Fuzzy de Sitter space from kappa-Minkowski space in matrix basis

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Abstract

We consider the Lie group $\mathbb{R}^D_\kappa$ generated by the Lie algebra of κ-Minkowski space. Imposing the invariance of the metric under the pull-back of diffeomorphisms induced by right translations in the group, we show that a unique right invariant metric is associated with $\mathbb{R}^D_\kappa$. This metric coincides with the metric of de Sitter space-time. We analyze the structure of unitary representations of the group $\mathbb{R}^D_\kappa$ relevant for the realization of the non-commutative κ-Minkowski space by embedding into $(2D-1)$-dimensional Heisenberg algebra. Using a suitable set of generalized coherent states, we select the particular Hilbert space and realize the non-commutative κ-Minkowski space as an algebra of the Hilbert-Schmidt operators. We define dequantization map and fuzzy variant of the Laplace-Beltrami operator such that dequantization map relates fuzzy eigenvectors with the eigenfunctions of the Laplace-Beltrami operator on the half of de Sitter space-time.

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I. INTRODUCTION

Recent analysis [1] of the matrix model that has been proposed as a nonperturbative formulation of type IIB superstring theory [2] showed that 3+1-dimensional non-commutative κ-Minkowski space is compatible with the solution discovered by a numerical analysis in [3] and interpreted as an expanding universe. In general, such non-commutative spaces represent a realization of an old idea, proposed by W. Heisenberg and realized by H. S. Snyder [4], that space-time has a structure which in the currently available experiments manifests itself as a smooth manifold, whereas its fundamental description needs some modified notion of space-time.

In the particular model [2], as well as in similar dynamical matrix models of Yang-Mills type, the structure of space-time is described using matrix geometry [5, 6] and emergent gravity [7–10]. While matrix geometry provides a tool to identify the geometry and field content of the model, the mechanism of emergent gravity gives rise to the couplings between these fundamental degrees of freedom. Specifically, coupling of the fields to geometry is defined by derivatives following from an expansion around a classical solution. Whenever derivatives span a Lie algebra, the matrix algebra can be viewed as a quantization of the algebra of functions on a certain homogeneous space. Then, in principle, a quantization map can be defined explicitly [11], which might be particularly beneficial in an attempt to formulate quantized/non-commutative counterparts of models used in cosmology or field theory on a curved background [12–15]. Therefore, understanding of the quantization map associated to the κ-Minkowski space in terms of matrix geometry is welcome.

Although κ-Minkowski space is one of the most studied non-commutative spaces, it has mainly been studied using the algebraic formalism and the structure of Hopf algebra (for an overview see [16]). In this context, several papers [17–19] pointed out that κ-Minkowski space is related to de Sitter space of momenta. However, the geometry and representation theory of the Lie group generated by the κ-Minkowski Lie algebra have not been studied in detail (as emphasized in [20], an exception being Ref. [21]).

The purpose of this paper is to fill this gap by considering κ-Minkowski space from the matrix-geometry point of view. A lot of the results needed for such an analysis are already known in the literature. We recall these results, interpret them in the context of matrix geometry and apply them to unveil the classical geometry associated to non-commutative

κ-Minkowski space.

II. CLASSICAL GEOMETRY OF THE LIE GROUP GENERATED BY THE LIE ALGEBRA OF κ-MINKOWSKI SPACE

Aiming at understanding of the classical geometry associated to κ-Minkowski space, in this section we discuss the geometry of the Lie group generated by the Lie algebra of κ-Minkowski space. More precisely, we consider the group denoted by $\mathbb{R}^D_\kappa$ with elements

$$g(x^\mu) = g(x^0, x^i) = e^{ix^\mu \hat{x}_\mu},\ x^\mu \in \mathbb{R}, \tag{1}$$

where $\hat{x}_\mu$ are generators of D-dimensional κ-Minkowski space

$$[\hat{x}_0, \hat{x}_k] = -i\hat{x}_k,\ [\hat{x}_j, \hat{x}_k] = 0, \tag{2}$$

while we use Greek letters for the indices which run from 0 to $(D-1)$ and Latin letters for the indices which run from 1 to $(D-1)$. Group multiplication, given by

$$g(a, b^k)g(c, d^k) = g\left(a + c, \frac{\phi(a)b^k + e^a\phi(c)d^k}{\phi(a + c)}\right), \tag{3}$$

$$\phi(x) = \frac{e^x - 1}{x}, \tag{4}$$

can be derived using the isomorphism of two-dimensional κ-Minkowski space with the group of affine transformations of the straight line [22, 23].

A glimpse at the upper central (derived) series reveals that $\mathbb{R}^D_\kappa$ is a solvable Lie group. It has an abelian subgroup $\mathbb{R}^{D-1}$ generated by the $(D-1)$-dimensional ideal of the κ-Minkowski Lie algebra spanned by $\hat{x}_k$'s. Consequently, the group $\mathbb{R}^D_\kappa$ is the semidirect product $\mathbb{R} \ltimes \mathbb{R}^{D-1}$ of two subgroups and each element of the group can be written in the form

$$g = g(t, 0)g(0, y^k),\ t, y^k \in \mathbb{R}, \tag{5}$$

which can be viewed as a choice of coordinates on the group manifold. For example, the two aforementioned coordinate patches are related by

$$t = x^0,\ y^k = x^k\phi(-x^0), \tag{6}$$

but with the particular choice (5) the fiber bundle structure of the group manifold is manifest. Namely, the right translations by elements of subgroup H, i.e. multiplication of the Lie group elements by elements of the given subgroup H from the right, generate nonintersecting orbits and induce foliation of the group manifold. The particular leaf of the foliation contains all elements which generate the same orbit reflecting the equivalence relation $g \sim gh,\ h \in H,\ g \in G$. Leaves are mutually diffeomorphic and, as a result, the group manifold can be regarded as a principal bundle with H being the structure group. The base space is identified as the space G/H of left cosets gH, while the fiber is identical to the subgroup H as a homogeneous space. In the case of $\mathbb{R}^D_\kappa$, the bundle is trivial.

Having convenient coordinates at hand, we can explicitly express the basis of the cotangent space of $\mathbb{R}^D_\kappa$ at each point of the cotangent bundle, that is a frame, in terms of right invariant one-forms, i.e. one-forms invariant under the pull-back induced by the right translations $L_{g'} : g \mapsto gg'$. The basis of the space of the right invariant one-forms itself and the dual basis of the space of the right invariant vector fields are given by

$$\theta^0_R = dt,\ \theta^k_R = e^t dy^k, \qquad (7)$$
$$e^R_0 = \partial_t,\ e^R_k = e^{-t}\partial_k.$$

Efficiently, one-forms (7) can be obtained from the Maurer-Cartan form $\theta_R = (dg)g^{-1} = \theta^\mu_R \hat{x}_\mu$, a Lie algebra valued right invariant one-form, which satisfies the Maurer-Cartan equation

$$d\theta_R = \theta_R \wedge \theta_R, \qquad (8)$$

and provides an isomorphism between the right invariant vector fields and the Lie algebra of the group. Consequently, the right invariant vector fields (7) close the same Lie algebra commutation relations as the defining generators of κ-Minkowski space.

Similarly, basis of the space of the left invariant one-forms and dual basis of the space of the left invariant vector fields follow from the left invariant Maurer-Cartan form

$$\theta^0_L = dt,\ \theta^k_L = dy^k + y^k dt, \qquad (9)$$
$$e^L_0 = \partial_t - y^k\partial_k,\ e^L_k = \partial_k.$$

Since left and right translations commute, taking into account that variation of the vector field V under the diffeomorphisms generated by vector field W is given by Lie derivative $\mathcal{L}_W V = [W, V]$, it is easy to see that vector fields which generate right translations are left invariant and vice versa. For arbitrary Lie group, left and right invariant vector fields are related by the push-forward of the map $g \to g^{-1}$ which is, for the particular case of the group $\mathbb{R}^D_\kappa$ and coordinates (5), given by

$$e^{it\hat{x}_0}e^{iy^i\hat{x}_i} \to e^{-iy^i\hat{x}_i}e^{-it\hat{x}_0} = e^{-it\hat{x}_0}e^{-ie^t y^i\hat{x}_i} \Rightarrow t \to -t,\ y^i \to -e^t y^i. \tag{10}$$

Considering $\mathbb{R}^D_\kappa$ as a differentiable manifold, it can be supplemented with arbitrary metric, but taking into account the group structure it would be natural to demand certain invariance conditions. Due to the Leibnitz rule for Lie derivative, invariance on the right(left) translations is inbuilt in the tensor product of right(left) invariant one-forms and vector fields. For example, one can define right invariant or left invariant volume form ω_R and ω_L, respectively

$$\omega_R = e^{(D-1)t}dt \wedge \bigwedge_{k=1}^{D-1} dy^k,\ \omega_L = dt \wedge \bigwedge_{k=1}^{D-1} dy^k, \tag{11}$$

which reveals that $\mathbb{R}^D_\kappa$ in not unimodular. Generally, for higher rank tensors it is even possible to impose bi-invariance, i.e. left invariance and right invariance at the same time. Although bi-invariance of the the metric is natural to demand for semi-simple Lie groups which posses non-degenerate Killing form, in accordance with Cartan's criterion of solvability, Killing form related to $\mathbb{R}^D_\kappa$ is degenerate and cannot be used to define a suitable metric. Instead, only one type of invariance, right or left, of the metric can be required, which on an arbitrary Lie group does not select unique metric [24]. For example, for any regular symmetric matrix G with elements $g_{\mu\nu}$ there is right invariant metric

$$ds^2_R = g_{\mu\nu}\theta^\mu_R\theta^\nu_R, \tag{12}$$

and, in principle, for different choices of the matrix G metrics (12) are not equivalent, but in the case of $\mathbb{R}^D_\kappa$, any right invariant metric can be recast into the metric assigned to the de Sitter space written in planar coordinates

$$ds^2_R = dt^2 - e^{2t}\delta_{kl}dy^k dy^l. \tag{13}$$

Metric (13) can be viewed as the metric induced by the embedding of de Sitter space into $(D+1)$-dimensional Minkowski space

$$\begin{aligned} X^0 &= -\sinh t - e^t\delta_{kl}\frac{y^k y^l}{2}, \\ X^k &= y^k e^t, \\ X^D &= -\cosh t - e^t\delta_{kl}\frac{y^k y^l}{2}, \end{aligned} \tag{14}$$

where the constraint $X^0 - X^D > 0$ implies that coordinates $(t, y^i),\ i \in 1, \cdots, (D-1)$ cover just half of de Sitter space [25].

Transformation from (12) to (13) can be understood if we restrict discussion on the two-dimensional κ-Minkowski space for which any right invariant metric with Lorentzian signature, up to an irrelevant scale, can be written in the form

$$ds_R^2 = \frac{1}{\tau^2}\left(\cos 2\theta d\tau^2 + 2\sin 2\theta d\tau dx - \cos 2\theta dx^2\right), \qquad (15)$$
$$\theta \in < -\frac{\pi}{4}, \frac{3\pi}{4}],\ \tau = \exp(-t).$$

Then, the metric (15) can be recast into the desired form (13) by the transformation

$$\tau' = \frac{1}{\sqrt{|\cos 2\theta|}}\tau,\ t' = -\ln\tau' \qquad (16)$$
$$x' = x\sqrt{|\cos 2\theta|} - \tau\frac{\sin 2\theta}{\sqrt{|\cos 2\theta|}},\ \theta \in < -\frac{\pi}{4}, \frac{\pi}{4} >,$$
$$x' = x\sqrt{|\cos 2\theta|} + \tau\frac{\sin 2\theta}{\sqrt{|\cos 2\theta|}}, \theta \in < \frac{\pi}{4}, \frac{3\pi}{4} > .$$

In the exceptional cases $\theta = \pi/4, 3\pi/4$, the metric is flat and will not be discussed here.

In order to prove this result in the D-dimensional case one writes the metric as a symmetric matrix with elements $m_{\mu\nu}$. Choosing suitable $SO(D-1)$ rotation of y^k's, the "vector" $(m_{01}, m_{02}, \ldots, m_{0D-1})$ can be put to the form $(\tilde{m}_{01}, 0, \ldots, 0)$. Then, an analogous transformation as in the two-dimensional case followed by a $SO(D-1)$ rotation of space-like coordinates diagonalizes the metric and appropriate scaling of spacelike coordinates gives (13) up to an irrelevant scale. Besides, we note that the Euclidean variant can be treated similarly.

Finally, due to the map (10), left invariant metric

$$ds_L^2 = \left(1 - \delta_{kl}y^k y^l\right) dt^2 - 2\sum_{k=1}^{D-1} dy^k dt - \delta_{kl}dy^k dy^l, \qquad (17)$$

can be viewed as a pull-back of the right invariant metric (13) with roles of the left and right invariant fields interchanged, i.e. left invariant metric is given in terms of a tensor product of left invariant forms, while the Killing vectors coincide with the right invariant vector fields.

To summarize, for $\mathbb{R}^D_\kappa$ there is a unique right invariant metric which is equivalent to the unique left invariant metric via the pull-back of the map $g \to g^{-1}$. With this metric $\mathbb{R}^D_\kappa$ is equivalent to the half of de Sitter space-time. The form of the metric (13) indicates that $\mathbb{R}^D_\kappa$ can be viewed as a homogeneous space with respect to $SO(1, D-1)$ group.

For later purpose, recall that eigenfunctions of the invariant Laplace-Beltrami operator in planar coordinates:

$$\Box_g = \frac{1}{\sqrt{|g|}}\partial_\mu \sqrt{|g|} g^{\mu\nu} \partial_\nu, \tag{18}$$

are given in terms of Hankel functions [26]:

$$\phi^i_{\mu,\lambda_k} = e^{i\lambda_k x^k} e^{\frac{1-D}{2}t} H^{(i)}_{\sqrt{\left(\frac{D-1}{2}\right)^2 - \mu^2}}(\lambda e^{-t}), \tag{19}$$

where g is defined by (13), $i = 1, 2$ and $\lambda = \sqrt{\delta^{kl}\lambda_k\lambda_l}$, while μ^2 is the eigenvalue of $\Box_g$.

III. REPRESENTATION THEORY AND HARMONIC ANALYSIS ON THE GROUP $\mathbb{R}^D_\kappa$

We define the fuzzy κ-Minkowski space as an algebra $\mathcal{A}$ of Hilbert-Schmidt operators acting on the suitable Hilbert space supplemented with the suitable derivatives such that there exists a limit in which $\mathcal{A}$ reproduces algebra of functions on the principal homogeneous space of the group $\mathbb{R}^D_\kappa$ [11]. Assuming that a regular representation, defined as a representation on the Hilbert space of functions over principal homogeneous space, is decomposable into irreducible modules, we are interested in the unitary irreducible representations of the group $\mathbb{R}^D_\kappa$. These can be built using the method of induced representations and utilizing the aforementioned fiber bundle structure of the Lie group.

Having at hand a representation $\rho(h),\ h \in H$ of the subgroup H one considers the space $G \times \mathcal{H}$ and nonintersecting orbits $(g, \psi) \to (gh, \rho(h^{-1})\psi),\ h \in H$, with $\mathcal{H}$ being carrier of ρ. With respect to the defined action of H, the space $G \times \mathcal{H}$ splits into equivalence classes, i.e. the transitivity classes that contain points in the same orbit. As a result, the space of transitivity classes can be identified with an associated vector bundle $G/H \times \mathcal{H}$. Moreover, the equivalence of points in $G \times \mathcal{H}$ implies that any section of $G/H \times \mathcal{H}$ defines a section of $G \times \mathcal{H}$ such that $\psi(g) = \rho(h)\psi(gh),\ \forall h \in H$, that is a function $\psi(g)$ on G with values in $\mathcal{H}$ equivariant with respect to the right action of H. Then, the action of the group G on $\psi(g)$ induced by left translations gives the Mackey's representation of the group G induced by the representation ρ of the subgroup H.

The group $\mathbb{R}^D_\kappa$ is a semidirect product of a one-parameter subgroup and abelian normal subgroup, and since unitary irreducible representations of both subgroups are one-dimensional, the relevant induced representations are defined on the sections of appropriate line bundles and can be viewed as a representation on the space of complex functions on the G/H.

Unitary representations ρ^Λ_*, $\Lambda \in \mathbb{R}^{D-1}$, induced by one-dimensional unitary irreducible representations of the maximal abelian normal subgroup

$$\rho^\Lambda(e^{iy^k \hat{x}_k})|\Lambda\rangle = e^{-i\lambda_k y^k}|\Lambda\rangle, \tag{20}$$

in analogy to two-dimensional case discussed in details in [21–23, 27], are realized on the Hilbert space $L^2(\mathbb{R})$ of square integrable functions on the real line

$$\rho^\Lambda_*(\hat{x}_0) = -i\partial_Q,\ \rho^\Lambda_*(\hat{x}_k) = \lambda_k e^Q. \tag{21}$$

By the same reasoning as in [22], one concludes that for $\Lambda \in \mathbb{R}^{D-1}/\{0\}$ classes of inequivalent unitary irreducible representations are labeled by points on the unit sphere S^{D-2} in accordance with [20]. Specifically, sphere S^0 is defined as a set with two points in this context. For $\Lambda = 0$, the normal subgroup is mapped trivially to the unit and therefore induced representation can be viewed as a regular representation of the one-dimensional subgroup generated by $\hat{x}_0$. Since regular representation of any one-parameter Lie group is not irreducible, it follows that induced representation for $\Lambda = 0$ is not irreducible as well. In order to verify these results one should note that the kernel of the homomorphism (20) is isomorphic to $\mathbb{R}^{D-2}$. More precisely, it is orthogonal complement in $\mathbb{R}^{D-1}$ of the one-dimensional space generated by vector Λ and therefore the problem of construction of the irreducible representations of $\mathbb{R}^D_\kappa$ reduces to an equivalent problem for $\mathbb{R}^2_\kappa$. Representations characterized by collinear vectors Λ are equivalent [22].

Furthermore, following [22], it can be shown that unitary representation l^ω_* induced by the one-dimensional unitary irreducible representation l^ω of the one-dimensional subgroup generated by $\hat{x}_0$

$$l^\omega(e^{it\hat{x}_0})|\omega\rangle = e^{-it\omega}|\omega\rangle \tag{22}$$

contains all inequivalent irreducible unitary representations ρ^Λ_*:

$$l^\omega_* = \int^{\Lambda \in S^{D-2}} \rho^\Lambda_*, \tag{23}$$

where trivial representation is omitted from the sum. For $D = 2$, integral in (22) has to be replaced by sum over two inequivalent unitary irreducible representations. Denoting the carrier of the one-dimensional representation l^ω as $\mathcal{H}_\omega$, the described identification of points of the space $G \times \mathcal{H}_\omega$ that belong to the same orbit under the action of the one-parameter group generated by $\hat{x}_0$ selects only Mackey's functions with property

$$\psi(e^{it\hat{x}_0}e^{ix^i\hat{x}_i}) = e^{-it\omega}\psi(e^{ie^t x^i\hat{x}_i}). \tag{24}$$

Consequently, the representation induced by the left translations as the pull-back $\phi_{\tilde{g}*}\psi$ of the function ψ

$$(\phi_{\tilde{g}*}\psi)(g) = \psi(\tilde{g}^{-1}g),\ \forall \tilde{g} \in G, \tag{25}$$

on the restricted functions $\psi(e^{x^i\hat{x}_i})$ is given by

$$\begin{aligned} l^\omega_*(e^{it\hat{x}_0}e^{i\lambda^k\hat{x}_k})\psi(e^{ix^i\hat{x}_i}) &= e^{i\omega t}\psi(e^{ie^{-t}(x^i-\lambda^i)\hat{x}_i}), \\ l^\omega_*(\hat{x}_0) = \omega + ix^k\partial_k,\ l^\omega_*(\hat{x}_i) &= i\partial_i. \end{aligned} \tag{26}$$

In order to verify the decomposition (23), one introduces the Fourier transformation:

$$\mathcal{F}\psi(k_i) = \int d^{(D-1)}x^i e^{ik_l x^l}\psi(x^i), \tag{27}$$

and finds the action of the group on the dual space that follows from (26)

$$\begin{aligned} \mathcal{F}l^\omega_*(e^{it\hat{x}_0}e^{i\lambda^k\hat{x}_k})\psi(e^{ix^i\hat{x}_i}) &= \int d^{(D-1)}x^i e^{ik_l x^l}e^{i\omega t}\psi(e^{ie^{-t}(x^i-\lambda^i)\hat{x}_i}) = \\ &= e^{i\omega t}e^{ik_l\lambda^l}e^{t(D-1)}\mathcal{F}\psi(e^t k_i). \end{aligned} \tag{28}$$

After redefinition

$$\mathcal{F}\psi(k_i) = \prod_{l=1}^{D-1} k_l^{-i\omega-D+1}\phi(k_i), \tag{29}$$

the corresponding action of the group on the functions ϕ is given by

$$e^{it\hat{x}_0}e^{ix^k\hat{x}_k} \triangleright \phi(k_i) = e^{ik_l x^l}\phi(e^t k_i) = e^{i|k|\left(\frac{k_l x^l}{|k|}\right)}\phi(e^t k_i),\ |k| = \sqrt{\sum_{l=1}^{D-1} k_l^2}. \tag{30}$$

Hence, we obtained representation on the space of functions on $\mathbb{R}^{D-1}$ given by

$$l^0_{\mathcal{F}*}(\hat{x}_0) = -ik_i\tilde{\partial}^i,\ l^0_{\mathcal{F}*}(\hat{x}_i) = k_i,\ \tilde{\partial}^i = \frac{\partial}{\partial k_i}, \tag{31}$$

and denoted by $l^0_{\mathcal{F}*}$ to indicate that it is related to l^0_* by Fourier transform. Comparison of (30) with the action in the irreducible representation (21)

$$\rho^\Lambda_*(e^{it\hat{x}_0}e^{ix^k\hat{x}_k})\psi(x) = e^{i\lambda_l x^l x}\psi(e^t x),\ x = e^Q > 0, \tag{32}$$

shows that to each ray, i.e. half-line diffeomorphic to $\mathbb{R}_+$ defined by $(D-1)$-tuple $(k_i,\ |k| \neq 0)$, a unique unitary irreducible representation $(\rho^\Lambda_*, \Lambda \in S^{D-2})$ with $\lambda_i = k_i/|k|$ is assigned. Point $|k| = 0$ corresponds to a trivial representation.

Finally, for H being the trivial subgroup that contains only the unit element, the prescribed procedure results in the left regular representation L_* which can be decomposed as

$$L_* = \int_\oplus^{\omega\in\mathbb{R}} l^\omega_*. \tag{33}$$

As has already been explained at the beginning of this section, regular representation and its structure is of the main interest for our purpose and once again, in order to verify decomposition (33), one can follow [22]. Having at hand a function on principal homogeneous space, written as $\psi(g) = \psi(e^{x^0}, x^i)$, define function $\psi(te^{x^0}, tx^i)$, $t \in \mathbb{R}_+$ and its Mellin transform with respect to the t dependence:

$$\mathcal{M}\psi(te^{x^0}, tx^i) \equiv \psi^\sigma(e^{x^0}, x^i) = \int_0^\infty dt\psi(te^{x^0}, tx^i)t^{-\sigma-1}. \tag{34}$$

Those definitions ensure that the action of the group $\mathbb{R}^D_\kappa$ on (34) derived from the left regular representation coincides with the representation (26) with $\sigma = i\omega$. Accordingly, the Mellin transform (34) restricts the regular representation to particular components (26), while the inverse of the Mellin transform

$$\psi(te^{x^0}, tx^i) = \frac{1}{2\pi i}\int_{c-\infty}^{c+\infty} d\sigma\psi^\sigma(e^{x^0}, x^i)t^\sigma, \tag{35}$$

for the particular value $t = 1$, explicitly reproduces (33). The same result can be inferred by considering the expansion of the scalar field on the principal homogeneous space of the group $\mathbb{R}^D_\kappa$ into the eigenfunctions (19). There, the two roots of the eigenvalue $\mu^2 = (\pm\omega)^2$ correspond to the two-fold degeneracy $(i = 1, 2)$ of Hankel functions, while spacelike plane waves correspond to the decomposition (23).

IV. FUZZY κ-MINKOWSKI SPACE

In an attempt to construct a noncommutative variant of homogeneous spaces it is convenient, as in [11], to realize the regular representation L_* by the action of the group on the group algebra, i.e. an algebra with elements of the form

$$F[f] = \int \mu_L(g) f(g) g, \tag{36}$$

where $\mu_L(g)$ is left invariant measure, while $f(g)$ is any finite distribution with compact support over smooth functions on the group manifold. The action of the group on (36) is defined by left multiplication and the structure of the algebra is ensured by defining the multiplication of elements (36) using the multiplication in the group

$$F_1 F_2 = \int \mu_L(g) \mu_L(g') f_1(g) f_2(g') g g'. \tag{37}$$

This multiplication can be realized on the algebra of functions/distributions by replacing the usual pointwise product with the convolution

$$(f_1 \star f_2)(g) = \int \mu_L(g') f_1(g') f_2(g'^{-1} g). \tag{38}$$

Supplemented with the involution defined by modular function Δ

$$f^*(g) = \bar{f}(g^{-1}) \Delta(g^{-1}), \tag{39}$$

where bar denotes complex conjugation, the completion of the group algebra in the $L^1(G)$ norm is a Banach $*$-algebra, isomorphic to the $L^1(G)$ as a vector space.

The structure of the $\mathbb{R}^2_\kappa$-group algebra has been studied in Refs.[21, 27], where it has been shown that, although generically $f(g)$ in (36) is regarded as an element of the $L^1(\mathbb{R}^2_\kappa)$, the set $\mathcal{B} \cap L^2(\mathbb{R}^2_\kappa)$ is dense in $L^2(\mathbb{R}^2_\kappa)$, where Schwartz space $\mathcal{B} \subset L^1(\mathbb{R}^2_\kappa)$ is defined as a Fourier dual of the Schwartz space $\mathcal{S}_c$ of the functions with compact support in the time-like variable. This result, extensible to $\mathbb{R}^D_\kappa$, enabled the authors of [27] to prove that an element of the group algebra is of the Hilbert-Schmidt type if and only if the function $f(g)$ is a Fourier transform of some square integrable function. Moreover, it enabled them to define a star product of the functions on $L^2(\mathbb{R}^2_\kappa)$, as well as a quantization map which resembles Weyl quantization of phase space. However, we are not only interested in the convolution/star product of functions, but also in the construction of fuzzy variant of the Laplace-Beltrami operator (18). Therefore we need to define fuzzy variants of vector fields and, seemingly, the easiest way to achieve this is in the coherent states approach elaborated in [11]. In the following, we restrict discussion on the subset of the $\mathbb{R}^D_\kappa$-group algebra that contains only elements of the Hilbert-Schmidt type.

Having at hand a Lie group G, coherent states are defined as an overcomplete set of states that belong to an unitary irreducible representation π of the group [28]. The set of coherent states can be built by the action of the group

$$|\phi_g\rangle = \pi(g)|\phi_0\rangle, \tag{40}$$

on any admissible initial state $|\phi_0\rangle$ from the Gårding domain of representation. If there is a so-called stability subgroup H of the group G characterized by

$$\pi(h)|\phi_0\rangle = e^{i\alpha(h)}|\phi_0\rangle, \ \alpha(h) \in \mathbb{C}, \ \forall h \in H, \tag{41}$$

then states $|\phi_g\rangle$ and $|\phi_{hg}\rangle$ are equivalent. Moreover, the representation of the group has an obvious extension to the representation of the group algebra and the set of coherent states topologically coincides with G/H. This enables to assign a function on the homogeneous space G/H to any element of the group algebra (see e.g. [11] and references therein)

$$\tilde{f}(P) = \mathcal{Q}^{-1}(F) = \langle\phi_P|\pi(F)|\phi_P\rangle, \ P \in G/H, \tag{42}$$

where the state $|\phi_P\rangle$ is obtained by the action of any representative of the coset gH on the initial state $|\phi_0\rangle$. Notation $\mathcal{Q}^{-1}$ suggests that the map has to be interpreted as the inverse of the quantization map. Using the dequantization map $\mathcal{Q}^{-1}$, one can define a star product of the functions on the homogeneous space G/H

$$\mathcal{Q}^{-1}(\pi(F_1))\hat{\star}\mathcal{Q}^{-1}(\pi(F_2)) = \mathcal{Q}^{-1}(\pi(F_1F_2)). \tag{43}$$

If the representation π is faithful, i.e. if the subgroup H is trivial, then the product $\hat{\star}$ is equivalent to the convolution (38). Otherwise, the space of functions on G/H obtained by the dequantization map supplemented with $\hat{\star}$-product has to be interpreted as the representation of the convolution algebra.

To get better insight in the application of the described formalism to $\mathbb{R}^D_\kappa$ group, first we restrict discussion to $D = 2$ and then we extend results to higher dimensions incorporating appropriate modifications. Of the particular interest is the embedding of the group algebra of $\mathbb{R}^2_\kappa$ into the group algebra of the Heisenberg group.

Recall that two-dimensional κ-Minkowski space admits only two inequivalent unitary irreducible representations. Both of them can be realized on the unitary irreducible representation of the Heisenberg group by the Jordan-Schwinger map (21) [21, 29]. More precisely, the carrier of the Schröedinger representation of Heisenberg group can be viewed as a representation space of the representation $l^0_{\mathcal{F}*}$ described in previous section. We showed that the representation $l^0_{\mathcal{F}*}$ can be decomposed into the direct sum of the trivial one, assigned to the point $x = 0$ on the real line, and two inequivalent irreducible representations, defined on the space of functions non-vanishing either only on the strictly positive or on the strictly negative half-line. On the other hand, assuming an irreducible representation of the Heisenberg group, a convenient matrix basis for the Hilbert-Schmidt operators is defined in terms of states generated by a successive action of the creation operators $a^\dagger$ on the "ground" state $|0\rangle$ annihilated by a, where $[a^\dagger, a] = -1$ as usual (see for example [30] and references therein). Therefore, due to the aforementioned decomposition, elements of the $\mathbb{R}^2_\kappa$-group algebra, considered as infinite-dimensional matrices embedded into the group algebra of the Heisenberg group, split into the two infinite-dimensional blocks. Choosing an initial normalizable state $|\phi_0\rangle$ from the carrier of an non-trivial irreducible component, we define the collection of states

$$|t, y^1\rangle = e^{it\hat{x}_0} e^{iy^1\hat{x}_1} |\phi_0\rangle. \tag{44}$$

In the case of the Heisenberg group, it is convenient to specify a unique initial state by imposing holomorphicity, but for κ-Minkowski space such a condition is not suitable as has been discussed in [31, 32] where an acceptable choice is suggested. Since the collection of states (44) is defined with respect to an irreducible component of the representation $l^0_{\mathcal{F}*}$, it should be clear that states (44) represent an overcomplete set of states with respect to the half of the representation space of $l^0_{\mathcal{F}*}$. Furthermore, it is well known that Schröedinger representation does not admit normalizable eigenstate of any element of the Lie algebra of the Heisenberg group. Similarly, in a unitary irreducible representation of $\mathbb{R}^2_\kappa$ there is no normalizable eigenstate of any non-trivial element of the Lie algebra generators of $\mathbb{R}^2_\kappa$ as well. Therefore, the stability group related to the collection of states (44) is trivial, thus implying that the representation of the group $\mathbb{R}^2_\kappa$ on the space spanned by the collection of states (44) is faithful.

Bearing in mind results obtained for $D = 2$, in higher dimensional case we define coherent states with respect to the representation $l^0_{\mathcal{F}*}$. Therefore, assuming the representation $l^0_{\mathcal{F}*}$, the representation space is the space of functions on $\mathbb{R}^{D-1}$. It can be viewed as a tensor product of $(D-1)$ copies of the representation space of the Schröedinger representation of Heisenberg group, where each copy admits the representation of $\mathbb{R}^2_\kappa$ group decomposable into two non-equivalent unitary irreducible components. In analogy to $\mathbb{R}^2_\kappa$, we select one of these non-equivalent representations for each component in the tensor product. For the sake of definition, in each component we select space of functions non-vanishing for strictly positive half-line. Thus we obtain the representation on the space of functions non-vanishing only on the subset $x_i > 0,\ i = 1, \cdots, D-1$ of $\mathbb{R}^{D-1}$. Furthermore, assuming a suitable initial state $|\phi_0\rangle$, we define a collection of states by the action of the group $\mathbb{R}^D_\kappa$

$$|t, y^1, \cdots, y^{D-1}\rangle = e^{it\hat{x}_0} e^{iy^i\hat{x}_i}|\phi_0\rangle. \tag{45}$$

In order to show that the collection of states (45) represents an overcomplete set of states with respect to the selected representation space, we define the operator

$$B = \int \mu_L(t, y^1, \cdots, y^{D-1})|t, y^1, \cdots, y^{D-1}\rangle\langle t, y^1, \cdots, y^{D-1}|, \tag{46}$$

which commutes with all operators $l^0_{\mathcal{F}*}(g)$. By virtue of (30) and (31), $l^0_{\mathcal{F}*}$ is decomposable into irreducible components

$$l^0_{\mathcal{F}*} = \int^{\Lambda \in S^{D-2}} \rho^\Lambda_*, \tag{47}$$

and it follows that B is non-vanishing only on the portion $(1/2)^{D-1}$ of the S^{D-2} sphere in (47). Moreover, if we impose the condition $\phi_0(x_i) = \phi_0(|x|)$ on the function that corresponds to the initial state $|\phi_0\rangle$, then the operator B is proportional to unit on the portion of domain on which it does not vanish, thus confirming that the collection of states (45) is an overcomplete set of states. As a final remark on the properties of the representation constructed by embedding of $\mathbb{R}^D_\kappa$-group algebra into the group algebra of Heisenberg group, we point out that the representation on the Hilbert space spanned by the collection of states (45) is faithful.

Finally, with the overcomplete collection of states which span the faithful representation at hand, we define an isomorphism from the group algebra to the space of functions on $\mathbb{R}^D_\kappa$

$$\mathcal{Q}^{-1}(F) = \langle\phi_0|e^{-iy^i\hat{x}_i} e^{-it\hat{x}_0} F e^{it\hat{x}_0} e^{iy^i\hat{x}_i}|\phi_0\rangle. \tag{48}$$

Supplemented with the product

$$\mathcal{Q}^{-1}(F_1)\hat{\star}\mathcal{Q}^{-1}(F_2) = \mathcal{Q}^{-1}(F_1F_2), \tag{49}$$

the space of functions on $\mathbb{R}^D_\kappa$ has a structure of an algebra isomorphic to the convolution algebra. Isomorphism depends on the choice of the initial state, as is evident from

$$\mathcal{Q}^{-1}(F) = \int \mu_L(g)f(g)\langle\phi_0|e^{-iy^i\hat{x}_i}e^{-it\hat{x}_0}ge^{it\hat{x}_0}e^{iy^i\hat{x}_i}|\phi_0\rangle. \tag{50}$$

Furthermore, we define derivatives as commutators with the Lie algebra generators, thus ensuring Leibnitz rule. Taking into account the explicit form of the Laplace-Beltrami operator (18)

$$(\partial_t^2 + (D-1)\partial_t + \mu^2 - e^{2t}\delta^{ij}\partial_i\partial_j)f(t,y^i) = 0 \tag{51}$$

and the identities

$$\begin{aligned} i\partial_t\mathcal{Q}^{-1}(F)(t,x^1,\cdots,x^{D-1}) &= \mathcal{Q}^{-1}([\hat{x}_0,F])(t,x^1,\cdots,x^{D-1}),\\ ie^{-t}\partial_j\mathcal{Q}^{-1}(F)(t,x^1,\cdots,x^{D-1}) &= \mathcal{Q}^{-1}([\hat{x}_0,F])(t,x^1,\cdots,x^{D-1}), \end{aligned} \tag{52}$$

easily derived using (48) and (2), we define fuzzy the Laplace-Beltrami operator as

$$\hat{\Box}_g \equiv -[\hat{x}_0,[\hat{x}_0,\ ]] - i(D-1)[\hat{x}_0,\ ] + \mu^2 + \delta^{ij}[\hat{x}_i,[\hat{x}_j,\ ]] = 0. \tag{53}$$

It follows that the map $\mathcal{Q}^{-1}$ assigns to an eigenvector of fuzzy Laplacian $\hat{\Box}_g$ an eigenfunction of the classical Laplace-Beltrami operator $\Box_g$.

V. CONCLUSION

We considered the Lie group $\mathbb{R}^D_\kappa$ generated by the Lie algebra of κ-Minkowski space. Imposing the invariance of the metric under the pull-back of diffeomorphisms induced by right translations in the group $\mathbb{R}^D_\kappa$, we found that a unique right invariant metric is associated with $\mathbb{R}^D_\kappa$. This metric coincides with the metric of de Sitter space-time in planar coordinates. Consequently, principal homogeneous space of the group $\mathbb{R}^D_\kappa$ endowed with right invariant metric coincides with the half of de Sitter space-time. The automorphism $g \to g^{-1}$ of the group $\mathbb{R}^D_\kappa$ implies the same conclusion for the left invariant metric.

Furthermore, we presented an analysis of the structure of unitary representations of the Lie group $\mathbb{R}^D_\kappa$ which are relevant for the formulation of the non-commutative κ-Minkowski space in matrix basis. From the practical point of view, a suitable matrix basis can be selected using an embedding of the $\mathbb{R}^D_\kappa$-group algebra into the group algebra of $(2D-1)$-dimensional Heisenberg group while faithfully represented on the space $L^2(\mathbb{R}^{D-1})$, as suggested in [29].

An insight in the particular properties of the embedding resulted in the construction of the specific collection of states and an associated operator B dependent on the choice of an initial state $|\phi_0\rangle$. With a suitable conditions imposed on the initial state, the operator B can be interpreted as a projector from the space $L^2(\mathbb{R}^{D-1})$ to the subspace $L^2(\mathbb{R}^{D-1}_+)$ for which the constructed collection of states realizes an overcomplete set of states. Space $L^2(\mathbb{R}^{D-1}_+)$ carries a reducible representation of the group $\mathbb{R}^{D-1}_\kappa$ and therefore the constructed collection of states can be considered as a generalization of the coherent states defined in [28].

Using the constructed overcomplete set of states and the fact that aforementioned reducible representation of the group $\mathbb{R}^{D-1}_\kappa$ on $L^2(\mathbb{R}^{D-1}_+)$ is faithful, we defined the dequantization map, that is an isomorphism from the space of Hilbert-Schmidt operators acting on $L^2(\mathbb{R}^{D-1}_+)$ onto to space of functions $L^2(\mathbb{R}^{D-1})$. Although the dequantization map itself depends on the choice of initial state $|\phi_0\rangle$, a star product naturally induced by dequantization map is equivalent to convolution.

Finally, we defined the fuzzy variant of the Laplace-Beltrami operator and we showed that dequantization map relates the fuzzy eigenfunctions with the eigenfunctions that correspond to the same eigenvalue of the classical Laplace-Beltrami operator on de Sitter space-time. In order to define the dequantization map more explicitly it remains to solve the fuzzy eigenvalue equations or to choose aforementioned initial state in some particular way which will be discussed elsewhere together with the analysis of the semi-classical limit and an action for the field theory on the fuzzy κ-Minkowski space.

Comparing the presented approach with the similar approaches to the construction of non-commutative spheres and their pseudo-Riemaniann counterparts [33–42], one should note that the presented approach determines a map only onto the half of de Sitter space.

Finally, apart from providing an insight in the structure of the non-commutative κ-Minkowski space from the matrix geometry point of view the presented construction might be implemented into the group theory approach to quantum field theory on de Sitter space-time [43, 44].

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