

“Winner takes it all”: strongest node rule for evolution of scale free networks

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We study a novel model for evolution of complex networks. We introduce information filtering for reduction of the number of available nodes to a randomly chosen sample, as stochastic component of evolution. New nodes are attached to the nodes that have maximal degree in the sample, which is a deterministic component of network evolution process. This fact is a novel for evolution of scale free networks and depicts a possible new route for modeling network growth. We present both simulational and theoretical results for network evolution. The obtained degree distributions exhibit an obvious power-law behavior in the middle with the exponential cut off in the end. This highlights the essential characteristics of information filtering in the network growth mechanisms.

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I. INTRODUCTION

Recently, there have been a number of extensive investigations in the field of complex networks. With such an extensive effort a number of important theoretical and practical results have been reported [1, 2, 3]. Many real world systems can be described as complex networks: www [4], internet routers [5, 6, 7], proteins [8], scientific collaborations [9], among others. The main features that separate complex networks from “ordinary” networks are the famous small world effect [10] and the scale free degree distribution [11].

The first and simplest model for the scale free distribution of degrees in complex network was proposed by Albert and Barabasi [12] (thereafter referred to as AB model). This model is based on a simple principle of preferential attachment. The network grows in such a way that at each time step t a new node is introduced into the network and attaches itself to some of older nodes designated by the moment s when they entered the network. The probability that the node t will attach itself to a node s is linearly proportional to the degree k_s of the older node $P_{t \rightarrow s} \sim k_s$. Using this simple principle a scale free network of exponent 3 is easily reconstructed. Although very appealing because of its simplicity the AB model cannot correctly reproduce all characteristics of real world networks. First, it produces a temporally correlated network in the sense that older nodes tend to have more edges than younger ones, which was not observed in real data [13]. Second, it assumes that every new node has the complete information about the whole network, which is unrealistic for real network formations [14, 15]. Third, in its original form, it reproduces only networks with degree distribution characterised by exponent $\gamma = 3$. Nevertheless the AB model has triggered a huge number of models that try to avoid these shortcomings, but are also a natural extension of the original. Among others there are models with nonlinear preferentiality [16], with rewiring of edges at later times [17], with a fitness parameter as an intrinsic value of a node [18, 19], etc. Although novel and more complex approaches, that describe a va-

riety of degree distributions and have more support in the real data, have been studied recently [20, 21], we believe that it is also of fundamental importance to examine “as simple as possible” processes that capture essential behavior of real world networks.

In this paper we present a novel model which exhibits power-law-like degree distribution of an undirected network or the in-degree power-law-like distribution of a directed network. The purpose of the model is to test information filtering as a stochastic component of the network evolution process, while using a simple deterministic rule for attachment of new nodes. The results we report in this article clearly show that our model can reproduce power-law distributions but also power-laws with a cut off, similarly to some real data reported recently [22].

II. MODEL

Our model introduces two crucial features that make it different from the Albert-Barabasi model. A new node is introduced into the network at each time step. For simulation purposes, we first generate a network of 1100 nodes which are completely randomly connected to each other. Each new node in this core is connected to one of the older ones with uniform probability, until a core is formed. The size of the core is taken to be 1100 because we chose to monitor filtration subsets up to 1000 nodes. After the core is formed, the following procedure takes place. Each new node attaches itself to the network with ω links. To choose to which of the already present nodes in the network it will attach itself, the following rule is applied. i) A sample of the already present nodes of fixed size m is randomly chosen from the network which contains t nodes. The probability of choosing any node in the sample equals m/t . ii) Chosen nodes are sorted by their degree in the decreasing order. For the nodes with the same degree no additional rearrangement is applied. iii) From such a sorted sample, a new node is attached to the first ω nodes that have the highest degree. The third rule is a simple deterministic “winner takes it all” algorithm, which combined with the first two rules produces

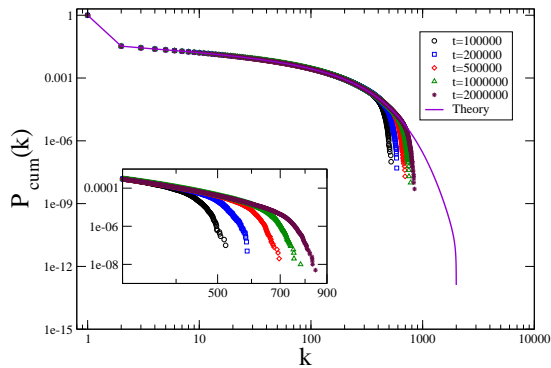


FIG. 1: Simulated cumulative probability functions with $m = 100$ and $\omega = 1$ for different final network sizes n_{max} are compared to the theoretically obtained one. The figure clearly depicts the asymptotic approach of simulation curves to the theoretical result. This implies that analytical results are precise and that they sufficiently well describe the behavior of the system when $n_{max} \rightarrow \infty$. The inset gives the enlarged section with the tails of the simulated cumulative probability distributions to better illustrate the effects of the finite network size.

very interesting macroscopic effects, as will be presented in this paper.

The nodes are numbered from 0, and the network is grown to the size n_{max} . We averaged over 100 simulations for every investigated ω , m , and n_{max} in order to get a statistically relevant ensemble of network realizations. We also performed a scaling investigation presented in Fig. 1 to see how a simulated distribution behaves for different network sizes.

III. THEORY

In the theoretical treatment of the node degree distribution we decide to limit ourselves to the description of network with $\omega = 1$. The reason for such an approach is a cumbersome analytical study for the case of $\omega > 1$, which would include many more summation terms that are analytically almost unentangleable. We use the master equation approach of Dorogovtsev and Mendes [23]. In this approach a new node enters the network at every moment s and is therefore denoted by s . It connects with one edge to the node with maximum degree in the randomly selected sample of size m . Nodes in sample are selected from t nodes that are already present in the network, so that every existing node has the probability $\frac{m}{t}$ of entering the sample.

The probability that the node s with degree k will enter the sample of size m at time t and will be chosen for the attachment of the new node is

$$v(k, m, t-1) = \sum_{l=0}^{m-1} \binom{\hat{B}(k, t-1)}{l} \times \frac{\binom{N(k, t-1) - 1}{m-l-1}}{(m-l) \binom{t}{m}}. \quad (1)$$

Here the first binomial coefficient in the numerator represents number of possible ways to chose l nodes with degree smaller than k into the sample, and

$$\hat{B}(k, t-1) = \sum_{q=1}^{k-1} N(q, t-1), \quad (2)$$

where $N(k, t)$ is the number of nodes with degree k at time t . The second binomial coefficient counts the number of possible ways to chose $m-1-l$ nodes with the same degree as node s into the sample. This part of expression (1) accounts for the possibility that in the selected sample exist other nodes with the same maximal degree as s . Using the fact that $N(q, t) = P(q, t) \cdot t$, together with approximation that for large t one can approximate $\binom{t}{m}$ with $t^m/m!$, we reduce the expression (1) to the following form:

$$v(k, m, t-1) \simeq \frac{1}{t} \sum_{l=0}^{m-1} \binom{m}{l} \Pi(k, t-1)^l P(k, t-1)^{m-l-1}, \quad (3)$$

where

$$\Pi(k, t-1) \equiv \sum_{q=1}^{k-1} P(q, t-1). \quad (4)$$

Using the well-established Dorogovtsev-Mendes master equation approach for calculating the node degree distribution, for $k \geq 2$ we write

$$p(k, s, t) = v(k-1, m, t-1)p(k-1, s, t-1) + (1 - v(k, m, t-1))p(k, s, t-1). \quad (5)$$

To calculate the probability distribution $P(k, t)$ that a randomly chosen node has k edges at time t , we average the probability distribution of all nodes s , i.e.

$$P(k, t) = \frac{1}{t+1} \sum_{s=0}^t p(k, s, t). \quad (6)$$

Thus we obtain

$$P(k, t) = \frac{\zeta(k-1, t-1)}{t+1} P(k-1, t-1) + \left(\frac{t}{t+1} - \frac{\zeta(k, t-1)}{t+1} \right) P(k, t-1), \quad (7)$$

where

$$\zeta(k, t-1) \equiv \sum_{l=0}^{m-1} \binom{m}{l} \Pi(k, t-1)^l P(k, t-1)^{m-l-1}. \quad (8)$$

Assuming that Eq. (7) has a stable asymptotic solution for $t \gg 1$ thus changing the time-dependent probability distribution into time independent $P(k, t) = P(k)$, we obtain the following closed form:

$$P(k) = \zeta(k-1)P(k-1) - \zeta(k)P(k). \quad (9)$$

Equations (9) are a polynomials of order m and hold for all $k \geq 2$. Written as polynomials, they adopt the following form:

$$a(0)P(k)^m + a(1)P(k)^{m-1} + \dots + a(l)P(k)^{m-l} + \dots + (1 + a(m-1))P(k) - \sum_{l'=0}^{m-1} \binom{m}{l'} \left(\sum_{q=1}^{k-2} P(q) \right)^{l'} P(k-1)^{m-l'} = 0, \quad (10)$$

where the coefficients $a(l)$ are

$$a(l) = \binom{m}{l} \left(\sum_{q=1}^{k-1} P(q) \right)^l. \quad (11)$$

For theoretical treatment of $P(1)$ as our boundary condition the following equation holds:

$$p(1, s, t) = \delta_{s,t} + (1 - \delta_{s,t}) (1 - v(1, m, t-1)) p(1, s, t-1), \quad (12)$$

with an obvious relation for probability that a node with one edge at time $t-1$ will adopt a new edge at time t :

$$v(1, s, t-1) = \frac{P(1, t-1)^{m-1}}{t}. \quad (13)$$

Using a procedure similar to that already mentioned above, we obtain the asymptotic value for $P(1)$:

$$P(1) = 1 - P(1)^m. \quad (14)$$

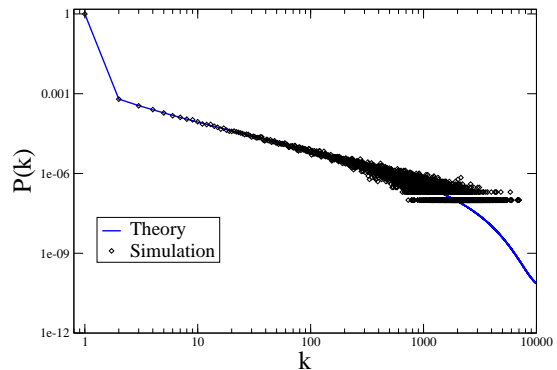


FIG. 2: Theoretical probability distribution (solid line) nicely follows simulation data (black diamonds) for $m = 1000$. Scattering in the tail is a consequence of low probability fluctuations induced by finite size effects. The reader should also note a big jump of probability for $P(k = 1)$.

Unfortunately, the set of Eqs. (10) and (14) is analytically unsolvable and is therefore solved numerically. The solutions of these polynomial equations show excellent agreement with numerical simulations as can be seen in Figs. 2, 3, 4, and 5. These findings further vindicate the master equation approach followed in this paper.

IV. DISCUSSION

As we have mentioned in the preceding section, a master equation approach yields a chain of the polynomial equations (10) and (14). Note the fact that $P(k^*)$ representing the probability that a randomly chosen node will have a degree k^* depends only on degree probabilities that are equal or less than $k^* - 10$. We have calculated the roots of the system to get a degree probability distribution.

All simulated data and analytical roots of polynomial equations exhibit a big jump from $P(k = 1)$ to $P(k = 2)$ of order of a magnitude or more. The difference $P(k = 1) - P(k = 2)$ depends strongly on the size of a chosen sample m . If the size of the sample is larger, then there is higher probability that a node of degree larger than 1 will enter the sample, and collect the new link. The smaller the sample the greater the probability that only nodes of degree one will be chosen in the sample, thus lowering the overall amount of nodes of degree one. The obtained analytical solutions from Eq. (14) are in excellent agreement with simulation results regarding to this jump. The average relative error for $m \in \{10, 100, 1000\}$ simulation and theory is $4.3 \cdot 10^{-5}$, and gets smaller as the sample size m grows larger for $n_{max} = 10^6$.

All simulated data exhibit a strong scattering in the tail. The scattering is a consequence of low probability fluctuations and makes the comparison between theory and simulation more difficult, Fig.2. In order to straighten

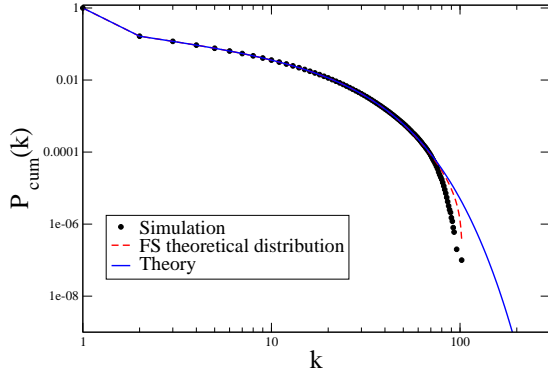


FIG. 3: For $m = 10$ the theoretical distribution (solid line) nicely follows simulation data (black dots). The disagreement in the very tail is explained by finite size effects of simulated data. However, a FS theoretical distribution obtained by transformation (16) shows even better agreement with simulation

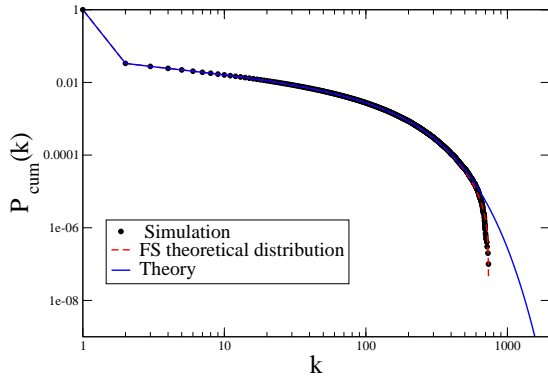


FIG. 4: For $m = 100$ it is easy to see that the theoretical distribution follows simulation data very well, and FS theoretical distribution even better.

up the data and compare theory and simulation, it is possible to use exponential binning or to transform probability distribution into the cumulative probability distribution. We implemented the second approach and produced a cumulative degree probability distribution P_{cum} .

$$P_{cum}(k) = \sum_{q=k}^{\infty} P(q). \quad (15)$$

This distribution contains the same system information as the degree distribution, but is much smoother in the tail. We compared our theoretical curve with the simulated one and found an excellent match between theory and simulations. The results of the comparison between simulation and theory are presented in Figs. 3, 4, and 5. The relative disagreement observed in the tails is a consequence of finite size effects, Fig.1. Since our theoretical curve falls down relatively slowly, as can be seen

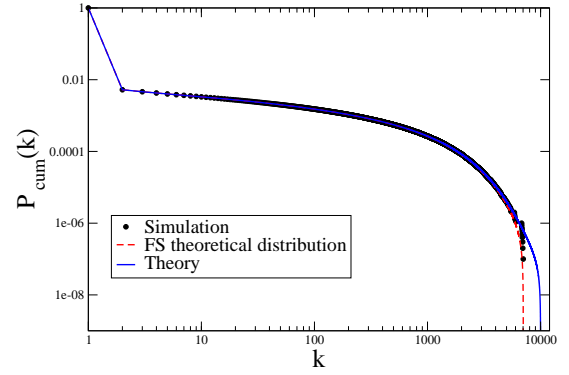


FIG. 5: $m = 1000$ is the largest monitored sample size but is still small enough compared to simulation number of nodes $n_{max} = 10^6$. Theory is in excellent agreement with simulation.

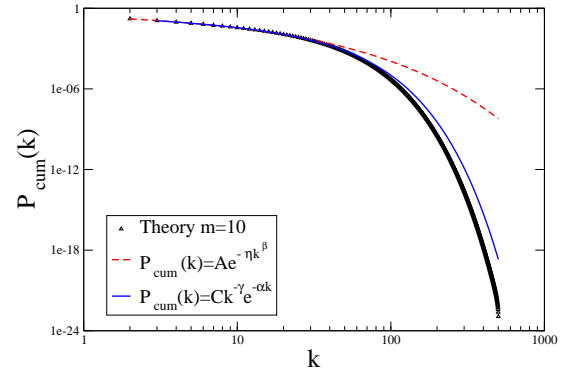


FIG. 6: Two different functions: i) stretched exponential and ii) power law with the cut off are fitted on theoretical data for sample size $m = 10$. This figure clearly shows that the power law with the exponential cut off better describes the tail of the theoretical distribution.

in table I, the summation of probabilities for $k > k_{max}$ in Eq. (15) contributes strongly to the cumulative degree probability in the tail. To get an even better match, we calculated “renormalized” cumulative probability distribution

$$\tilde{P}_{cum}(k) = \frac{\sum_{q=k}^{k_{max}} P(q)}{1 - \sum_{q=k_{max}+1}^{\infty} P(q)}. \quad (16)$$

This Finite Size cumulative probability distribution (hereafter denoted as FS theoretical distribution) is even better in describing finite size effects, as shown in the Figs.3, 4, and 5.

To obtain a description of the degree distribution in the thermodynamical limit, we fitted theoretical cumulative degree distribution (theoretical and not simulation) distribution was also used since it does not suffer from finite size effects) with the stretched exponential (17) and

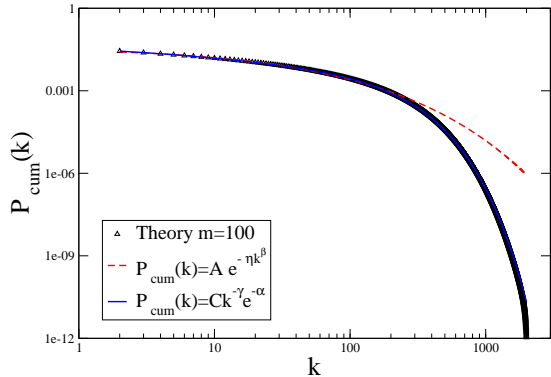


FIG. 7: Fitted curves for $m = 100$. Power law with the exponential cut off represents the theoretical distribution very well.

power-law distribution with the exponential cut off (18), [15]:

$$P_{cum} \sim e^{-\eta k^\beta}, \quad (17)$$

$$P_{cum} \sim k^{-\gamma} e^{-\alpha k}. \quad (18)$$

For fitting purposes we used all theoretical $P_{cum}(k)$ values, except $P_{cum}(1)$, because its value is clearly not determined by the scale-free-like behavior as opposed to all other k values. Both distributions fit our overall results very well, as presented in Table I, and Figs. 6, 7, and 8. The correlation coefficients of the fitted distributions are all above 0.99 margin, proving that both fitting models are capable of describing theoretically obtained curves very well. The power law with the exponential cut off always has just a slightly higher correlation coefficients than stretched exponential for the same sample size m . Figures clearly show that the reason for this behavior is much better description of the tail, which power law with exponential cut off exhibits. Stretched exponential is clearly not suitable for the description of the tail properties.

It is worthy to mention that the power-law distribution with the exponential cut off has already been obtained in a similar model [15], which has shown that exponential parameter α is trivially connected with the sample size m by the relation $\alpha = \frac{1}{m}$. Although one cannot expect this relation to be valid for this model also, the parameter α is very close to $1/m$, and this coincidence is better for larger m , as can be seen in table I. In our opinion, it would be interesting to measure α in some observed network distributions of a similar shape and compare it with the expected sizes of samples on which the new node has the possibility of creating a link.

Finally, let us briefly discuss simulational results for $\omega > 1$. Simulational results for the cumulative probability distribution (without $P(\omega)$) are displayed in Fig. 9.

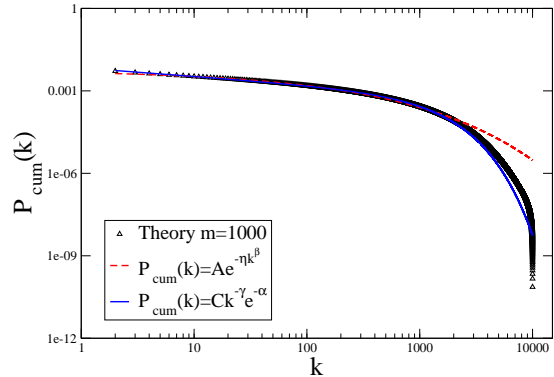


FIG. 8: Excellent agreement of fitted and theoretical distributions for $m = 1000$.

	$P_{cum} \sim k^{-\gamma} e^{-\alpha k}$			$P_{cum} \sim e^{-\eta k^\beta}$		
	γ	α	corr	η	β	corr
m=10	0.5501	0.0765	0.9980	0.9829	0.4718	0.9976
m=100	0.4026	0.0092	0.9995	0.3894	0.4385	0.9987
m=1000	0.3067	0.0011	0.9994	0.2230	0.3823	0.9978

TABLE I: Fitted distribution parameters for different sample sizes. Correlation coefficients show excellent agreement between the theoretical distribution data and the presented fits.

The typical characteristics of the distribution are equivalent to the $\omega = 1$ case. The degree $k = \omega$ has substantially larger probability compared to all other degrees. The cumulative probability distribution for $k \geq 2$ obtained in the simulations displays the scale-free-like properties. These simulational distributions can be well fitted with the power law with the exponential cut off (18) as shown in Fig. 9.

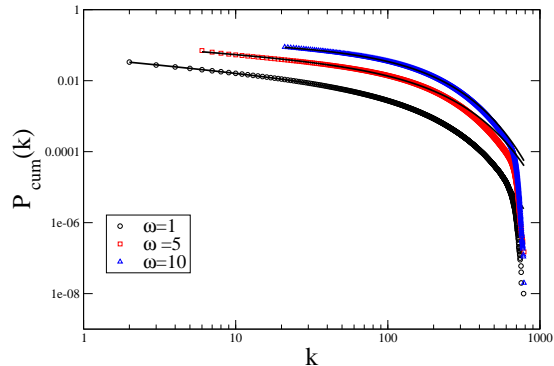


FIG. 9: Evidence that the distributions for $\omega > 1$ fall in the same class as the distributions studied analytically. The situation with $m = 100$ is presented.

V. CONCLUSION

We have shown that using a simple “winner takes it all” algorithm, together with the fact that nodes do not possess complete information on network structure, a macroscopic node-degree power law is created. Evolution of real networks is still an open question and we have shown that realistic imperfect knowledge can have a substantial effect on network growth. Although the field of complex networks has made great progress during the last few years, there is still much open space for research of microscopic models that describe the formation of complex networks with certain expected features. Our results clearly show that stochastic-deterministic pro-

cesses even as simple as that described in this paper can be used to reproduce some macroscopic effects of complex networks. Moreover, in this paper as well as in [15], we have demonstrated that the power law with the exponential cut off can be a significant distribution for types of networks in which information filtering is performed. New findings in social contact networks [22] lead us to believe that the power law with the exponential cut off and stretched exponentials should be studied more intensively in the future.

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