infty} \sum_{k_1, \dots, k_n=1}^M [x_n(jk_1 \cdots k_n)^\dagger + y_n(jk_1 \cdots k_n k_{n-1})^\dagger](ik_1 \cdots k_n), \end{aligned} \quad (25)$$

and for $|q| < 1$,

$$\begin{aligned} x_1 &= -\frac{q}{1-q^2}, \quad z_1 = \frac{-2q+q^3}{1-q^2}, \quad y_1 = \frac{1}{1-q^2}, \\ x_2 &= q, \quad y_2 = -1. \end{aligned} \quad (26)$$

For $n \geq 2$, x_n and y_n satisfy the recursion relations

$$\begin{aligned} x_n + qy_n &= -(x_{n-1} + qy_{n-1}) - \frac{2}{n!}(q + nz_1) - 2 \sum_{k=1}^{n-2} \frac{x_k}{(n-k)!}, \\ qx_n + y_n &= -(qx_{n-1} + y_{n-1}) - \frac{2}{n!}(-1 + ny_1) - 2 \sum_{k=1}^{n-2} \frac{y_k}{(n-k)!}. \end{aligned} \quad (27)$$

When $q = \pm 1$, the expansion in eq.(25) reduces to $a_i a_j^\dagger = \delta_{ij} \pm a_j^\dagger a_i$ and all other terms vanish identically.

The partition function of a free M -level system defined in eq.(17) is (for $|q| < 1$)

$$Z(x_1, \dots, x_M) = \prod_{i=1}^M \frac{1}{1-x_i} + \prod_{i=1}^M (1-x_i) - \sum_{i=1}^M x_i - 1 \quad (28)$$

and the number of independent N -particle states is the sum of Bose and Fermi counting rules

$$D(M, N) = \binom{M+N-1}{N} + \binom{M}{N} \quad N \geq 2. \quad (29)$$

We point out that our simple interpolation defined by eq.(17) is equivalent to the statistics introduced by Wu *et al.* [7] and Scipioni [8]. The construction of Wu *et*

al. is based on two vacuums $|\pm\rangle$ and on the commutation rules containing the g -operator:

$$\begin{aligned} a_i a_j^\dagger - g a_j^\dagger a_i &= \delta_{ij} , \\ g|\pm\rangle &= \pm|\pm\rangle . \end{aligned} \tag{30}$$

They introduced the ϕ -vacuum as a linear combination of the $|\pm\rangle$ vacuums, $|\phi\rangle = \cos\phi|+\rangle + \sin\phi|-\rangle$ and defined the corresponding Fock representation built on $|\phi\rangle$. The q parameter in eq.(17) is then related to the angle ϕ through $\cos\phi = \sqrt{(1+q)/2}$. In our approach we start with one vacuum from the beginning and (except a_i, a_i^\dagger), no additional operators appear in eq.(25).

The physical consequences of the minimal interpolation follow from the grand partition function, eq.(28). It consists of the bosonic and fermionic partition functions from which one-particle states are subtracted. Such a partition function mainly has a Bose character since, for a large number of particles, the symmetric (bosonic) subspace is much larger than the antisymmetric (fermionic) subspace. Therefore, the whole spectrum of bosonic phenomena can be found here: Bose condensation [7], black body radiation [8]. The effects of the antisymmetric states keep trace only in corrections to the ordinary Bose phenomena, disappearing completely in the high temperature limit.

It is worth mentioning that the statistics of the type described in this letter can be discussed from the point of view of possible violation of Bose statistics. Some analysis has been done [16], comparing the experimental limits on the Z boson decay into two photons with the theoretical consideration based on a general phenomenological model of Bose symmetry violation. In the model of minimal mixing between bosons and fermions, the q -parameter would be $q < 10^{-2}$.

For comparison, we mention that another simple interpolation between Bose and Fermi statistics [5]

$$a_i a_j^\dagger - q a_j^\dagger a_i = \delta_{ij} , \quad |q| < 1 , \tag{31}$$

corresponds to infinite quon statistics, i.e. to the maximal interpolation in which every IRREP μ of S_N contributes with the multiplicity $n(\mu) = K_{\mu,1^N} = \dim(\mu)$. The number of independent N -particle states is $D(M, N) = M^N$.

Minimal interpolation between parabosons and parafermions

Para-Bose and para-Fermi statistics of a given order $p \in N$ generalize the Pauli exclusion principle and also belong to the class of permutation invariant statistics. The N -particle state of para-Bose (para-Fermi) statistics of order p cannot be anti-symmetrized (symmetrized) in more than p indices, which means that the allowed Young tableaux are restricted to those with at most p rows (columns). They are defined through trilinear relations:

$$[(a_i^\dagger a_j \pm a_j a_i^\dagger), a_k^\dagger] = \frac{2}{p} \delta_{jk} a_i^\dagger, \quad i, j, k = 1, 2, \dots, M \quad (32)$$

with the unique vacuum $|0\rangle$ and the following conditions: $\langle 0|0\rangle = 1$, $a_i|0\rangle = 0$, $a_i a_j^\dagger|0\rangle = \delta_{ij}|0\rangle$. The upper (lower) sign in eq.(32) corresponds to the para-Bose (para-Fermi) algebra and p is an integer.

It was shown that no interpolation between para-Bose and para-Fermi statistics through deformed trilinear relations was possible [6], since states of the negative norm appeared. However, in [14] it is suggested that such an interpolation is possible through a continuous family of generic matrices. The concrete construction was performed through deformed Green's oscillators obeying infinite quon statistics. The corresponding statistics belongs to the class of infinite statistics and is similar to that of Greenberg [5] and reduces to it for $p = 1$ and $p = \infty$.

Here we suggest a new family of generic matrices interpolating between para-Bose and para-Fermi generic matrices defined by eq.(15). It follows from eq.(32) that the coefficients $f^{p,\epsilon}(\pi)$ for parastatistics, $\epsilon = +$ (parafermions)/ $-$ (parabosons) of order p , satisfy recursion relations [6,11], and that

$$f^{p,-\epsilon}(\pi) = (-)^{I(\pi)} f^{p,\epsilon}(\pi). \quad (33)$$

This recursion relation implies that para-Bose and para-Fermi statistics of order p are dual to each other (see eq.(10) and equations following eq.(11)).

The generic matrices $A_{1^N}^{p,\epsilon}$, eqs.(6),(33), are hermitian and if p is a positive integer their eigenvalues are non-negative [3]. Using the results of [12], we find that the eigenvalues $(\Lambda^{p,\epsilon})_\mu$, corresponding only to one of equivalent IRREP's μ of S_N , are

$$\Lambda_\mu^{p,\epsilon} = \sum_{\pi \in S_N} f^{p,\epsilon}(\pi) \chi^\mu(\pi), \quad (34)$$

where χ^μ is the character of the IRREP μ . We point out that the eigenvalues corresponding to all other (except one) equivalent IRREP's μ are identically zero.

Applying the interpolation between dual statistics, eq.(15), to para-Bose and para-Fermi statistics, we have

$$A_{1^N}^{p,q} = \frac{1+q}{2} A_{1^N}^{p,-} + \frac{1-q}{2} A_{1^N}^{p,+}, \quad (35)$$

where $|q| \leq 1$. If $|q| < 1$, there are at most two positive eigenvalues corresponding to the equivalent IRREP's μ , ($\mu \neq 1^N, N$) of S_N and if $\mu = 1^N, N$, then

$$\Lambda_\mu^{p,q} = \frac{1+q}{2} \Lambda_\mu^{p,-} + \frac{1-q}{2} \Lambda_\mu^{p,+}, \quad (36)$$

where $\Lambda_\mu^{p,\epsilon}$ are given by eq.(34). The $\Lambda_\mu^{p,\epsilon} = 0$ if the number of rows of μ , $l(\mu)$, is $l(\mu) > p$ for $\epsilon = -$ and $l(\mu^T) > p$ for $\epsilon = +$.

Note that the eigenvalues corresponding to all (except at most two) equivalent IRREP's μ vanish identically. Hence, at most two equivalent IRREP's μ ($\mu \neq 1^N, N$) contribute to eqs. (7),(9). We therefore call the above interpolation minimal since $n(\mu) \leq 2$.

The probability of finding a generic state $a_{i_1}^\dagger \cdots a_{i_N}^\dagger |0\rangle$ with mutually different indices, in all equivalent IRREP's μ of S_N , is

$$w(\mu) = \frac{K_{\mu,1^N}}{N!} \left[\frac{1-q}{2} \Lambda_\mu^{p,+} + \frac{1+q}{2} \Lambda_\mu^{p,-} \right], \quad (37)$$

which generalizes the result for mixing of bosons and fermions. The relations (3)-(5) for the minimal interpolation between para-Bose and para-Fermi statistics of order p can be obtained similarly as in [11].

Let us point out that the above minimal interpolation can be obtained by generalizing the statistics of Wu *et al.* [7] and Scipioni [8]. The relations in (32) become

$$[(a_i^\dagger a_j + g a_j a_i^\dagger), a_k^\dagger] = \frac{2}{p} \delta_{j,k} a_i^\dagger, \quad (38)$$

where $i, j, k = 1, 2, \dots, M$ and $g|\pm\rangle = \pm|\pm\rangle$. Choosing one vacuum $|\phi\rangle = \cos\phi|+\rangle + \sin\phi|-\rangle$, one obtains the same statistics as in eq.(35) with $\cos\phi = \sqrt{(1+q)/2}$, $\phi \in [0, \pi/2]$.

Simple interpolation between anyonic-like statistics

Finally, let us mention that the above consideration on duality and a simple interpolation can be extended to anyonic-like generalized statistics, which are not invariant under the permutation group. We call them anyonic-like statistics by analogy with the anyonic interpolation between Bose and Fermi statistics, where we interpolate between any two permutation invariant dual statistics.

The anyonic-like generic matrix A_{1N}^ϕ can be obtained from any permutation invariant generic matrix A_{1N} , eq.(6), in the following way:

$$A_{1N}^\phi = D_{1N}^\phi A_{1N} D_{1N}^{-\phi}, \quad (39)$$

where D_{1N}^ϕ is the $N! \times N!$ diagonal matrix with matrix elements,

$$(D_{1N}^\phi)_{\pi\sigma} = e^{i\phi(\pi)} \delta_{\pi\sigma} \quad (40)$$

and $\phi(\pi)$ are real coefficients (i.e. function from S_N to real numbers). If the generic matrix A_{1N} is hermitian, then the corresponding anyonic generic matrix A_{1N}^ϕ in eq.(40) is hermitian with the same spectrum as A_{1N} . If $A_{1N} = \sum_{\pi \in S_N} f(\pi) R(\pi)$, then $A_{1N}^\phi = \sum_{\pi \in S_N} f(\pi) R^\phi(\pi)$, where $R^\phi = D^\phi R D^{-\phi}$, i.e. $R_{\alpha\beta}^\phi(\pi) = e^{i[\phi(\alpha) - \phi(\beta)]} R_{\alpha\beta}(\pi)$ is equivalent to the regular representation. The non-generic matrix $A_\lambda^\phi(i_1, \dots, i_N)$, $\lambda \neq 1^N$, $|\lambda| = N$, is similarly defined as A_λ , see the relation preceding eq.(6). However, the phases in D^ϕ , eq.(40), are of a more general form $\phi(\pi; \lambda)$ depending

on permutation π and the multiplicities of equal indices. Thus, we point out that generally no simple redefinition of states by an insignificant phase factor is possible. It could be done for the generic matrix A_{1N} alone, but not for A_λ^ϕ for all partitions $\lambda \neq 1^N$, $|\lambda| = N$ simultaneously. There is a large class of anyonic-like algebras which can be obtained by non-linear transformation on a_i^\dagger , a_i from permutation invariant algebras, but this is not true for every anyonic-like statistics in general.

The dual generic matrix is defined by

$$(A_{1N}^\phi)^d = (A_{1N}^d)^\phi = D_{1N}^\phi A_{1N}^d D_{1N}^{-\phi} = \sum_{\pi \in S_N} f^d(\pi) R^\phi(\pi). \quad (41)$$

The anyonic-like generic matrices A_{1N}^ϕ and $(A_{1N}^\phi)^d$ are hermitian and have the same eigenvectors and eigenvalues.

The simple interpolation between the A_{1N}^ϕ and $(A_{1N}^\phi)^d$, defined by eq.(15), has non-negative eigenvalues for $|q| \leq 1$.

The simplest examples of anyonic-like statistics related to permutation invariant statistics by eq.(40) are obtained as special cases $|q_{ij}| = 1$, $i, j = 1, 2, \dots, M$, of the operator algebra defined by $a_i a_j^\dagger - q_{ij} a_j^\dagger a_i = \delta_{ij}$, $q_{ij}^* = q_{ji}$, $i, j = 1, 2, \dots, M$, investigated in [17],[18]. They can be obtained by regular non-linear mapping from fermions and/or bosons. However, anyonic generic matrices are not permutation invariant, although they are in a simple way related to Bose and Fermi generic matrices, eq.(39).

Conclusion

We have considered general permutation invariant statistics in the second quantized approach. Particularly we have investigated generic matrices A_{1N} and their dual matrices A_{1N}^d . Then we have suggested a simple interpolation between these two types of statistics. The permutation invariant statistics considered is completely determined (including the probabilities of finding IRREP's μ of S_N in all decompositions) by the functions $f(\pi)$, $\pi \in S_N$, for all N that lead to non-negative eigenvalues

of A_{1^N} .

Particularly, we have presented new minimal interpolations between (para)bosons and (para)fermions of order p and established a connection with the mixing of bosons and fermions proposed by Wu *et al.* [7] and Scipioni [8].

Finally, we have proposed an extension of our analysis to anyonic-like statistics, which are related to permutation invariant statistics by eq.(40).

Symbolically, we can write

$$\begin{array}{ccccc}
 A_{1^N} & \Leftarrow & A_{1^N}^q & \Longrightarrow & A_{1^N}^d \\
 \Downarrow & & \Downarrow & & \Downarrow \\
 A_{1^N}^\phi & \Leftarrow & (A_{1^N}^\phi)^q & \Longrightarrow & (A_{1^N}^\phi)^d,
 \end{array} \tag{42}$$

where the interpolating generic matrices are defined by eq.(15).

The physical properties of the minimal mixing of bosons and fermions were investigated in [7],[8] and the properties of the minimal mixing between parabosons and parafermions, as well as mixing between anyons, are under investigation.

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