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Double field theory algebroid and curved L_∞-algebras

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Abstract

A double field theory algebroid (DFT algebroid) is a special case of the metric (or Vaisman) algebroid, shown to be relevant in understanding the symmetries of double field theory. In particular, a DFT algebroid is a structure defined on a vector bundle over doubled spacetime equipped with the C-bracket of double field theory. In this paper we give the definition of a DFT algebroid as a curved L_∞-algebra and show how implementation of the strong constraint of double field theory can be formulated as an L_∞-algebra morphism. Our results provide a useful step towards coordinate invariant descriptions of double field theory and the construction of the corresponding sigma-model.

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1 Introduction and overview

Double field theory (DFT) is a field theory defined on a double, $2d$-dimensional configuration space where the extra coordinates are introduced in an effort to realise T-duality of string theory as a symmetry of field theory [1–4]. DFT enjoys global $O(d,d)$ symmetry, and all objects in the theory belong to some representation of the $O(d,d)$ duality group. In particular, coordinates and their duals form generalised coordinates $X^A = (x^a, \tilde{x}_a)$ in the fundamental representation of $O(d,d)$, and similarly derivatives $\partial_A = (\partial_a, \tilde{\partial}^a)$, while indices are raised and lowered by the $O(d,d)$ metric:

$$\eta_{AB} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^b_a & 0 \end{pmatrix} \ .$$

Throughout the paper $A, B, \ldots = 1, \ldots, 2d$ and $a, b, \ldots = 1, \ldots, d$ are indices corresponding to double and standard spacetime respectively, while latin indices from the middle of the alphabet $I, J, \ldots = 1, \ldots, 2d$ and $i, j, \ldots = 1, \ldots, d$ are reserved for bundle indices. The field content of the theory, in its simplest form, consists of the dilaton, the d-dimensional metric g_{ab} and the 2-form Kalb-Ramond field B_{ab}, corresponding to the universal gravitational massless bosonic sector of closed string theory. In principle, all fields in the theory depend on generalised coordinates. The metric g_{ab} and Kalb-Ramond field B_{ab} are packaged together into a symmetric *generalised metric* $\mathcal{H}_{AB}$

$$\mathcal{H}_{AB} = \begin{pmatrix} g_{ab} - B_{ac}\, g^{cd}\, B_{db} & -B_{ac}\, g^{cb} \\ g^{ac}\, B_{cb} & g^{ab} \end{pmatrix} \ ,$$

or, in the frame formalism, the d-dimensional bein $e^m{}_a$ and B_{ab} are collected into a generalised bein $\mathcal{E}^M{}_A$. The dilaton is combined with the determinant of the metric g_{ab} into an $O(d,d)$ scalar. However, from now on we shall drop the dilaton field for simplicity, see [5] for a geometric description of the dilaton field.

DFT also enjoys local, gauge symmetry combining standard diffeomorphisms and gauge transformations of the Kalb-Ramond field, generated by an $O(d,d)$ vector $\epsilon^A = \mathcal{E}_M{}^A \lambda^M = (\epsilon^a, \tilde{\epsilon}_a)$. However, the theory is gauge invariant only if the physical fields satisfy a set of constraints which can be written in a local basis as:

$$\eta^{AB}\partial_A \partial_B(\cdots) = 0 \ ,$$

where dots in the bracket denote a product of different fields. This *strong constraint* (as opposed to the weak constraint of DFT that corresponds to the level-matching condition in string theory) also enforces closure of the symmetry algebra of the theory [6]. Namely, the algebra of gauge transformation, $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] = \delta_{\epsilon_{12}}$, defined by the C-bracket, $\epsilon_{12} = -[\![\epsilon_1, \epsilon_2]\!]$

$$[\![\epsilon_1, \epsilon_2]\!]^B = (\epsilon_1^A \partial_A \epsilon_2^B - \tfrac{1}{2}\eta_{AD}\epsilon_1^A \partial^B \epsilon_2^D) - (\epsilon_1 \leftrightarrow \epsilon_2) \ ,$$

closes only for fields and gauge parameters obeying the strong constraint. After imposing the strong constraint, the C-bracket of DFT reduces to the Courant bracket, the properties of which are captured by Courant algebroids [7–9].

The question that naturally arises is if one can provide a geometric description of DFT symmetries based on the C-bracket, before reducing the theory by imposing the strong constraint. In Ref. [10] the authors suggested that the relevant geometric structure is a pre-NQ manifold. This structure is defined on non-negatively (N) graded manifolds with a degree 1 vector field (Q) which does not square to zero, with the obstruction controlled by the strong constraint. The relevant pre-NQ manifold was obtained as a half-dimensional submanifold from the Vinogradov algebroid defined over a doubled space.

Similarly, motivated by the observation that in double field theory one doubles the configuration space, while in a Courant algebroid one extends the bundle, in Ref. [11] the starting point was a "large" Courant algebroid defined over a doubled target space. Using a specific projection from the large Courant algebroid one arrives at a DFT algebroid, the structural data of which are related to fluxes and the generalised bein of double field theory. After imposing the strong constraint, the DFT algebroid reduces to a Courant algebroid over an undoubled target space. It was further shown that there exists a more general structure (dubbed a pre-DFT algebroid in [11]) that corresponds to the metric or Vaisman algebroid [12], of which the DFT algebroid is a special case.

In this paper we shall analyse the structure of a DFT algebroid from a different perspective, giving its definition in terms of a *curved* L_∞-algebra [13, 14]. This naturally extends the results of [10], and moreover, it implies that one should be able to formulate a DFT algebroid in terms of a Q structure. This becomes especially important when constructing the corresponding sigma-model, but this we leave for future work. Here we shall start by recalling the definition of a DFT algebroid and analysing its properties in more detail. The important observation is that the symmetric bilinear on the bundle induces a symmetric pairing on the doubled configuration space, which allows for the appropriate geometric description of a DFT algebroid. This pairing is actually a para-Hermitan metric on the doubled configuration space, see Refs. [15–19] for the description of double field theory in terms of para-Hermitian manifolds, and more generally in terms of Born geometry. Next, in section 3 we introduce curved L_∞-algebras in our convention and as a motivating example we review the Courant algebroid L_∞-algebra [20, 21]. Thereafter, in subsections 3.2 and 3.3 we construct the curved L_∞-algebra for a DFT algebroid on two different graded spaces underlying the L_∞-algebra. Section 4 is dedicated to the understanding of the strong constraint on the DFT algebroid as an L_∞-morphism. We begin by recalling the definition of L_∞-morphisms and then explicitly construct the map from a DFT algebroid to a Courant algebroid. Appendix A is intended as a reminder to the reader of the explicit relation between DFT algebroid structures and double field theory data. Finally, appendices B and C provide completeness for longer calculations of sections 3.3 and 4.2.

2 DFT algebroid

A DFT algebroid was constructed in Ref. [11] using a specific projection from a Courant algebroid defined over doubled target space. In particular, it was shown that the projected bracket is the C-bracket of double field theory, and the projected bundle map ρ is related to the generalised bein. The explicit relation between DFT algebroid structures and the flux formulation of double field theory using local basis is reviewed in Appendix A. Here we discuss global properties of a DFT algebroid, starting with the following definition:

Definition 2.1 (**DFT algebroid** [11]) *Let $\mathcal{M}$ be a 2d-dimensional manifold. A DFT algebroid is a quadruple $(L, [\![\,\cdot\,,\,\cdot\,]\!], \langle\,\cdot\,,\,\cdot\,\rangle, \rho)$, where L is a vector bundle of rank 2d over $\mathcal{M}$ equipped with a skew-symmetric bracket $[\![\,\cdot\,,\,\cdot\,]\!] : \Gamma(L)\otimes\Gamma(L)\to\Gamma(L)$, a non-degenerate symmetric form $\langle\,\cdot\,,\,\cdot\,\rangle : \Gamma(L)\otimes\Gamma(L)\to C^\infty(\mathcal{M})$, and a smooth bundle map $\rho : L\to T\mathcal{M}$, such that:*

1. $\langle \mathcal{D}f, \mathcal{D}g\rangle = \frac{1}{4}\langle \mathrm{d}f, \mathrm{d}g\rangle$;
2. $[\![e_1, f\, e_2]\!] = f\,[\![e_1, e_2]\!] + \big(\rho(e_1)f\big)\, e_2 - \langle e_1, e_2\rangle\, \mathcal{D}f$;
3. $\langle [\![e_3, e_1]\!] + \mathcal{D}\langle e_3, e_1\rangle, e_2\rangle + \langle e_1, [\![e_3, e_2]\!] + \mathcal{D}\langle e_3, e_2\rangle\rangle = \rho(e_3)\langle e_1, e_2\rangle$;

for all $e_i \in \Gamma(L)$ and $f, g \in C^\infty(\mathcal{M})$, where $\mathcal{D} : C^\infty(\mathcal{M}) \to \Gamma(L)$ is the derivative defined through $\langle \mathcal{D}f, e\rangle = \frac{1}{2}\,\rho(e)f$.

From property 1 in Definition 2.1 it follows that a pairing in the bundle L induces a symmetric pairing on $T\mathcal{M}$:

$$\begin{aligned} &\eta : T\mathcal{M}\times T\mathcal{M} \to C^\infty(\mathcal{M}) \\ &\eta = \tfrac{1}{2}\eta_{AB}\mathrm{d}X^A \vee \mathrm{d}X^B, \qquad A, B = 1, \dots, 2d\ , \end{aligned} \tag{2.2}$$

or, in other words, an $O(d, d)$ metric with components:

$$\eta_{AB} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^b_a & 0 \end{pmatrix}, \qquad a, b = 1, \dots, d\ . \tag{2.3}$$

Here we introduced the symmetric tensor product $u\vee v = u\otimes v + v\otimes u$, in analogy with the more standard wedge product. Since η_{AB} is invertible it also defines a symmetric 2-vector:

$$\begin{aligned} &\eta^{-1} : T^*\mathcal{M}\times T^*\mathcal{M} \to C^\infty(\mathcal{M}) \\ &\eta^{-1} = \tfrac{1}{2}\eta^{AB}\partial_A \vee \partial_B, \qquad A, B = 1, \dots, 2d\ . \end{aligned} \tag{2.4}$$

The action on functions is defined via the natural contraction with 1-forms:

$$\eta^{-1}(df) = \iota_{\eta^{-1}} df = \eta^{AB}\partial_A f \partial_B\ . \tag{2.5}$$

Additionally, we define the action of the symmetric 2-vector η^{-1} on a section $v = v^A\partial_A$ of $\Gamma(T\mathcal{M})$ using the Schouten-Nijenhuis bracket for symmetric vectors [22] as follows:

$$\eta^{-1}(v) := [\eta^{-1}, v]_{SN} = \eta^{AB}\partial_A v^C \partial_C \vee \partial_B\ . \tag{2.6}$$

We would now like to further explore the structural data of the DFT algebroid. First, one notices that $\mathrm{Im}(\mathcal{D})$ is not in the kernel of map ρ:

$$(\rho\circ\mathcal{D})f = \tfrac{1}{2}\eta^{-1}(df)\ . \tag{2.7}$$

Next, one can show that map ρ is not a homomorphism:

$$\begin{aligned} \rho([\![e_1, e_2]\!])(f) - [\rho(e_1), \rho(e_2)](f) &= -\,\mathsf{SC}_\rho(e_1, e_2)f\ , \\ \mathsf{SC}_\rho(e_1, e_2)f &:= \tfrac{1}{2}\eta(\rho(e_1), \eta^{-1}(df)(\rho(e_2))) - (e_1 \leftrightarrow e_2)\ , \end{aligned} \tag{2.8}$$

where the second bracket on the lhs in the first line is the standard Lie bracket of vector fields, and we introduced a shorthand notation for the rhs. Furthermore, the bracket does not satisfy the Jacobi identity, in fact we have:

$$\mathrm{Jac}(e_1, e_2, e_3) := [\![[\![e_1, e_2]\!], e_3]\!] + \text{cyclic} = \mathcal{DN}(e_1, e_2, e_3) + \mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3) \ , \tag{2.9}$$

where $\mathcal{N}$ is

$$\mathcal{N}(e_1, e_2, e_3) := \tfrac{1}{3} \langle [\![e_1, e_2]\!], e_3 \rangle + \text{cyclic} \ , \tag{2.10}$$

and we introduced the shorthand notation:

$$\begin{aligned} &\mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3) := \\ &\tfrac{1}{2}\rho^{-1} \left\{ (-\eta^{-1}(\rho(e_2))(\eta(\rho(e_1)) + \eta^{-1}(\rho(e_1))(\eta(\rho(e_2)) + [\rho(e_1), \rho(e_2)])\rho(e_3) \right\} + \text{cyclic} \ . \end{aligned}$$

Compared with definitions in [11], the definition of $\mathsf{SC}_{\mathrm{Jac}}$ is extended, see App. A for more details. Here we defined the inverse of the anchor map $\rho^{-1} : T\mathcal{M} \to L$ as shown in the following commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\hat{\eta}} & L^* \\ \uparrow{\scriptstyle \rho^{-1}} & & \uparrow{\scriptstyle \rho^*} \\ T\mathcal{M} & \xrightarrow{\eta} & T^*\mathcal{M} \end{array} \qquad : \qquad \hat{\eta} \circ \rho^{-1} = \rho^* \circ \eta \ ,$$

where the map $\hat{\eta} : L \to L^*$ is induced by the DFT algebroid symmetric form. With a slight abuse of notation, we denote the bilinear form and the map it induces with the same letter, both for $\hat{\eta}$ and η. The existence of the inverse map ρ^{-1} is due to property 1 of Definition 2.1 and the relation of the anchor to the generalised bein in DFT, as reviewed in appendix A.

Finally, in analogy to three further properties of a Courant algebroid, see Prop. 4.2 and Lemmas 5.1 and 5.2 of [20], three additional properties of a DFT algebroid will prove useful in the ensuing analysis.

Lemma 2.11 *The following identities hold in a DFT algebroid:*

1. $2\langle e_1, \mathcal{D}\langle e_2, \mathcal{D}f \rangle\rangle - 2\langle e_1, [\![e_2, \mathcal{D}f]\!]\rangle = \rho(\mathcal{D}f)\langle e_1, e_2\rangle + \mathsf{SC}_\rho(e_1, e_2)f$,
2. $\mathcal{N}(e_1, e_2, \mathcal{D}f) - \frac{1}{4}\rho[\![e_1, e_2]\!] f = -\frac{1}{4}\mathsf{SC}_\rho(e_1, e_2)f$,
3. $\mathcal{N}([\![e_1, e_2]\!], e_3, e_4) + \langle \mathcal{DN}(e_1, e_2, e_3), e_4\rangle + \textit{antisymm.}(1,2,3,4) =$
   $= \frac{1}{2}\langle \mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3), e_4\rangle + \textit{antisymm.}(1,2,3,4)$,
   where antisymm.$(1,2,3,4)$ *indicates all terms needed for the antisymmetrisation of* e_1, e_2, e_3 *and* e_4.

The proof is easily obtained by direct calculation using the defining properties of a DFT algebroid. Note that in the case of a Courant algebroid the rhs of all three properties above vanishes.

3 L_∞-algebra for the DFT algebroid

Our aim in this section is to show that a DFT algebroid can be understood as a *curved* L_∞-algebra. We shall start, however, with a brief introduction to curved L_∞-algebras.

3.1 On (curved) L_∞-algebras

L_∞-algebras are generalizations of Lie algebras with infinitely-many higher brackets, related to each other by higher homotopy versions of the Jacobi identity [13,14], defined as follows.

Definition 3.1 (L_∞**-algebra [14]**) *A L_∞-algebra (L, μ_i) is a graded vector space L together with a collection of multilinear maps that are graded totally antisymmetric:*

$$\mu_i : \mathsf{L}^{\times i} \to \mathsf{L},$$

of degree $2-i$ where $i \in \mathbb{N}_0$ and satisfy the homotopy Jacobi identities:

$$\sum_{j+k=n} \sum_{\sigma} \chi(\sigma; l_1, \ldots, l_n)(-1)^k \mu_{k+1}(\mu_j(l_{\sigma(1)}, \ldots, l_{\sigma(j)}), l_{\sigma(j+1)}, \ldots, l_{\sigma(n)}) = 0 \ ;$$

for all $l_i \in \mathsf{L}$, $n \in \mathbb{N}_0$. Here $\chi(\sigma; l_1, \ldots, l_n)$ indicates the graded Koszul sign including the sign from the parity of the permutation of $\{1, \ldots, n\}$ that is ordered as: $\sigma(1) < \cdots < \sigma(j)$ and $\sigma(j+1) < \cdots < \sigma(n)$ (such permutations are also known as unshuffles).

The convention used is that totally graded antisymmetric means the following:

$$\mu_i(\ldots, l_r, l_s, \ldots) = (-1)^{|l_r||l_s|+1} \mu_i(\ldots, l_s, l_r, \ldots) \ ,$$

with $|l_r|$ the L degree of homogeneous element $l_r \in \mathsf{L}$. When $\mu_0 \neq 0$ this algebra is called a curved L_∞-algebra, while the name L_∞-algebra usually refers to the case $\mu_0 = 0$. Importantly, when $\mu_0 = 0$, the map μ_1 is a differential and the elements of the graded vector space $\mathsf{L} = \bigoplus_i \mathsf{L}_i$ form a cochain complex:

$$\cdots \xrightarrow{\mu_1} \mathsf{L}_i \xrightarrow{\mu_1} \mathsf{L}_{i+1} \xrightarrow{\mu_1} \cdots$$

Here we would like to present two examples, relevant in the context of this paper.

Example 3.2 (Curved dgla [23]) A curved differential graded Lie algebra is a triple $(\mathfrak{g}, d, R)$ where $\mathfrak{g}$ is a graded Lie algebra, d is a derivation with degree 1, and R is a curvature element of degree 2 such that $dR = 0$ and $d^2 x = [R, x]$ for all $x \in \mathfrak{g}$. In the L_∞ framework we identify the map μ_0 with the constant curvature R, μ_1 with the derivation d and μ_2 with the graded Lie bracket, satisfying the homotopy relations

$$\mu_1 \mu_0 = 0 \ ,$$
$$\mu_1(\mu_1(l)) = \mu_2(\mu_0, l) \ .$$

Example 3.3 (Courant algebroid [20, 21]) A Courant algebroid is a quadruple $(E \to M, [\cdot\,,\cdot]_C, \langle\cdot\,,\cdot\rangle_C, a)$ where E is a vector bundle of rank $2d$ over a d-dimensional manifold M with a skew-symmetric bracket defined on its sections, a symmetric bilinear and a bundle map a to TM. All these structural data satisfy a certain number of properties or compatibility conditions, see e.g. [7–9]. Alternatively, a Courant algebroid can be understood as an L_∞-algebra where structural data are encoded in maps on graded spaces, and the properties and compatibility conditions follow from the homotopy relations. We can choose the underlying graded vector space of the L_∞-algebra as either the one in [20] (ignoring the space of constants L_{-2}):

$$\begin{array}{ccc} \mathsf{L}_{-1} & \xrightarrow{D} & \mathsf{L}_0 \\ f \in C^\infty(M) & & e \in \Gamma(E) \end{array}$$

or the one proposed in [21]:

$$\begin{array}{ccccc} \mathsf{L}_{-1} & \xrightarrow{D} & \mathsf{L}_0 & \xrightarrow{a} & \mathsf{L}_1 \\ f \in C^\infty(M) & & e \in \Gamma(E) & & h \in \mathfrak{X}(M) \end{array}$$

Here, $h = h^b\partial_b$, $b = 1, \ldots, d$ and D is the derivative defined through $\langle Df, e\rangle_C = \frac{1}{2}\,a(e)f$. Maps that make this graded vector space an L_∞-algebra are given by the set (with $i \geqslant 0$):

$$\begin{aligned} \mu_1(f) &= Df \\ \mu_2(e_1, e_2) &= [e_1, e_2]_C \\ \mu_2(e, f) &= \langle e, Df\rangle_C \\ \mu_3(e_1, e_2, e_3) &= \mathcal{N}_c(e_1, e_2, e_3) \\ \hline \mu_{i+1}(h_1, \ldots, h_i, e) &= h_1^{a_1}\cdots h_i^{a_i}\partial_{a_1}\cdots\partial_{a_i}a(e)^b\partial_b \end{aligned} \tag{3.4}$$

where $\mathcal{N}_c$ is defined in analogy to (2.10) and the maps below the dashed line are the needed extension in the latter case, as shown in [21]. The extended structure is more natural in case we want to define a Courant algebroid given the L_∞-algebra, while Roytenberg and Weinstein have shown in [20] that starting from the Courant algebroid one can omit the L_1 subspace from the graded space and uniquely define the L_∞ structure. Furthermore, the properties of a Courant algebroid

$$\boxed{\begin{aligned} (a \circ D)f &= 0 \\ a[e_1, e_2]_C - [a(e_1), a(e_2)] &= 0 \\ \mathrm{Jac}(e_1, e_2, e_3) - D\mathcal{N}_c(e_1, e_2, e_3) &= 0 \end{aligned}} \tag{3.5}$$

come out in this form from the homotopy relations of the extended L_∞-algebra. Note that for the Courant algebroid $\mu_0 = 0$, therefore the map μ_1 is a differential and one can define the chain complex underlying the L_∞-algebra structure [20].

3.2 Curved L_∞-algebra for the DFT algebroid

The first proposal for an L_∞ structure relevant for DFT was given in Ref. [10] based on the graded geometry of a pre-NQ manifold. In that case, the homotopy relations of the proposed L_∞-algebra were satisfied only up to the strong constraint. Here we wish to extend this result by constructing proper, albeit curved, L_∞-algebra. We begin by defining the relevant graded vector space:

$$\begin{array}{ccccc} \mathsf{L}_{-1} & \oplus & \mathsf{L}_0 & \oplus & \mathsf{L}_2 \\ f \in C^\infty(\mathcal{M}) & & e \in \Gamma(L) & & \mu_0 \end{array}$$

where L_2 is a 1-dimensional vector space spanned by the constant element μ_0. In general, there is no chain complex underlying the graded vector space of curved L_∞-algebras, thus we use $\oplus$ symbol connecting the spaces. The maps that do not involve space L_2 are taken in analogy with the Courant algebroid maps:

$$\begin{aligned} \mu_1(f) &= \mathcal{D}f \ , \\ \mu_2(e_1, e_2) &= [\![e_1, e_2]\!] \ , \ \mu_2(e, f) = \langle e, \mathcal{D}f \rangle \ , \\ \mu_3(e_1, e_2, e_3) &= \mathcal{N}(e_1, e_2, e_3) \ . \end{aligned} \tag{3.6}$$

The maps involving L_2 are going to be constructed from the homotopy relations. Before we begin with our construction, it is useful to see which homotopy relations will be non-trivial. To this end we prove the following

Lemma 3.7 *For every $l_1, l_2 \in \mathsf{L}$ such that $\mu_i(l_1, l_2, \ldots) = 0$, the homotopy Jacobi identities of Definition 3.1 can be written in the following way:*

$$\sum_{j+k=i} \sum_{\sigma'} \chi(\sigma'; l_1, \ldots, l_i)(-1)^k \Big(\mu_{k+1}(\mu_j(l_1, l_{\sigma'(2)}, \ldots, l_{\sigma'(j)}) l_2, l_{\sigma'(j+2)}, \ldots, l_{\sigma'(i)}) + \\ + (-1)^{1 + l_1 l_2 + (l_1 - l_2) \sum_{m=2}^{j} l_{\sigma'(m)}} \mu_{k+1}(\mu_j(l_2, l_{\sigma'(2)}, \ldots, l_{\sigma'(j)}) l_1, l_{\sigma'(j+2)}, \ldots, l_{\sigma'(i)}) \Big) = 0 \ .$$

The proof of the Lemma follows from the fact that since unshuffles are ordered and l_1 and l_2 must be in different products, all unshuffles will necessarily have l_1 and l_2 for the first and $j+1$-st element or vice versa. This implies we can split the homotopy relation into two sums, those that have unshuffles that begin with l_1 and those that begin with l_2. Then it is simply a matter of connecting the graded Koszul signs of these two unshuffles.

Additionally one can observe that in our case:

$$\mu_{i+1}(\mu_0, \mu_0, \ldots) = 0, \qquad \forall i \in \mathbb{N} \ , \tag{3.8}$$

holds due to the graded antisymmetry of the maps and the fact that $|\mu_0| = 2$. Therefore, all homotopy relations of two μ_0 arguments must be trivial, which reduces the number of identities to be calculated significantly.

We proceed by constructing the maps involving the space L_2 from the homotopy relations. The homotopy identity for $n = 0$ is trivial in our case, so we move on to $n = 1$:

$$\mu_1 \mu_1(l) = \mu_2(\mu_0, l) \ .$$

This contains one non-trivial identity:

$$\mu_2(\mu_0, e) = 0 \ . \tag{3.9}$$

For $n = 2$ the homotopy identity:

$$\mu_1(\mu_2(l_1, l_2)) - \mu_2(\mu_1(l_1), l_2) - (-1)^{1+|l_1||l_2|}\mu_2(\mu_1(l_2), l_1) = -\mu_3(\mu_0, l_1, l_2) \ ,$$

contains three non-trivial cases: $(l_1, l_2) = \{(\mu_0, f), (e, f), (f_1, f_2)\}$. The first produces the condition:

$$\mu_2(\mathcal{D}f, \mu_0) = 0 \ ,$$

which is automatically satisfied by (3.9). The second and third are simply the definitions of higher brackets:

$$\begin{aligned} \mu_3(\mu_0, e, f) &= [\![e, \mathcal{D}f]\!] - \mathcal{D}\langle e, \mathcal{D}f\rangle \qquad (3.10)\\ &= -\tfrac{1}{2}\rho^{-1}\eta^{-1}(df)(\rho(e)) \ , \\ \mu_3(\mu_0, f_1, f_2) &= 2\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle \qquad (3.11)\\ &= \tfrac{1}{2}\eta^{-1}(df_1, df_2) \ . \end{aligned}$$

In the case of $n = 3$,

$$\begin{aligned} &\mu_1(\mu_3(l_1, l_2, l_3)) - \mu_2(\mu_2(l_1, l_2), l_3) + (-1)^{|l_2||l_3|}\mu_2(\mu_2(l_1, l_3), l_2) - \\ &\quad -(-1)^{|l_1|(|l_2|+|l_3|)}\mu_2(\mu_2(l_2, l_3), l_1) + \mu_3(\mu_1(l_1), l_2, l_3) - \\ &\quad -(-1)^{|l_1||l_2|}\mu_3(\mu_1(l_2), l_1, l_3) + (-1)^{|l_3|(|l_1|+|l_2|)}\mu_3(\mu_1(l_3), l_1, l_2) = \mu_4(\mu_0, l_1, l_2, l_3) \ , \end{aligned}$$

there are four different identities to be satisfied: $(l_1, l_2, l_3) = \{(\mu_0, e_1, e_2), (\mu_0, f_1, f_2), (e_1, e_2, e_3), (e_1, e_2, f)\}$. The first case is a consistency condition satisfied due to (3.9) once one takes into account (3.8). For the next case, the identity is:

$$2\mathcal{D}\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle = -\mu_3(\mu_0, \mathcal{D}f_1, f_2) - \mu_3(\mu_0, \mathcal{D}f_2, f_1) \ ,$$

which is directly satisfied by use of (3.10). For choice (e_1, e_2, e_3), the corresponding homotopy identity is a definition:

$$\mathcal{D}\mathcal{N}(e_1, e_2, e_3) - \mathrm{Jac}(e_1, e_2, e_3) = \mu_4(\mu_0, e_1, e_2, e_3) \ ,$$

or by use of (2.9):

$$\mu_4(\mu_0, e_1, e_2, e_3) = -\mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3) \ . \tag{3.12}$$

The last of the $n = 3$ expressions defines $\mu_4(\mu_0, e_1, e_2, f)$:

$$\begin{aligned} \mu_4(\mu_0, e_1, e_2, f) &= -\langle [\![e_1, e_2]\!], \mathcal{D}f\rangle - \langle e_2, \mathcal{D}\langle e_1, \mathcal{D}f\rangle\rangle + \langle e_1, \mathcal{D}\langle e_2, \mathcal{D}f\rangle\rangle + \mathcal{N}(\mathcal{D}f, e_1, e_2) \\ &= 0 \ , \qquad (3.13) \end{aligned}$$

where the second equality holds by (2.8) and the first identity of Lemma 2.11.

Next is $n=4$ with three non-trivial conditions: $(l_1,l_2,l_3,l_4)=\{(\mu_0,e_1,e_2,f),(e_1,e_2,e_3,e_4),(\mu_0,e,f_1,f_2)\}$. The first case (μ_0,e_1,e_2,f) with condition:

$$\mu_4(\mu_1(f),\mu_0,e_1,e_2)+\mu_4(\mu_2(e_1,e_2),\mu_0,f)-\mu_3(\mu_2(e_1,f),\mu_0,e_2)+ \\ +\mu_3(\mu_2(e_2,f),\mu_0,e_1)+\mu_2(\mu_3(\mu_0,e_1,f),e_2)-\mu_2(\mu_3(\mu_0,e_2,f),e_1)=0\ ,$$

produces, after plugging in (3.6) and (3.10):

$$\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,\mathcal{D}f)+\mathcal{D}\langle[\![e_1,e_2]\!],\mathcal{D}f\rangle+[\![\mathcal{D}f,[\![e_1,e_2]\!]]\!]+ \\ +\mathcal{D}\langle e_2,\mathcal{D}\langle e_1,\mathcal{D}f\rangle\rangle-\mathcal{D}\langle e_1,\mathcal{D}\langle e_2,\mathcal{D}f\rangle\rangle-[\![[\![\mathcal{D}f,e_1]\!],e_2]\!]+[\![[\![\mathcal{D}f,e_2]\!],e_1]\!]=0\ ,$$

which vanishes by use of (2.9) and (3.13). The second identity in $n=4$ is the definition:

$$\mathcal{N}([\![e_1,e_2]\!],e_3,e_4)+\langle\mathcal{D}\mathcal{N}(e_1,e_2,e_3),e_4\rangle+\text{antisymm.}(1,2,3,4)=-\mu_5(\mu_0,e_1,e_2,e_3,e_4)\ .$$

Using the third identity of Lemma 2.11, this can be rewritten as:

$$\mu_5(\mu_0,e_1,e_2,e_3,e_4)=-\tfrac{1}{2}\langle\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3),e_4\rangle+\text{antisymm.}(1,2,3,4)\ . \tag{3.14}$$

The last identity of $n=4$ is the compatibility:

$$\mu_3(\mu_2(e,f_1),\mu_0,f_2)+\mu_3(\mu_2(e,f_2),\mu_0,f_1)-\mu_2(\mu_3(\mu_0,e,f_1),f_2)- \\ -\mu_2(\mu_3(\mu_0,e,f_2),f_1)-\mu_2(\mu_3(\mu_0,f_1,f_2),e)=0\ .$$

This can easily be shown to hold using Lemma 2.11.

Moving on to $n=5$ with two non-trivial identities for: $(l_1,l_2,l_3,l_4,l_5)=\{(\mu_0,e_1,e_2,e_3,f),(\mu_0,e_1,e_2,e_3,e_4)\}$. The first is:

$$0=\tfrac{1}{2}\langle\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3),\mathcal{D}f\rangle+2\langle\mathcal{D}\mathcal{N}(e_1,e_2,e_3),\mathcal{D}f\rangle+ \\ +\Big(\tfrac{1}{2}\langle\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,\mathcal{D}f),e_3\rangle+\mathcal{N}([\![e_1,\mathcal{D}f]\!],e_2,e_3)-\mathcal{N}(\mathcal{D}\langle e_1,\mathcal{D}f\rangle,e_2,e_3)+\text{cyclic}(1,2,3)\Big)\ ,$$

where by utilising properties (2.8), (2.9) and Lemma 2.11, one obtains:

$$\tfrac{1}{6}\Big(\langle\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,\mathcal{D}f),e_3\rangle+\mathsf{SC}_\rho(e_2,e_3)(\rho(e_1)f)+\text{cyclic}(1,2,3)\Big)+ \\ +\tfrac{1}{2}\langle\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3),\mathcal{D}f\rangle=0\ .$$

This relation can be shown to be identically satisfied by direct calculation. The second and last identity of $n=5$, after plugging in all the appropriate definitions, states:

$$-\mathsf{SC}_{\mathrm{Jac}}([\![e_1,e_2]\!],e_3,e_4)-[\![e_4,\mathcal{D}\mathcal{N}(e_1,e_2,e_3)]\!]+\mathcal{D}\langle e_4,\mathcal{D}\mathcal{N}(e_1,e_2,e_3)\rangle+ \\ +[\![\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3),e_4]\!]-\tfrac{1}{2}\mathcal{D}\langle\mathsf{SC}_{\mathrm{Jac}}(e_1,e_2,e_3),e_4\rangle+\text{antisymm.}(1,2,3,4)=0\ .$$

Properties (2.9) and the third identity of Lemma 2.11 produce:

$$[\![\mathrm{Jac}(e_1,e_2,e_3),e_4]\!]-\mathrm{Jac}([\![e_1,e_2]\!],e_3,e_4)+\text{antisymm.}(1,2,3,4)=0\ ,$$

that is satisfied by direct calculation.

Finally, $n = 6$ has only one non-trivial relation $(l_1, l_2, l_3, l_4, l_5, l_6) = (\mu_0, e_1, e_2, e_3, e_4, e_5)$ where, by the definitions given above, one obtains:

$$\tfrac{1}{2}\langle \mathsf{SC}_{\text{Jac}}([\![e_1, e_2]\!], e_3, e_4), e_5\rangle - \mathcal{N}(\mathsf{SC}_{\text{Jac}}(e_1, e_2, e_3), e_4, e_5) - \\ -\tfrac{1}{2}\langle \mathcal{D}\langle \mathsf{SC}_{\text{Jac}}(e_1, e_2, e_3), e_4\rangle, e_5\rangle + \text{antisymm.}(1,2,3,4,5) = 0 \ .$$

A straightforward but lengthy and rather tedious direct calculation shows this holds.

All higher homotopy identities vanish and we summarise our findings in the following boxed set of equations.

$$\boxed{\begin{aligned} \mu_1(f) &= \mathcal{D}f \\ \mu_2(e_1, e_2) &= [\![e_1, e_2]\!] \\ \mu_2(e, f) &= \langle e, \mathcal{D}f\rangle \\ \mu_3(e_1, e_2, e_3) &= \mathcal{N}(e_1, e_2, e_3) \\ \mu_3(\mu_0, e, f) &= [\![e, \mathcal{D}f]\!] - \mathcal{D}\langle e, \mathcal{D}f\rangle \\ \mu_3(\mu_0, f_1, f_2) &= 2\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle \\ \mu_4(\mu_0, e_1, e_2, e_3) &= \mathcal{D}\mathcal{N}(e_1, e_2, e_3) - \text{Jac}(e_1, e_2, e_3) \\ \mu_5(\mu_0, e_1, e_2, e_3, e_4) &= \tfrac{1}{2}\langle \mathcal{D}\mathcal{N}(e_1, e_2, e_3), e_4\rangle - \tfrac{1}{2}\langle \text{Jac}(e_1, e_2, e_3), e_4\rangle + \\ &\quad + \text{antisymm.}(1,2,3,4) \end{aligned}} \tag{3.15}$$

All non-zero maps that include the constant element μ_0 of the space L_2 are controlled by the pairing on $T\mathcal{M}$ (2.2) and its inverse (2.4), as can be seen from (2.7) and Lemma 2.11. Here we choose to represent the space L_2 as the space spanned by the constant symmetric bivector η^{-1}.

3.3 Extending the curved L_∞-algebra for the DFT algebroid

A better understanding of the DFT algebroid that arises from the L_∞ structure can be obtained if we extend the underlying vector space by adding L_1, containing sections of $T\mathcal{M}$. In that way, the anchor map is included in the L_∞ maps, the choice of representation of L_2 as the space spanned by the constant symmetric bivector η^{-1} is natural, and the homotopy relations reproduce the defining properties of a DFT algebroid. Therefore, we shall start with the following graded vector space:

$$\begin{array}{ccccccc} \mathsf{L}_{-1} & \oplus & \mathsf{L}_0 & \oplus & \mathsf{L}_1 & \oplus & \mathsf{L}_2 \\ f \in C^\infty(\mathcal{M}) & & e \in \Gamma(L) & & h \in \mathfrak{X}(\mathcal{M}) & & \mu_0 \end{array}$$

the boxed maps (3.15) and

$$\mu_{i+1}(h_1, \ldots, h_i, e) = h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \rho(e)^B \partial_B \ , \qquad i \geqslant 0 \ , \tag{3.16}$$

a choice based on the analogous relation for Courant algebroids. Additional maps are constructed from the homotopy identities as follows.

As in the previous subsection we begin our construction with the $n = 1$ homotopy identity since the $n = 0$ case is trivial. This case has two non-trivial possibilities $l = f$ and $l = e$. The first produces:

$$\mu_2(\mu_0, f) = \mu_1(\mu_1(f)) = \rho \circ \mathcal{D}f = \tfrac{1}{2}\eta^{AB}\partial_B f \partial_A \ , \tag{3.17}$$

whereas the second:

$$\mu_1\rho(e) = \mu_2(\mu_0, e) \in \mathsf{L}_2 \ ,$$

must be trivial since L_2 is by construction spanned by the constant element μ_0 and cannot, therefore, non-trivially depend on an arbitrary section e of L. Thus the following must hold:

$$\mu_1(h) = 0 \quad \text{and} \quad \mu_2(\mu_0, e) = 0 \ . \tag{3.18}$$

It is interesting to note that, since there is an L_1 space in this extension, one can explicitly see the curving of our "differential" μ_1 on functions. The same reasoning implies that all homotopy identities in the space L_2 must be trivially satisfied:

$$\begin{aligned} \mu_i(h_1, \ldots, h_i) &= 0 \ , \\ \mu_{i+2}(h_1, \ldots, h_i, \mu_0, e) &= 0 \ . \end{aligned} \tag{3.19}$$

Using Lemma 3.7 one can show that in general we can have at most 15 non-trivial identities for each n.

Moving on to $n = 2$, we find 4 non-trivial identities, however, only three of these are different from the $\mathsf{L}_1 = \varnothing$ case above: $(l_1, l_2) = \{(e, f), (e_1, e_2), (h, f)\}$. These give, respectively:

$$\begin{aligned} \mu_2(\rho(e), f) &= 0 \ , \\ \rho[\![e_1, e_2]\!] - [\rho(e_1), \rho(e_2)] &= -\mu_3(\mu_0, e_1, e_2) \ , \\ \rho\mu_2(h, f) + \tfrac{1}{2}\eta^{BC}h^A\partial_A\partial_C f\partial_B &= -\mu_3(\mu_0, h, f) \ , \end{aligned}$$

that result in:

$$\begin{aligned} \mu_2(h, f) &= 0 \ , \\ \mu_3(\mu_0, e_1, e_2) &= \mathsf{SC}_\rho(e_1, e_2) \ , \\ \mu_3(\mu_0, h, f) &= -\tfrac{1}{2}\eta^{BC}h^A\partial_A\partial_C f\partial_B \ . \end{aligned} \tag{3.20}$$

Continuing to the $n = 3$ case, one has 8 non-trivial identities, of these only 5 are new in comparison to the previous subsection. They are: $(l_1, l_2, l_3) = \{(h, f_1, f_2), (h, e, f), (h_1, h_2, f), (e_1, e_2, h), (\mu_0, e, f)\}$ with the corresponding homotopy expressions:

$$\begin{aligned} \mu_4(\mu_0, h, f_1, f_2) &= 0 \ , \\ \mu_4(\mu_0, h, e, f) &= 0 \ , \\ \mu_3(\mathcal{D}f, h_1, h_2) &= \mu_4(\mu_0, h_1, h_2, f) \ , \\ \mu_2(h, [\![e_1, e_2]\!]) + \mu_2(\mu_2(e_1, h), e_2) + \mu_3(\rho(e_1), e_2, h) - e_1 \leftrightarrow e_2 &= \mu_4(\mu_0, e_1, e_2, h) \ , \\ \rho[\![e, \mathcal{D}f]\!] - i_{\eta^{-1}}d\langle e, \mathcal{D}f\rangle + \tfrac{1}{2}\mu_2(i_{df}\eta^{-1}, e) - & \\ -\mu_2(\langle e, \mathcal{D}f\rangle, \mu_0) - \mu_3(\rho(e), \mu_0, f) + \mu_3(\mathcal{D}f, \mu_0, e) &= 0 \ . \end{aligned}$$

The first four are definitions of higher maps:

$$\begin{aligned} \mu_4(\mu_0, h_1, h_2, f) &= \tfrac{1}{2}\eta^{BC} h_1^{A_1} h_2^{A_2} \partial_{A_1}\partial_{A_2}\partial_C f \partial_B \ , \\ \mu_4(\mu_0, e_1, e_2, h) &= -h^A \partial_A \mathsf{SC}_\rho(e_1, e_2)^B \partial_B \ , \end{aligned} \tag{3.21}$$

whereas the last is a condition satisfied by use of (2.8), and the maps defined thus far. Definitions (3.20) and (3.21) suggest, in the spirit of (3.16), the following Ansatz for the non-vanishing maps:

$$\begin{aligned} \mu_{i+2}(h_1, \ldots, h_i, \mu_0, f) &= \tfrac{1}{2}\eta^{BC} h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_C f \partial_B \ , \\ \mu_{i+3}(h_1, \ldots, h_i, \mu_0, e_1, e_2) &= h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \mathsf{SC}_\rho(e_1, e_2)^B \partial_B \ . \end{aligned} \tag{3.22}$$

Using this Ansatz one can show that all higher identities, which are infinite in number, are satisfied, see appendix B. We collect the maps for the extended L_∞-algebra corresponding to a DFT algebroid in the following list (where $i \geqslant 0$).

$$\begin{aligned} \mu_1(f) &= \mathcal{D}f \\ \mu_2(e_1, e_2) &= [\![e_1, e_2]\!] \\ \mu_2(e, f) &= \langle e, \mathcal{D}f \rangle \\ \mu_3(e_1, e_2, e_3) &= \mathcal{N}(e_1, e_2, e_3) \\ \mu_3(\mu_0, e, f) &= [\![e, \mathcal{D}f]\!] - \mathcal{D}\langle e, \mathcal{D}f \rangle \\ \mu_3(\mu_0, f_1, f_2) &= 2\langle \mathcal{D}f_1, \mathcal{D}f_2 \rangle \\ \mu_4(\mu_0, e_1, e_2, e_3) &= \mathcal{D}\mathcal{N}(e_1, e_2, e_3) - \mathrm{Jac}(e_1, e_2, e_3) \\ \mu_5(\mu_0, e_1, e_2, e_3, e_4) &= \tfrac{1}{2}\langle \mathcal{D}\mathcal{N}(e_1, e_2, e_3), e_4 \rangle - \tfrac{1}{2}\langle \mathrm{Jac}(e_1, e_2, e_3), e_4 \rangle + \\ &\quad + \text{antisymm.}(1,2,3,4) \\ \mu_{i+1}(h_1, \ldots, h_i, e) &= h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \rho(e)^B \partial_B \\ \mu_{i+2}(h_1, \ldots, h_i, \mu_0, f) &= \tfrac{1}{2}\eta^{BC} h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \partial_C f \partial_B \\ \mu_{i+3}(h_1, \ldots, h_i, \mu_0, e_1, e_2) &= h_1^{A_1} \cdots h_i^{A_i} \partial_{A_1} \cdots \partial_{A_i} \mathsf{SC}_\rho(e_1, e_2)^B \partial_B \end{aligned} \tag{3.23}$$

The homotopy relations reproduce the defining properties of a DFT algebroid as discussed in Sect. 2:

$$\boxed{\begin{aligned} (\rho \circ \mathcal{D}) f &= \tfrac{1}{2}\eta^{-1}(df) \\ \rho[\![e_1, e_2]\!]_C - [\rho(e_1), \rho(e_2)] &= -\mathsf{SC}_\rho(e_1, e_2) \\ \mathrm{Jac}(e_1, e_2, e_3) - \mathcal{D}\mathcal{N}(e_1, e_2, e_3) &= \mathsf{SC}_{\mathrm{Jac}}(e_1, e_2, e_3) \end{aligned}} \tag{3.24}$$

and their higher derivatives.

4 L_∞-morphism as the strong constraint

In order to complete the description of a DFT algebroid in terms of an L_∞-algebra, we would also like to include the strong constraint in this framework. Since we know that on the solution of the strong constraint the C-bracket of double field theory reduces to the Courant bracket, we are looking for a relation between the L_∞-algebra for a DFT algebroid and the one for a Courant algebroid. The natural relation between L_∞-algebras is an L_∞-algebra morphism or L_∞-morphism for short. In the following, we explicitly construct an L_∞-morphism from DFT to a Courant algebroid implementing the strong constraint.

4.1 On (curved) L_∞-morphisms

Before we start with the construction of mappings, we first recall the definition of an L_∞-morphism.

Definition 4.1 (**L_∞-morphism [24]**) *A collection of multilinear, totally graded antisymmetric homogeneous maps $\phi_i : \mathsf{L}^{\times i} \to \mathsf{L}'$ of degree $1-i$, $i \in \mathbb{N}_0$, is an L_∞-morphism between two L_∞-algebras (L, μ) and (L', μ') if they satisfy:*

$$\begin{aligned}
&\sum_{j+k=n} \sum_{\sigma \in \mathrm{Sh}(j;n)} (-1)^k \chi(\sigma; l_1, \ldots, l_n) \phi_{k+1}(\mu_j(l_{\sigma(1)}, \ldots, l_{\sigma(j)}), l_{\sigma(j+1)}, \ldots, l_{\sigma(n)}) = \\
&= \sum_{k_1+\cdots+k_j=n} \tfrac{1}{j!} \sum_{\sigma \in \mathrm{Sh}(k_1,\ldots,k_{j-1};n)} \chi(\sigma; l_1, \ldots, l_n) \zeta(\sigma; l_1, \ldots, l_n) \times \\
&\quad \times \mu'_j(\phi_{k_1}(l_{\sigma(1)}, \ldots, l_{\sigma(k_1)}), \ldots, \phi_{k_j}(l_{\sigma(k_1+\ldots+k_{j-1}+1)}, \ldots, l_{\sigma(n)})) \ ,
\end{aligned}$$

where $\chi(\sigma; l_1, \ldots, l_n)$ is the graded Koszul sign and $\zeta(\sigma; l_1, \ldots, l_n)$ the sign coming from the mixing of the degrees of the various maps ϕ_{k_i}, given for a $(k_1, \ldots, k_{j-1}; n)$-shuffle σ by:

$$\zeta(\sigma; l_1, \ldots, l_n) = (-1)^{\sum_{1 \leqslant r < s \leqslant j} k_r k_s + \sum_{r=1}^{j-1} k_r (j-r) + \sum_{r=2}^{j} (1-k_r) \sum_{k=1}^{k_1+\ldots+k_{r-1}} |l_{\sigma(k)}|} \ .$$

Similarly to the expression for the homotopy relations, the condition of Def. 4.1 is actually a possibly infinite series of relations, one for each $n \in \mathbb{N}_0$. Here we explicitly state the first three:

- $n = 0$

$$\phi_1(\mu_0) = \mu'_0 + \mu'_1(\phi_0) + \tfrac{1}{2!}\mu'_2(\phi_0, \phi_0) + \cdots$$

- $n = 1$

$$\phi_1(\mu_1(l)) - \phi_2(\mu_0, l) = \mu'_1(\phi_1(l)) + \mu'_2(\phi_0, \phi_1(l)) + \tfrac{1}{2!}\mu'_3(\phi_0, \phi_0, \phi_1(l)) + \cdots$$

- $n = 2$

$$\begin{aligned}
&\phi_3(\mu_0, l_1, l_2) - \phi_2(\mu_1(l_1), l_2) + (-1)^{l_1 l_2} \phi_2(\mu_1(l_2), l_1) + \phi_1(\mu_2(l_1, l_2)) = \\
&= \mu'_1(\phi_2(l_1, l_2)) + \mu'_2(\phi_0, \phi_2(l_1, l_2)) + \tfrac{1}{2!}\mu'_3(\phi_0, \phi_0, \phi_2(l_1, l_2)) + \cdots + \\
&\quad + \mu'_2(\phi_1(l_1), \phi_1(l_2)) + \mu'_3(\phi_0, \phi_1(l_1), \phi_1(l_2)) + \tfrac{1}{2!}\mu'_4(\phi_0, \phi_0, \phi_1(l_1), \phi_1(l_2)) + \cdots
\end{aligned}$$

In the case of non-vanishing ϕ_0 this is called a curved L_∞-morphism.

4.2 From a DFT algebroid to a Courant algebroid

To set the stage, we begin with L, a DFT algebroid over a doubled space $\mathcal{M}$, and L', a Courant algebroid over M, where M is a subspace of $\mathcal{M}$ and $\dim M = \dim \mathcal{M}/2$. Then introduce a mapping $\phi : \mathsf{L} \to \mathsf{L}'$ that projects the DFT algebroid to the Courant algebroid:

$$\begin{array}{lccccc}
\mathrm{DFT}: & \mathsf{L}_{-1} = C^\infty(\mathcal{M}) & \oplus & \mathsf{L}_0 = \Gamma(L) & \oplus & \mathsf{L}_2 \\
\phi \downarrow & \phi_1 \downarrow & & \phi_1 \downarrow & & \phi_1 \downarrow \\
\mathrm{CA}: & \mathsf{L}'_{-1} = C^\infty(M) & \oplus & \mathsf{L}'_0 = \Gamma(E) & \oplus & \varnothing \ .
\end{array}$$

This basically means that if we pick a coordinate chart on $\mathcal{M}$, $x^A = (x^a, \tilde{x}_a)$, such that the coordinates x^a correspond to coordinates of the manifold M and M is then implicitly defined by $\tilde{x}_a = \text{const.}$, all functions $f(x^A)$ on $\mathcal{M}$ upon restriction only depend on half the coordinates, namely $f(x^a, \tilde{x}_a = \text{const.})$. However, the fiber structure remains unchanged. To verify that such a mapping is indeed an L_∞-morphism one must check that it satisfies the conditions od Def. 4.1. We begin with $n = 0$ that implies only $\phi_1(\mu_0) = 0$ as a Courant algebroid does not include spaces L'_1 nor L'_2. Therefore $\phi_0 = 0$ and we are dealing with flat L_∞-morphism. For the case of $n = 1$ we make the following choice:

$$\phi_1(f) = \tfrac{1}{2} f\big|_M \ , \tag{4.2}$$

$$\phi_1(e) = e\big|_M \ , \tag{4.3}$$

where $e\big|_M$ means the component function is restricted and the section exists only over M. This case has only one non-trivial identity, the one corresponding to $l = f$:

$$\phi_1(\mu_1(f)) - \phi_2(\mu_0, f) = \mu'_1(\phi_1(f)) \ .$$

By plugging in the products from 3.2 and Example 3.3, this becomes:

$$\phi_1(\mathcal{D}f) - \phi_2(\mu_0, f) = D\phi_1(f) \ .$$

In $(x, \tilde{x})$ coordinates the DFT anchor splits into two: $\rho^A{}_I = (\rho^a{}_I, \tilde{\rho}_{aI})$, the first is the one we relate to the anchor $a^a{}_I$ of a Courant algebroid. This choice is consistent since for a DFT algebroid we have:

$$\rho^a{}_I \hat{\eta}^{IJ} \rho^b{}_J = 0 \ ,$$

according to property 1 of Def. 2.1 (see also (A.3)), meaning $\rho^a{}_I$ satisfies the first identity in (3.5). Therefore, the DFT derivative splits into two: $\mathcal{D} = \frac{1}{2}D + \frac{1}{2}\tilde{D}$, the first of which we associate with the Courant algebroid differential as its image is in the kernel of a. One may wonder about this extra factor of 1/2, it stems from the different definitions of the derivative and pairing in DFT and in a Courant algebroid. Whereas in DFT the derivative carries the factor, in a Courant algebroid the pairing does instead. By using (4.2) and the fact that $(\partial_a f)\big|_M = \partial_a(f\big|_M)$ and $(\tilde{\partial}^a f)\big|_M \neq \tilde{\partial}^a(f\big|_M)$, one obtains:

$$\phi_2(\mu_0, f) = \tfrac{1}{2}(\tilde{D}f)\big|_M \ . \tag{4.4}$$

The case of $n = 2$ has two non-trivial possibilities: $(l_1, l_2) = \{(f, e), (e_1, e_2)\}$. To keep calculations as simple as possible we shall make the Ansatz that all components ϕ_i for $i > 1$ not including μ_0 vanish. The first produces:

$$\phi_3(\mu_0, f, e) - \phi_1(\langle e, \mathcal{D}f\rangle) = -\langle \phi_1(e), D\phi_1(f)\rangle_C \ ,$$

that reduces to the definition:

$$\phi_3(\mu_0, f, e) = \tfrac{1}{4}\langle e, \tilde{D}f\rangle\big|_M \ . \tag{4.5}$$

The second identity is (again, after the choice of $\phi_2(e_1, e_2) = 0$):

$$\phi_3(\mu_0, e_1, e_2) + \phi_1([\![e_1, e_2]\!]) = [\phi_1(e_1), \phi_1(e_2)]_C \ .$$

Here we shall make the identification of $\hat{T}$ of DFT with the twist of the Courant algebroid, therefore trivially one has:

$$\phi_3(\mu_0, e_1, e_2) = [e_1, e_2]_C\big|_M - [\![e_1, e_2]\!]\big|_M \ . \tag{4.6}$$

In the case of $n = 3$ we have three possibilities: $(l_1, l_2, l_3) = \{(e_1, e_2, e_3), (\mu_0, f, e), (\mu_0, f_1, f_2)\}$, the first being a definition as before and the other two being consistency checks. The first produces the definition of $\phi_4(\mu_0, e_1, e_2, e_3)$:

$$\phi_4(\mu_0, e_1, e_2, e_3) = \left(\tfrac{1}{2}\mathcal{N}(e_1, e_2, e_3) - \mathcal{N}_c(e_1, e_2, e_3)\right)\big|_M \ , \tag{4.7}$$

The next case is the following consistency condition:

$$\begin{aligned} -\phi_3(\mathcal{D}f, \mu_0, e) + \phi_2(\langle e, \mathcal{D}f\rangle, \mu_0) - \phi_1([\![e, \mathcal{D}f]\!] - \mathcal{D}\langle e, \mathcal{D}f\rangle) = \\ = D\phi_3(\mu_0, f, e) - [\phi_1(e), \phi_2(\mu_0, f)]_C \ , \end{aligned}$$

which, by use of (4.5) and (4.6), transforms into:

$$D\langle e, Df\rangle_C - [e, Df]_C = 0 \ .$$

This is valid by Prop. 4.2 of [20] as all present structures correspond to a Courant algebroid. The third non-vanishing condition gives:

$$\begin{aligned} \phi_3(\mu_0, \mathcal{D}f_1, f_2) + \phi_3(\mu_0, \mathcal{D}f_2, f_1) + \phi_1(2\langle \mathcal{D}f_1, \mathcal{D}f_2\rangle) = \\ = \langle \phi_2(\mu_0, f_2), D\phi_1(f_1)\rangle_C + \langle \phi_2(\mu_0, f_1), D\phi_1(f_2)\rangle_C \ , \end{aligned}$$

that vanishes by virtue of $\langle Df_1, Df_2\rangle_C = 0$.

For $n = 4$ there are two non-trivial possibilities for the selection of elements: $(l_1, l_2, l_3, l_4) = \{(\mu_0, e_1, e_2, e_3), (\mu_0, e_1, e_2, f)\}$, both producing compatibility conditions. The former combination yields condition:

$$\begin{aligned} \phi_3([\![e_1, e_2]\!], \mu_0, e_3) + \text{cyclic(1,2,3)} + \phi_2(\mathcal{N}(e_1, e_2, e_3), \mu_0) - \phi_1(\mathsf{SC}_{\text{Jac}}(e_1, e_2, e_3)) = \\ = D\phi_4(\mu_0, e_1, e_2, e_3) - ([\phi_1(e_1), \phi_3(\mu_0, e_2, e_3)]_C + \text{cyclic(1,2,3)}) \ , \end{aligned}$$

that is satisfied by use of (2.9) and the third relation in (3.5). The condition corresponding to the latter selection of elements is:

$$\begin{aligned} &\phi_3(\langle e_2, \mathcal{D}f\rangle, \mu_0, e_1) - e_1 \leftrightarrow e_2 + \phi_3([\![e_1, e_2]\!], \mu_0, f) + \phi_4(\mathcal{D}f, \mu_0, e_1, e_2) = \\ &= \langle \phi_1(e_2), D\phi_3(\mu_0, e_1, f)\rangle_C - e_1 \leftrightarrow e_2 + \\ &\quad + \langle \phi_3(\mu_0, e_1, e_2), D\phi_1(f)\rangle_C + \mathcal{N}_c(\phi_1(e_1), \phi_1(e_2), \phi_2(\mu_0, f)) \ , \end{aligned}$$

reducing to:

$$2\mathcal{N}_c(Df, e_1, e_2)\big|_M = \langle [e_1, e_2]_C, Df\rangle_C\big|_M \ ,$$

satisfied by Lemma 5.1 of [20].

Finally, $n = 5$ has only one condition to consider, $(l_1, l_2, l_3, l_4, l_5) = (\mu_0, e_1, e_2, e_3, e_4)$:

$$\begin{aligned} &\phi_4(\mu_0, [\![e_1, e_2]\!], e_3, e_4) + \phi_3(\mu_0, \mathcal{N}(e_1, e_2, e_3), e_4) + \phi_1(\langle \mathsf{SC}_{\text{Jac}}(e_1, e_2, e_3), e_4\rangle_+) + \\ &+ \text{antisymm.(1,2,3,4)} = \\ &= \langle \phi_1(e_1), D\phi_4(\mu_0, e_2, e_3, e_4)\rangle_C + \mathcal{N}_c(\phi_1(e_1), \phi_1(e_2), \phi_3(\mu_0, e_3, e_4)) + \text{antisymm.(1,2,3,4)} \ , \end{aligned}$$

satisfied by the last identity of Lemma 2.11 and Lemma 5.2 of [20]. To summarise we present all non-vanishing morphism components in the following boxed set of equations.

$$\boxed{\begin{aligned}\phi_1(f) &= \tfrac{1}{2}f\big|_M\\ \phi_1(e) &= e\big|_M\\ \phi_2(\mu_0,f) &= \tfrac{1}{2}(\tilde{D}f)\big|_M\\ \phi_3(\mu_0,f,e) &= \tfrac{1}{4}\langle e,\tilde{D}f\rangle\big|_M\\ \phi_3(\mu_0,e_1,e_2) &= [e_1,e_2]_C\big|_M - [\![e_1,e_2]\!]\big|_M\\ \phi_4(\mu_0,e_1,e_2,e_3) &= \big(\tfrac{1}{2}\mathcal{N}(e_1,e_2,e_3)-\mathcal{N}_c(e_1,e_2,e_3)\big)\big|_M\end{aligned}}$$

We finish this section by presenting a minimal extension of the morphism above in order to encompass the algebras of subsection 3.3 and the extended Courant algebroid L_∞-algebra of [21]. To accomplish this we must make certain assumptions about this morphism. Our choice is the following:

- ϕ_1 morphism components are:

$$\phi_1 : \begin{cases} C^\infty(\mathcal{M}) \to C^\infty(M), & f \mapsto \tfrac{1}{2}f\big|_M \\ \Gamma(L) \to \Gamma(E), & e \mapsto e\big|_M \\ \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(M), & h^A\partial_A \mapsto h^a\big|_M\partial_a \end{cases},$$

- all morphism components constructed above remain unchanged,
- the only non-vanishing ϕ_i without μ_0 as an argument are ϕ_1,
- the morphism is "flat" i.e. $\phi_0 = 0$.

Details of the calculation of the morphism conditions can be found in appendix C. Here we simply state the maps that constitute an L_∞-morphism from a DFT algebroid to a Courant algebroid (both viewed as L_∞-algebras) in the following boxed set of expressions.

$$\boxed{\begin{aligned}\phi_1(f) &= \tfrac{1}{2}f\big|_M\\ \phi_1(e) &= e\big|_M\\ \phi_2(\mu_0,f) &= \tfrac{1}{2}(\tilde{D}f)\big|_M\\ \phi_3(\mu_0,f,e) &= \tfrac{1}{4}\langle e,\tilde{D}f\rangle\big|_M\\ \phi_3(\mu_0,e_1,e_2) &= [e_1,e_2]_C\big|_M - [\![e_1,e_2]\!]\big|_M\\ \phi_4(\mu_0,e_1,e_2,e_3) &= \big(\tfrac{1}{2}\mathcal{N}(e_1,e_2,e_3)-\mathcal{N}_c(e_1,e_2,e_3)\big)\big|_M\\ \hline \phi_1(h) &= h^a\big|_M\partial_a\\ \phi_{i+2}(h_1,\dots,h_i,\mu_0,e) &= \big(h_1^{A_1}\cdots h_i^{A_i}\partial_{A_1}\cdots\partial_{A_i}\rho(e)^b - h_1^{a_1}\cdots h_i^{a_i}\partial_{a_1}\cdots\partial_{a_i}a(e)^b\big)\big|_M\partial_b\end{aligned}}$$

5 Concluding remarks

A DFT algebroid is a geometric structure describing properties of the C-bracket relevant for the gauge symmetry of double field theory. Here we discussed its global properties and gave a formulation in terms of an L_∞-algebra. As indicated in [25], this is relevant because an L_∞-algebra can be extended to full, classical on-shell field theory content including dynamical fields, equations of motion, Noether identities (see e.g. [26]), by adding corresponding vector spaces. This is, of course, not surprising having in mind that an L_∞-algebra is actually the geometric structure underlying the BV-BRST complex. Furthermore, it also can accommodate the off-shell formulation of the theory given by the appropriate action functional, provided one can define a compatible, non-degenerate, graded symmetric cyclic inner product needed for defining the variational principle.

In [11], a sigma-model based on the structure of a DFT algebroid was proposed, which, however, was shown to be gauge invariant only up to the strong constraint. Although this result is in accord with standard double field theory, we would like to go one step further. Our motivation for finding a gauge invariant sigma-model without constraints is based on recent proposals suggesting that there exist physically relevant closed strings backgrounds which cannot be obtained as a solution of the strong constraint [27, 11]. Using the L_∞-algebra framework one could bootstrap consistent gauge theories by choosing the initial data of the theory in the form of 1- and 2-brackets and construct the appropriate higher brackets using the homotopy relations [28]. This is akin to the deformation of a free gauge theory into an interacting one in the BV-BRST approach, see e.g. [29].

In the language of graded geometry, an L_∞-algebra is fully defined by a Q-structure, an odd homological vector field on a graded manifold. Here, the graded vector space of a Q-manifold fibered over a point corresponds precisely to the Chevalley-Eilenberg algebra of an L_∞-algebra. The inner product of *cyclic* L_∞-algebras becomes a compatible symplectic structure on this Q-manifold, resulting in a QP-manifold describing the corresponding (classical) field theory. In this paper we gave the definition of a DFT algebroid in terms of a *curved* L_∞-algebra, implying that one should be able to formulate a DFT algebroid in terms of a Q-structure, going beyond the results of [10]. Still, the existence of a compatible symplectic structure, or cyclic inner product is an open problem we shall address in future work. A promising route to tackle this issue consists of allowing for a *degenerate* symplectic structure following the construction in [30]. In that case one obtains a presymplectic generalisation of the BV formalism, which reduces to the standard one after factorising out the zero modes of the presymplectic form. However, for most applications one can employ the presymplectic structure without ever performing factorization explicitly. Thus, one expects to obtain an unconstrained gauge invariant theory relevant for understanding the implications of T-duality in a field theory setting.

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Data availability. The data that support the findings of this study are available within the article.

A Relation of the DFT algebroid with the flux formulation of DFT

Here we review the correspondence between the structural data of a DFT algebroid and double field theory [11, 31], using a local basis. Starting from Def. 2.1, we relate the $2d$-dimensional base manifold $\mathcal{M}$ spanned by $\{X^A\}$, $A = 1, \ldots, 2d$ with the doubled configuration space of double field theory spanned by $\{x^a, \tilde{x}_a\}$, $a = 1, \ldots, d$. Next, we introduce a local basis for the sections of L, e_I where $I = 1, \ldots, 2d$, with the following operations

$$
\begin{aligned}
[\![e_I, e_J]\!] &= \hat{\eta}^{KM} \hat{T}_{IJK}\, e_M \ , & \langle e_I, e_J \rangle &= \hat{\eta}_{IJ} \ , \\
\rho(e_I) f &= \rho^A{}_I\, \partial_A f \ , & \mathcal{D} f &= \mathcal{D}^I f\, e_I = \tfrac{1}{2}\, \rho^A{}_J\, \partial_A f\, \hat{\eta}^{JI}\, e_I \ ,
\end{aligned}
\tag{A.1}
$$

where the generalised 3-form $\hat{T}$ is introduced as a twist of the bracket and $\hat{\eta}$ corresponds to the symmetric bilinear form on L, with components

$$
\hat{\eta}^{IJ} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix} , \qquad i, j = 1, \ldots, d \ . \tag{A.2}
$$

Property 1 of Def. 2.1 implies:

$$
\hat{\eta}^{IJ} \rho^A{}_I \rho^B{}_J = \eta^{AB} \ . \tag{A.3}
$$

Using the Leibniz rule, i.e., property 2 of Def. 2.1, one can show that the bracket on a general section $E^I(X) e_I \in \Gamma(L)$ is the C-bracket of double field theory:

$$
[\![E_1, E_2]\!]^J = \rho^A{}_I \left(E_1^I\, \partial_A E_2^J - \tfrac{1}{2}\, \hat{\eta}^{IJ}\, E_1^K\, \partial_A E_{2K} - E_1 \leftrightarrow E_2 \right) + \hat{\eta}^{JM} \hat{T}_{MIK}\, E_1^I\, E_2^K \ , \tag{A.4}
$$

while property 3 evaluated in a local basis imposes the antisymmetry of $\hat{T}$ in all three indices.

Relation (A.3) enables us to identify the components of the anchor map ρ with the generalised bein $\mathcal{E}^A{}_I$ of the flux formulation of DFT, see e.g. [32]:

$$
\hat{\eta}^{IJ} \mathcal{E}^A{}_I \mathcal{E}^B{}_J = \eta^{AB} \ .
$$

Moreover, properties of the bracket (2.8) and (2.9) written in a local basis produce:

$$
\begin{aligned}
2\rho^B{}_{[I} \partial_{\underline{B}} \rho^A{}_{J]} - \rho^A{}_M \hat{\eta}^{MN} \hat{T}_{NIJ} &= \eta_{BC} \rho^C{}_{[I} \partial^A \rho^B{}_{J]} \ , \\
3\hat{\eta}^{MN} \hat{T}_{M[JK} \hat{T}_{IL]N} + 4\rho^A{}_{[L} \partial_{\underline{A}} \hat{T}_{JKI]} &= \mathcal{Z}_{JKIL} \ ,
\end{aligned}
$$

where:

$$
\mathcal{Z}_{IJKL} = 3\eta_{AD}\eta_{BE}\eta^{CF} \rho^D{}_{[I} \partial_{\underline{F}} \rho^A{}_J\, \rho^E{}_K \partial_{\underline{C}} \rho^B{}_{L]} \ .
$$

By direct comparison with the expression for fluxes and their Bianchi identities in double field theory, given as (Ref. [32], Eqs.(1.2) and (1.5)):

$$
\begin{aligned}
&\mathcal{F}_{IJK} = 3\mathcal{E}_{[I}{}^{A}\partial_{\underline{A}}\mathcal{E}_{J}{}^{B}\mathcal{E}_{K]B} \ , \\
&3\hat{\eta}^{MN}\mathcal{F}_{M[JK}\mathcal{F}_{IL]N} + 4\mathcal{E}_{[L}{}^{M}\partial_{\underline{M}}\mathcal{F}_{JKI]} = 4\hat{\mathcal{Z}}_{JKIL} \ ,
\end{aligned}
$$

we observe that the twist of the bracket $\hat{T}$ can be identified with the 3-form flux $\mathcal{F}$ of double field theory and $\mathcal{Z}_{JKIL} = 4\hat{\mathcal{Z}}_{JKIL}$. The origin of the totally antisymmetric tensor $\mathcal{Z}_{IJKL}$ has been explained in [11], where we have shown that at the level of the corresponding 3d DFT sigma-model one can realise this term as a Wess-Zumino term on an extension of the membrane worldvolume to four dimensions, as in [33]. However, this distinction is not crucial in the present context, and in the rest of this paper $\mathcal{Z}_{IJKL}$ is packaged into SC_{Jac} together with the rest of the strong-constraint breaking terms appearing in the expression for the Jacobiator of the C-bracket (2.9).

B Homotopy conditions for a DFT algebroid L_∞-algebra when $n \geqslant 4$

Taking into account all of the argumentation in section 3.3, for arbitrary n one can have only 4 non-trivial distinctly new types of homotopy relations, those that are in space L_1. They will be higher orders of expressions found in the $n = 4$ case: $(l_1, \dots, l_n) = \{(h_1, \dots, h_{n-1}, f), (h_1, \dots, h_{n-2}, e_1, e_2), (\mu_0, h_1, \dots, h_{n-3}, e, f), (\mu_0, h_1, \dots, h_{n-4}, e_1, e_2, e_3)\}$, the first two being the definitions of (3.22) and the last two consistency conditions. In order we have:

$$
\mu_n(h_1, \dots, h_{n-1}, \mathcal{D}f) = (-1)^{n+1}\mu_{n+1}(\mu_0, h_1, \dots, h_{n-1}, f) \ ,
$$

from which one immediately sees the first line of (3.22), and:

$$
\begin{aligned}
(-1)^n\mu_{n-1}(\mu_2(e_1, e_2), h_1, \dots, h_{n-2}) +& \\
+\cdots -& \\
-(\mu_{n-m}(\mu_{m+1}(h_1, \dots, h_m, e_1), h_{m+1}, \dots, h_{n-2}, e_2) - e_1 \leftrightarrow e_2) +& \\
+\cdots =& \\
= (-1)^{n+1}\mu_{n+1}(\mu_0, h_1, \dots, h_{n_2}, e_1, e_2)& \ ,
\end{aligned}
$$

where the dots indicate summation over m and terms of all unshuffles σ: $h_{\sigma(1)}, \dots, h_{\sigma(m)}$ and $h_{\sigma(m+1)}, \dots, h_{\sigma(n-2)}$. This summation is nothing more than the product rule expansion of differential operator $h_1^{A_1} \cdots h_{n-2}^{A_{n-2}}\partial_{A_1} \cdots \partial_{A_{n-2}}$ acting on $[\rho(e_1), \rho(e_2)]$ implying the second line of (3.22). Continuing on now to the conditions, the third combination of elements produces:

$$\begin{aligned}(-1)^{n-1}\mu_n(\mathcal{D}f,\mu_0,h_1,\dots,h_{n-3},e)-\mu_{n-1}(\langle e,\mathcal{D}f\rangle,\mu_0,h_1,\dots,h_{n-3})+\\+\mu_{n-2}(\mu_3(\mu_0,e,f),h_1,\dots,h_{n-3})+\\+\cdots+\\+(-1)^{m+1}\mu_{n-m}(\mu_{n+1}(h_1,\dots,h_m,e),\mu_0,h_{m+1},\dots,h_{n-3},f)+\cdots+\\+(-1)^{n-m}\mu_{n-m}(\mu_{n+1}(\mu_0,h_1,\dots,h_{m-1},f),h_m,\dots,h_{m-3},e)+\cdots+\\+\cdots=0\ ,\end{aligned}$$

where, just as previously, dots indicate summation over m and term of all unshuffles of h. Again, realising this summation can be resummed produces:

$$(-1)^n h_1^{A_1}\cdots h_{n-3}^{A_{n-3}}\partial_{A_1}\cdots\partial_{A_{n-3}}(-\mathsf{SC}_\rho(e,\mathcal{D}f)^D-\tfrac{1}{4}\eta^{DC}\partial_C(\rho(e)f)+\tfrac{1}{2}\eta_{CD}\rho(e)^B\partial_B\partial_C f)\partial_D=0\ ,$$

that vanishes in the exact same way it does in the $n=3,4$ cases. The final condition valid for $n\geqslant 4$ is:

$$\begin{aligned}(-1)^n\mu_{n-1}(\mu_2(e_1,e_2),\mu_0,h_1,\dots,h_{n-4},e_3)+\text{cyclic}+\\+\mu_{n-2}(\mu_3(e_1,e_2,e_3),\mu_0,h_1,\dots,h_{n-4})+\\+\mu_{n-3}(\mu_4(\mu_0,e_1,e_2,e_3),h_1,\dots,h_{n-4})+\\+\cdots+\\+(-1)^{n-m-1}\mu_{n-m}(\mu_{m+1}(\mu_0,h_1,\dots,h_{m-2},e_1,e_2),h_{m-1},\dots,h_{n-4},e_3)+\text{cyclic}+\cdots+\\+(-1)^m\mu_{n-m}(\mu_{m+1}(h_1,\dots,h_m,e_1),\mu_0,h_{m+1},\dots,h_{n-4},e_2,e_3)+\text{cyclic}+\cdots+\\+\cdots=0\ ,\end{aligned}$$

utilising the same logic as before it is easy to see this is nothing more than a higher derivative of the condition obtained for $n=4$:

$$\begin{aligned}(-1)^n h_1^{A_1}\cdots h_{n-4}^{A_{n-4}}\partial_{A_1}\cdots\partial_{A_{n-4}}\big(-\mathsf{SC}_\rho([\![e_1,e_2]\!],e_3)-[\mathsf{SC}_\rho(e_1,e_2),\rho(e_3)]+\text{cycl.}-\\-\rho\,\mathrm{Jac}(e_1,e_2,e_3)\big)=0\ ,\end{aligned}$$

again satisfied by the fact that $\mathrm{Jac}(\rho(e_1),\rho(e_2),\rho(e_3))=0$.

C Calculation of morphism conditions with degree 1 spaces

In this appendix we show the calculation of morphism conditions in the case of DFT and Courant algebroids with spaces L_1 (respectively L_1'). In order to calculate the remaining morphism components we turn to the condition of Def. 4.1 order by order, beginning with $n=0$ that does not change with respect to section 4.2 by the introduction of space L_1. Next, $n=1$, has only one new and non-trivial condition:

$$\phi_1(\rho(e))-\phi_2(\mu_0,e)=a(\phi_1(e))\ ,$$

implying:

$$\phi_2(\mu_0, e) = 0 \; . \tag{C.1}$$

In the case of $n = 2$, by taking into account the assumptions above we have three new and non-trivial morphism conditions. The first is a compatibility condition corresponding to $(l_1, l_2) = (\mu_0, f)$:

$$\phi_1\big(\tfrac{1}{2}\eta^{-1}(df)\big) = a\big(\tfrac{1}{2}\tilde{D}f\big|_M\big) \; ,$$

that is satisfied by (4.2) and (2.3). Next, for $(l_1, l_2) = (h, e)$, the morphism condition produces:

$$\phi_3(\mu_0, h, e) + \phi_1(h^A\partial_A\rho(e)^B\partial_B) = \phi_1(h)^a\partial_a a(\phi_1(e))^b\partial_b \; ,$$

by splitting capital indices one obtains:

$$\phi_3(\mu_0, h, e) = -(\tilde{h}_a\tilde{\partial}^a\rho(e)^b)\big|_M\partial_b \; .$$

Lastly for $n = 2$ we have $(l_1, l_2) = (h, f)$ that results in:

$$\phi_3(\mu_0, h, f) = 0 \; ,$$

directly from our assumptions and Def. 4.1. Before moving on to higher cases of n, guided by what we have obtained thus far for $n = 1, 2$ we shall make the following Ansatz for the new components to ϕ:

$$\phi_{i+2}(h_1, \ldots, h_i, \mu_0, e) = \big(h_1^{A_1}\cdots h_i^{A_i}\partial_{A_1}\cdots\partial_{A_i}\rho(e)^b - h_1^{a_1}\cdots h_i^{a_i}\partial_{a_1}\cdots\partial_{a_i}a(e)^b\big)\big|_M\partial_b \; , \tag{C.2}$$

or explicitly:

$$\begin{aligned}\phi_{i+2}(h_1, \ldots, h_i, \mu_0, e) = \big(&\tilde{h}_{1a_1}\cdots\tilde{h}_{ia_i}\tilde{\partial}^{a_1}\cdots\tilde{\partial}^{a_i}\rho(e)^b + h_1^{a_1}\tilde{h}_{2a_2}\cdots\tilde{h}_{ia_i}\partial_{a_1}\tilde{\partial}^{a_2}\cdots\tilde{\partial}^{a_i}\rho(e)^b + \\ &+\cdots\big)\big|_M\partial_b \; ,\end{aligned}$$

where the dots indicate all possible combinations of h and $\tilde{h}$ except the one with no $\tilde{h}$. All other possible new ϕ components vanish. Continuing on to $n = 3$ where after taking into account our assumptions and previous Ansatz one finds there are, in fact, three new and non-trivial identities to be satisfied. The first, $(l_1, l_2, l_3) = (h_1, h_2, e)$, is just the definition (C.2), however the second $(l_1, l_2, l_3) = (\mu_0, h, f)$ and third $(l_1, l_2, l_3) = (\mu_0, e_1, e_2)$ are consistency conditions of our Ansatz and yield, respectively:

$$\begin{aligned}-\phi_3(\mathcal{D}f, \mu_0, h) - \phi_1(\tfrac{1}{2}\eta^{BC}h^A\partial_A\partial_B f\partial_C) &= -\phi_1(h)^a\partial_a(\phi_2(\mu_0, f))^b\partial_b \; , \\ -\phi_3(\rho(e_1), \mu_0, e_2) + \phi_3(\rho(e_2), \mu_0, e_1) + \phi_1(\mathsf{SC}_\rho(e_1, e_2)) &= a(\phi_3(\mu_0, e_1, e_2)) \; .\end{aligned} \tag{C.3}$$

The latter is satisfied automatically once one takes into account $\rho \circ \tilde{D} = \tilde{\partial}$, whereas for the former one needs the homomorphism property of the Courant bracket and relations (2.8). It may be useful to note that since (C.2) for $i = 0$ vanishes as is in accord with (C.1) the second relation above does not have an extra term coming from (C.2) that will appear in higher identities.

The analysis so far enables us to move on to the case of a general n. As ϕ only has infinite components for one combination of elements, $\phi_{i+2}(h_1, \ldots, h_i, \mu_0, e)$, we need only look at identities involving this component since all others are taken care of in explicit n cases either above or in section 4.2 for identities that are equivalent. That withstanding we have three possibilities, starting with $(l_1, \ldots, l_n) = (h_1, \ldots, h_{n-1}, e)$ which is simply the definition (C.2). Next, $(l_1, \ldots, l_n) = (\mu_0, h_1, \ldots, h_{n-2}, f)$, is simply the higher derivative case of the first line in (C.3) vanishing for the same reason. Finally, the generalisation of the second line or $(l_1, \ldots, l_n) = (\mu_0, h_1, \ldots, h_{n-3}, e_1, e_2)$ and the only slightly non-trivial identity for a general n. Definition 4.1 implies:

$$
\begin{aligned}
&\cdots + \\
&+ (-1)^j \phi_{n-j+1}(\mu_j(h_1, \ldots, h_{j-1}, e_1), \mu_0, h_j, \ldots, h_{n-3}, e_2) - \\
&- (-1)^j \phi_{n-j+1}(\mu_j(h_1, \ldots, h_{j-1}, e_2), \mu_0, h_j, \ldots, h_{n-3}, e_1) + \\
&+ \cdots + \\
&+ (-1)^n \phi_{n-1}(\mu_2(e_1, e_2), \mu_0, h_1, \ldots, h_{n-3}) + \phi_1(\mu_n(\mu_0, h_1, \ldots, h_{n-3}, e_1, e_2)) = \\
&= \cdots + \\
&\quad + (-1)^{n-j-1} \mu'_{n-j+1}(\phi_j(\mu_0, h_1, \ldots, h_{j-2}, e_1), \phi_1(h_1), \ldots, \phi_1(h_{n-3}), \phi_1(e_2)) - \\
&\quad - (-1)^{n-j-1} \mu'_{n-j+1}(\phi_j(\mu_0, h_1, \ldots, h_{j-2}, e_2), \phi_1(h_1), \ldots, \phi_1(h_{n-3}), \phi_1(e_1)) + \\
&\quad + \cdots + \\
&\quad + \mu'_{n-2}(\phi_3(\mu_0, e_1, e_2), \phi_1(h_1), \ldots, \phi_1(h_{n-3}))\ ,
\end{aligned}
$$

where by resumming the partial derivatives and utilising the definitions of maps ϕ_i, μ_i, and μ'_i one obtains

$$
\begin{aligned}
&h_1^{A_1} \cdots h_{n-3}^{A_{n-3}} \partial_{A_1} \cdots \partial_{A_{n-3}} ([\rho(e_1), \rho(e_2)]^b - \mathsf{SC}_\rho(e_1, e_2)^b - \rho[\![e_1, e_2]\!]^b) \partial_b = \\
&= h_1^{a_1} \cdots h_{n-3}^{a_{n-3}} \partial_{a_1} \cdots \partial_{a_{n-3}} ([a(e_1), a(e_2)]^b - a[e_1, e_2]_C^b) \partial_b\ ,
\end{aligned}
$$

with $\mathsf{SC}_\rho(e_1, e_2)^b$ defined by the splitting $\partial_B = \partial_b + \tilde{\partial}^b$. Each side of this equality vanishes on its own, the lhs because of (2.8) and the rhs because of the homomorphism property of a Courant algebroid anchor map a.

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