Published for SISSA by Springer

Received: February 8, 2019
Accepted: April 22, 2019
Published: May 9, 2019

Fluxes in exceptional field theory and threebrane sigma-models

Athanasios Chatzistavrakidis,[a,b] **Larisa Jonke,**[a,b] **Dieter Lüst**[b,c] **and Richard J. Szabo**[d,e,f]

[a] *Division of Theoretical Physics, Ruđer Bošković Institute, Bijenička 54, 10000 Zagreb, Croatia*
[b] *Arnold Sommerfeld Center for Theoretical Physics, Department für Physik, Ludwig-Maximilians-Universität München, Theresienstraße 37, 80333 München, Germany*
[c] *Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, Föhringer Ring 6, 80805 München, Germany*
[d] *Department of Mathematics, Heriot-Watt University, Colin Maclaurin Building, Riccarton, Edinburgh EH14 4AS, U.K.*
[e] *Maxwell Institute for Mathematical Sciences, Edinburgh, U.K.*
[f] *The Higgs Centre for Theoretical Physics, Edinburgh, U.K.*

E-mail: Athanasios.Chatzistavrakidis@irb.hr, larisa@irb.hr, dieter.luest@lmu.de, R.J.Szabo@hw.ac.uk

Abstract: Starting from a higher Courant bracket associated to exceptional generalized geometry, we provide a systematic derivation of all types of fluxes and their Bianchi identities for four-dimensional compactifications of M-theory. We show that these fluxes may be understood as generalized Wess-Zumino terms in certain topological threebrane sigma-models of AKSZ-type, which relates them to the higher structure of a Lie algebroid up to homotopy. This includes geometric compactifications of M-theory with G-flux and on twisted tori, and also its compactifications with non-geometric Q- and R-fluxes in specific representations of the U-duality group SL(5) in exceptional field theory.

Keywords: Flux compactifications, M-Theory, Differential and Algebraic Geometry, Sigma Models

ArXiv ePrint: 1901.07775

https://doi.org/10.1007/JHEP05(2019)055

JHEP05(2019)055

Contents

JHEP05(2019)055

1 Introduction

A general string or M-theory background may carry non-vanishing vacuum expectation values for some of its NS-NS or RR field strengths, commonly known as background fluxes which are widely used in all modern attempts to relate string theory to low-energy phenomenology [1]. Apart from the standard geometric settings, where one may also include the possibility of non-vanishing torsion, the T- and U-dualities of closed string and M-brane theories reveal the existence of exotic fluxes that cannot be described in the context of standard geometry. These are commonly known as non-geometric fluxes; see [2] for a recent review in the context of string theory with a complete list of references, and also [3] for a review of some of the mathematical features in the setting of the present paper.

In the context of M-theory, the full set of fluxes for its seven-dimensional compactifications[1] was determined in [4] using SL(5) exceptional field theory [5], and studied further also for dimensions up to seven in [6–10]. The latter is the M-theory analogue of double field theory [11–13], in that both theories are proposals for a duality-invariant formulation, be it T-duality in the string theory case or U-duality in the M-theory case. This exceptional field

[1]For clarity, the seven dimensions here refer to the external spacetime.

theory is related by construction to a generalized geometry on the tangent bundle extended by 2-forms [14, 15]. This bundle can be equipped with a bracket [16–20], the higher analogue of the Courant bracket defined in [21] whose properties are collected in the structure of a Courant algebroid [22]. In string theory, the Courant bracket was used to systematically determine the general expressions for the full set of geometric and non-geometric fluxes together with their Bianchi identities [23, 24]; indeed, one can subsequently show that these expressions coincide with the local form of the axioms of a Courant algebroid.

On the other hand, the axioms of a Courant algebroid also coincide with the conditions for gauge (or BRST) invariance and on-shell closure of the algebra of gauge transformations for a first-order action functional for Wess-Zumino terms in three dimensions, called the Courant sigma-model [25–29]. This may be neatly phrased in the language of the BV field-antifield formalism [30, 31], where the structure of a general gauge theory is encoded in the master equation: the axioms of a Courant algebroid are equivalent to the classical master equation of the Courant sigma-model. The Courant sigma-model falls in the general class of topological sigma-models constructed geometrically in [32], called AKSZ sigma-models. The utility of membrane sigma-models as a fundamental microscopic description of closed strings in non-geometric flux backgrounds was originally suggested by [23], and further elucidated in [33–36].

The upshot is that the general expressions for the fluxes and Bianchi identities of a generic string compactification are in direct correspondence to the axioms of a Courant algebroid and to the generalized Wess-Zumino terms of Courant sigma-models. One may then wonder whether this "triple point" also exists for M-theory compactifications. The main purpose of this paper is to investigate this problem for the SL(5) case of M-theory flux compactifications to seven dimensions.

Our approach comprises two steps. In a first step, we consider the higher Courant bracket on the extended bundle $E_2 = TM \oplus \bigwedge^2 T^*M$ and identify its possible twists, which in the physical case where $\dim M = 4$ turn out to be

$$G_{ijkl}\,, \quad F_{ij}{}^{k}\,, \quad Q_i{}^{jkl} \qquad \text{and} \qquad \mathcal{R}^{i,jklm}\,. \tag{1.1}$$

These are the higher counterparts of the more familiar set of NS-NS fluxes H_{ijk}, $F_{ij}{}^{k}$, $Q_i{}^{jk}$ and R^{ijk} encountered in string compactifications. The first two are associated to geometric compactifications of M-theory, while the last two are non-geometric fluxes [37] sourced by exotic branes [38]. In particular, the last entry is the higher analogue of the locally non-geometric 3-vector R-flux, and presently it necessarily has the structure of a mixed-symmetry 5-vector of type $(1,4)$, in accord with [4]. Using the higher Courant bracket on E_2 we determine the general expressions for the fluxes (1.1) and their corresponding Bianchi identities in terms of a vielbein, a 3-form and a 3-vector.

In a second step, we employ the construction of topological field theories of AKSZ-type in four worldvolume dimensions (an open threebrane), studied in [39, 40]. The classical master equation for the corresponding threebrane sigma-model yields a set of conditions, which are then used to define the higher structure of a Lie algebroid up to homotopy [39] — see also [41] for a somewhat more general construction. Such threebrane sigma-models were already used in [44] to relate AKSZ theories of M2-branes in topological M-theory to exceptional generalized geometry and fluxes. Here we show that by choosing the extended bundle E_2 in which the fields of the sigma-model take values, accompanied by a projection to SL(5) tensors, the local coordinate expressions for the axioms of a specific Lie algebroid up to homotopy on E_2 reproduce the expressions for the M-theory fluxes and their Bianchi identities for compactifications to seven dimensions. Therefore we conclude that the same set of equations underlies the SL(5) M-theory fluxes and their Bianchi identities, the gauge structure of a topological sigma-model in four worldvolume dimensions, and the axioms of a specific Lie algebroid up to homotopy.

This paper is organized as follows. In section 2 we briefly recall the necessary background material on the higher Courant bracket for extended bundles of the type $E_p = TM \oplus \bigwedge^p T^*M$, and the construction of a generalized metric for the physically relevant case $p = 2$ which accounts for SL(5) exceptional generalized geometry. In section 3 we use the higher Courant bracket to determine the general expressions for the fluxes and their Bianchi identities, and explain how in the case $\dim M = 4$ they indeed yield the correct SL(5) tensor structures. In section 4, topological sigma-models on an open threebrane are discussed, together with their relation to the structure of a Lie algebroid up to homotopy; we further present some simple examples and their relation to M-theory backgrounds. In section 5 the relation between the general M-theory fluxes and generalized Wess-Zumino terms of the topological sigma-models is established. Section 6 contains a first step towards the generalization of our results to the extended geometry of exceptional field theory; here the dimensionality of the target space is extended from four to ten and the general expressions for the SL(5) exceptional field theory fluxes up to the section condition are determined, though we have not yet found the appropriate extension of our threebrane sigma-model whose Wess-Zumino terms support the exceptional field theory fluxes. Finally, section 7 contains our conclusions and an outlook toward some open problems related to our work.

2 Higher Courant algebroids

2.1 Higher Courant brackets

We begin with a brief description of higher analogues of Courant algebroids (see for example [16–20]), and their uses in defining a generalized metric associated to exceptional group structures on their underlying vector bundle [14] which extends the corresponding construction in generalized complex geometry [45, 46].

The starting point is a d-dimensional manifold M and a vector bundle $E_p \to M$, which is the extension of its tangent bundle by p-forms,

$$E_p = TM \oplus \textstyle\bigwedge^p T^*M \,, \tag{2.1}$$

where p is a non-negative integer. For $p = 1$ this is simply the vector bundle corresponding to a splitting of the exact sequence

$$0 \longrightarrow T^*M \xrightarrow{\rho^\top} E_1 \xrightarrow{\rho} TM \longrightarrow 0 \tag{2.2}$$

which gives rise to an exact Courant algebroid, where $\rho^\top : T^*M \to E_1$ denotes the transpose of the anchor map ρ. For any $p \geq 1$, sections of the vector bundle E_p correspond to a formal sum of vectors and p-forms,

$$\Gamma(E_p) \ni A = X + \eta \qquad \text{with} \quad X \in \Gamma(TM)\,,\ \eta \in \Gamma(\textstyle\bigwedge^p T^*M)\ . \tag{2.3}$$

The vector bundle E_p can be endowed with a non-degenerate symmetric fiber pairing

$$\langle\,\cdot\,,\,\cdot\,\rangle\,:\,E_p \times E_p \longrightarrow \textstyle\bigwedge^{p-1} T^*M\,, \tag{2.4}$$

which is given in terms of a symmetrization of contractions between vectors and p-forms,

$$\langle X+\eta, Y+\xi\rangle = \tfrac{1}{2}\,(\iota_X\xi + \iota_Y\eta)\,, \tag{2.5}$$

resulting in a $(p-1)$-form. In the special case $p = 1$, this is a map to $C^\infty(M)$ and it defines a metric with split signature (d, d), which is invariant under the continuous T-duality group $O(d, d)$.

A binary operation on sections of E_p can also be defined, either in terms of a higher Dorfman bracket, which we denote here as a circle product

$$(X+\eta)\circ(Y+\xi) = [X,Y] + \mathcal{L}_X\xi - \iota_Y \mathrm{d}\eta\,, \tag{2.6}$$

or in terms of its antisymmetrization, a higher Courant bracket

$$[X+\eta, Y+\xi] = [X,Y] + \mathcal{L}_X\xi - \mathcal{L}_Y\eta - \tfrac{1}{2}\,\mathrm{d}(\iota_X\xi - \iota_Y\eta)\ . \tag{2.7}$$

Here $\mathcal{L}_X$ denotes the Lie derivative along the vector field X. For $p = 1$ these are precisely the standard Dorfman and Courant brackets, and they are formally given in terms of the same expressions for higher p.

We further define a smooth bundle map $\rho : E_p \to TM$, which could be for instance the projection to the tangent bundle, such that the quadruple $(E_p, \langle\,\cdot\,,\,\cdot\,\rangle, [\,\cdot\,,\,\cdot\,], \rho)$ satisfies the following properties:

- Modified Jacobi identity:

$$[[A,B],C] + \mathrm{cyclic}(A,B,C) = \mathrm{d}\,\mathcal{N}(A,B,C)\ ; \tag{2.8}$$

- Homomorphism property:

$$\rho[A,B] = [\rho(A),\rho(B)]\ ; \tag{2.9}$$

- Modified Leibniz rule:

$$[A, f\,B] = f\,[A,B] + \big(\rho(A)f\big)\,B - \mathrm{d}f \wedge \langle A,B\rangle\ ; \tag{2.10}$$

- Compatibility condition:

$$\mathcal{L}_{\rho(C)}\langle A,B\rangle = \langle [C,A] + \mathrm{d}\langle C,A\rangle, B\rangle + \langle A, [C,B] + \mathrm{d}\langle C,B\rangle\rangle\,, \tag{2.11}$$

where the Nijenhuis operator is defined by

$$\mathcal{N}(A,B,C) = \frac{1}{3}\langle [A,B],C\rangle + \text{cyclic}(A,B,C)\,, \tag{2.12}$$

for any $A, B, C \in \Gamma(E_p)$ and $f \in C^\infty(M)$. The last condition, expressing the compatibility between the pairing and the bracket, is somewhat modified as compared to the $p = 1$ case, in the sense that a Lie derivative appears on the left-hand side. Evidently, this condition reduces to the standard one for a Courant algebroid in the $p = 1$ case.

When the anchor map ρ is the projection to TM, the higher Courant bracket may be twisted by a closed $(p+2)$-form H as

$$[X+\eta, Y+\xi]_H = [X+\eta, Y+\xi] + \iota_X\iota_Y H\ . \tag{2.13}$$

The twist yields an additional p-form term in the bracket. In string theory, where the relevant structure is $p = 1$, the twist is a closed 3-form; it is customarily identified with the NS-NS flux whose de Rham cohomology class is the Ševera class which classifies the exact Courant algebroids over M. In that case, one can also consider more general twists, usually denoted as (H, f, Q, R) corresponding to a 3-form H, a vector-valued 2-form f, a bivector-valued 1-form Q and a trivector R. The relevant bracket is then the Courant-Roytenberg bracket and the underlying structure corresponds to a general Courant algebroid where the anchor is not simply the projection to the tangent bundle, or equivalently to a proto-bialgebroid [34, 47, 48]. However, for $p > 1$, the twisting is more restricted. For instance, the analogue of a twisted Poisson structure (which would be a twisted Nambu-Poisson structure) does not exist [20]. Nevertheless, one can directly infer from the entries of the bracket what are the five possible additional types of twists that could in principle be considered apart from a $(p+2)$-form: a vector-valued 2-form, a $(p+1)$-vector-valued 1-form, a p-vector-valued $(p+1)$-form, a $2p$-vector-valued p-form, and a $(2p+1)$-vector. The precise structure of such twists will be explained in the ensuing sections.

For our purpose of investigating the flux content of M-theory for four-dimensional compactification manifolds, giving rise to dimensional reductions to seven dimensions, the relevant higher structure is simply the one with $p = 2$ which corresponds to extending string charges for $p = 1$ to M2-brane charges [14]. We shall also find it necessary to formulate these higher Courant algebroids more generally later on by replacing the tangent bundle TM and the cotangent bundle T^*M by a Lie algebroid E over M and its dual E^*. In order to investigate cases with $d > 4$ the above setting is not sufficient and additional ingredients, in the form of extended bundles, are necessary to account for the charges of higher-dimensional objects such as M5-branes and MKK6-monopoles [14]. Therefore, we mostly focus on the special case of $p = 2$ and $d = 4$, although some of our results will be valid for any compactification dimension d as we indicate.

2.2 SL(5) exceptional generalized geometry

Let us recall that in the more familiar $p = 1$ case, a generalized metric on E_1 that combines a Riemannian metric g and a 2-form B on TM may be defined [46]. One way to do this is to start from the $O(d,d)$-structure that preserves the fiber pairing of the Courant algebroid.

The $O(d,d)$ T-duality transformations split into three types. The first corresponds to smooth bundle automorphisms $f : TM \to TM$ and their inverse transpose $f^{-\top} : T^*M \to T^*M$, acting on a section of $E_1 = TM \oplus T^*M$ as $X + \eta \mapsto f(X) + f^{-\top}(\eta)$, or in $\mathrm{GL}(d) \subset O(d,d)$ matrix form

$$F \begin{pmatrix} X \\ \eta \end{pmatrix} = \begin{pmatrix} f & 0 \\ 0 & f^{-\top} \end{pmatrix} \begin{pmatrix} X \\ \eta \end{pmatrix} . \tag{2.14}$$

The second set of transformations are the B-transforms acting as $X+\eta \mapsto X+\eta+\iota_X B$, or

$$e^B \begin{pmatrix} X \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} X \\ \eta \end{pmatrix} . \tag{2.15}$$

The third type are the β-transforms which act via a bivector β on T^*M as $X + \eta \mapsto X + \eta + \iota_\eta \beta$, or

$$e^\beta \begin{pmatrix} X \\ \eta \end{pmatrix} = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ \eta \end{pmatrix} . \tag{2.16}$$

The geometric subgroup of these $O(d,d)$-transformations preserving the Courant bracket is $\mathrm{GL}(d) \ltimes \Omega^2_{\mathrm{cl}}$, the semi-direct product of diffeomorphisms of M with closed 2-form transformations which act as bundle automorphisms (2.15) preserving the pairing (2.5).

Then an $O(d,d)$-covariant generalized metric $\mathcal{H}_1$ may be parametrized in terms of g and B as

$$\mathcal{H}_1 = \begin{pmatrix} g - B\, g^{-1}\, B & -B\, g^{-1} \\ g^{-1}\, B & g^{-1} \end{pmatrix} , \tag{2.17}$$

which is a B-transform of the induced Riemannian metric $g \oplus g^{-1}$ on $TM \oplus T^*M$. This is not the most general parametrization of $\mathcal{H}_1$, since $O(d,d)$-transformations generate fractional linear transformations of $g + B$, while field redefinitions allow for instance an expression of the generalized metric in terms of a metric $\tilde{g}$ and a bivector β on T^*M [49]. The generalized metric for $p = 1$ yields a reduction of the structure group $O(d,d)$ of E_1 to its maximal compact subgroup $O(d) \times O(d)$, and thus the moduli space of such reductions is the d^2-dimensional coset $O(d,d)/O(d) \times O(d)$,

A similar prescription was followed in [14] to define a generalized metric for $p = 2$ and $d = 4$. In that case, the group of transformations acting on $E_2 = TM \oplus \bigwedge^2 T^*M$ is the U-duality group $\mathrm{SL}(5)$, and the geometric subgroup preserving the higher Courant bracket is the semi-direct product of diffeomorphisms of M with closed 3-form transformations Ω^3_{cl}. The action can be decomposed into four types, in a similar way as before. The 15-dimensional $\mathrm{SL}(4)$ subgroup acts separately on vectors, in the representation **4**, and on 2-forms, in the representation **6**, which combine into the antisymmetric representation **10** of $\mathrm{SL}(5)$. A 3-form $C \in \Gamma(\bigwedge^3 T^*M)$ acts as $X + \eta \mapsto X + \eta + \iota_X C$ (C-transform) and a 3-vector $\varOmega \in \Gamma(\bigwedge^3 TM)$ acts as $X + \eta \mapsto X + \eta + \iota_\eta \varOmega$ ($\varOmega$-transform). Both the 3-form and the 3-vector have four independent components in four dimensions. The matrix versions of these transformations are very similar to those of the $p = 1$ case above, thus we do not write them explicitly. Finally, a scaling transformation $X + \eta \mapsto \alpha^3\, X + \alpha^2\, \eta$ with $\alpha \in \mathbb{R}^\times$

guarantees closure of the group action. As before, a generalized metric $\mathcal{H}_2$ can then be parametrized in terms of a metric g and a 3-form C on TM in the form [14]

$$\mathcal{H}_2 = \begin{pmatrix} g + \frac{1}{2}\, C\, g^{-1}\wedge g^{-1}\, C & -\frac{1}{2}\, C\, g^{-1}\wedge g^{-1} \\ -\frac{1}{2}\, g^{-1}\wedge g^{-1}\, C & \frac{1}{2}\, g^{-1}\wedge g^{-1} \end{pmatrix} . \tag{2.18}$$

In the present case the structure group of E_2 is SL(5), its maximal compact subgroup is SO(5), and the moduli space of reductions by $\mathcal{H}_2$ is the corresponding 14-dimensional coset SL(5)/SO(5).

3 SL(5) M-theory fluxes

3.1 Fluxes from the higher Courant bracket

To determine the general form of the fluxes in M-theory for $7+4$ dimensions, where the four internal dimensions are compactified, we follow a strategy similar to the one suggested in [23] for non-geometric fluxes in generalized geometry. Recall that in generalized geometry the expressions for all the types of fluxes may be found upon acting with the twist operator $e^B\, e^\beta$ (which is an element of $O(d,d)$) on the local holonomic basis spanned by $\partial_i = \frac{\partial}{\partial x^i}$ and $\mathrm{d}x^i$ [24, 50]. This gives

$$\partial_i \xrightarrow{e^B\, e^\beta} e_i := \partial_i + B_{ij}\, \mathrm{d}x^j\,, \tag{3.1a}$$

$$\mathrm{d}x^i \xrightarrow{e^B\, e^\beta} e^i := \mathrm{d}x^i + \beta^{ij}\, \partial_j + \beta^{ij}\, B_{jk}\, \mathrm{d}x^k = \mathrm{d}x^i + \beta^{ij}\, e_j\ . \tag{3.1b}$$

Then, computing the *untwisted* Courant brackets of the new basis, one obtains

$$[e_i, e_j] = H_{ijk}\, e^k + F_{ij}{}^k\, e_k\,, \tag{3.2a}$$

$$[e_i, e^j] = F_{ik}{}^j\, e^k + Q_i{}^{jk}\, e_k\,, \tag{3.2b}$$

$$[e^i, e^j] = Q_k{}^{ij}\, e^k + R^{ijk}\, e_k\,, \tag{3.2c}$$

and the fluxes are identified with the generalized structure constants appearing on the right-hand side. Their explicit expressions are

$$H_{ijk} = 3\,\partial_{[i} B_{jk]}, \tag{3.3a}$$

$$F_{ij}{}^k = \beta^{kl}\, H_{lij}\,, \tag{3.3b}$$

$$Q_k{}^{ij} = \partial_k \beta^{ij} + \beta^{il}\, \beta^{jm}\, H_{lmk}\,, \tag{3.3c}$$

$$R^{ijk} = 3\,\beta^{[i\underline{l}}\, \partial_{\underline{l}} \beta^{jk]} + \beta^{il}\, \beta^{jm}\, \beta^{kn}\, H_{lmn}\,, \tag{3.3d}$$

where underlined indices do not participate in the antisymmetrization. The corresponding Bianchi identities may be obtained by using the Jacobi identity for this bracket [24].

JHEP05(2019)055

For later reference, let us recall that in a general non-holonomic basis the above expressions take the form

$$H_{abc} = 3\,\nabla_{[a}B_{bc]}, \tag{3.4a}$$

$$F_{ab}{}^{c} = f_{ab}{}^{c} + \beta^{cd}\,H_{abd}\,, \tag{3.4b}$$

$$Q_{a}{}^{bc} = \partial_a\beta^{bc} + \beta^{bd}\,f_{ad}{}^{c} - \beta^{cd}\,f_{ad}{}^{b} + \beta^{bd}\,\beta^{ce}\,H_{ade}\,, \tag{3.4c}$$

$$R^{abc} = 3\,\beta^{[a\underline{d}}\,\nabla_d\beta^{bc]} + \beta^{ad}\,\beta^{be}\,\beta^{cf}\,H_{def}\,, \tag{3.4d}$$

where ∇_a is the covariant derivative with respect to a vielbein $e_a = e_a{}^i\,\partial_i$, and its dual $e^a = e^a{}_i\,\mathrm{d}x^i$, acting as

$$\nabla_a B_{bc} = \partial_a B_{bc} - \mathit{\Gamma}_{ab}{}^{d}\,B_{dc} - \mathit{\Gamma}_{ac}{}^{d}\,B_{bd}\,, \tag{3.5}$$

and

$$f_{ab}{}^{c} = 2\,e^c{}_j\,e_{[a}{}^{i}\,\partial_i e_{b]}{}^{j} =: 2\,\mathit{\Gamma}_{[ab]}{}^{c} \tag{3.6}$$

is the purely geometric torsion flux, appearing for example in string compactifications on twisted tori [51].

Following the same prescription in the present case,[2] we consider a local basis $\{e_I\}, I = 1,\dots,\frac{d\,(d+1)}{2}$ $(I = 1,\dots,10)$ of sections of the vector bundle E_2, which may be split in a $d + \frac{d\,(d-1)}{2}$ $(4+6)$ fashion as

$$\{e_I\} = \{e_i, e^{ij}\}\,, \quad i,j = 1,\dots,d\ (i,j = 1,2,3,4)\,, \tag{3.7}$$

with $e^{ij} = -e^{ji}$. Then we compute the untwisted higher Courant bracket

$$[e_I, e_J] = T_{IJ}{}^{K}\,e_K\,, \tag{3.8}$$

and identify the corresponding M-theory fluxes with the local structure constants $T_{IJ}{}^{K}$. Explicitly, the brackets are generally given as

$$[e_i, e_j] = G_{ijkl}\,e^{kl} + F_{ij}{}^{m}\,e_m\,, \tag{3.9a}$$

$$[e_i, e^{jk}] = \widetilde{F}_{ilm}{}^{jk}\,e^{lm} + Q_i{}^{jkm}\,e_m\,, \tag{3.9b}$$

$$[e^{ij}, e^{kl}] = \widetilde{Q}_{mn}{}^{ij,kl}\,e^{mn} + R^{ij,kl,n}\,e_n\,. \tag{3.9c}$$

Following the terminology of the string theory case, we refer to G, F and $\widetilde{F}$ as geometric fluxes and to Q, $\widetilde{Q}$ and R as non-geometric fluxes. Compared to the string theory case there is a proliferation in that F and Q do not repeat in two different brackets, but rather two apparently new fluxes appear, $\widetilde{F}$ and $\widetilde{Q}$. We shall determine their relation to the fluxes F and Q below, where we will see that they are not independent. There is further a proliferation in the index structure; for instance, the previously trivector flux R now

[2] Everything that follows holds for an arbitrary dimensionality d of M, as indicated; however, the physically relevant case is only the one with $d = 4$, specified in parentheses. For higher d, the physical case requires a different extension of the tangent bundle, as already mentioned.

becomes a (mixed-symmetry) tensor with five indices (a 5-vector, but not fully antisymmetric — a completely antisymmetric tensor would anyway vanish in four dimensions) and index structure $(2,2,1)$. Recalling that previous studies of M-theory fluxes have revealed a mixed-symmetry 5-vector of type $(1,4)$ [4, 6–10], it is natural to ask what is the relation between the two. We shall return to this point after computing the explicit expressions for these fluxes.

As in the string theory case, we begin with the coordinate basis of sections of $E_2 = TM \oplus \bigwedge^2 T^*M$, spanned by ∂_i and $\frac{1}{2}\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, and twist them by SL(5) transformations corresponding to the action of a 3-form $C = \frac{1}{6}\,C_{ijk}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ and a 3-vector $\varOmega = \frac{1}{6}\,\varOmega^{ijk}\,\partial_i \wedge \partial_j \wedge \partial_k$ to get

$$\partial_i \xrightarrow{e^C\,e^\varOmega} e_i := \partial_i + \frac{1}{2}\,C_{ijk}\,\mathrm{d}x^j \wedge \mathrm{d}x^k\,, \tag{3.10a}$$

$$\frac{1}{2}\,\mathrm{d}x^i \wedge \mathrm{d}x^j \xrightarrow{e^C\,e^\varOmega} e^{ij} := \frac{1}{2}\,\mathrm{d}x^i \wedge \mathrm{d}x^j + \frac{1}{2}\,\varOmega^{ijk}\,e_k\ . \tag{3.10b}$$

We now compute the *untwisted* higher Courant brackets of the new basis $\{e_i, e^{ij}\}$ and comparing with (3.9a)–(3.9c) we formally obtain

$$G_{ijkl} = 4\,\partial_{[i}C_{jkl]}\,, \tag{3.11a}$$

$$F_{ij}{}^m = -\frac{1}{2}\,G_{ijkl}\,\varOmega^{klm}\,, \tag{3.11b}$$

$$\widetilde{F}_{ilm}{}^{jk} = \frac{1}{2}\,G_{ilmn}\,\varOmega^{njk}\,, \tag{3.11c}$$

$$Q_i{}^{jkm} = \frac{1}{2}\left(\partial_i\varOmega^{jkm} - \frac{1}{2}\,\varOmega^{jkn}\,G_{inps}\,\varOmega^{psm}\right), \tag{3.11d}$$

$$\widetilde{Q}_{mn}{}^{ij,kl} = -\frac{1}{4}\Big(\delta^l_{[m}\,\partial_{n]}\varOmega^{ijk} - \delta^k_{[m}\,\partial_{n]}\varOmega^{ijl} - \delta^j_{[m}\,\partial_{n]}\varOmega^{ikl} + \delta^i_{[m}\,\partial_{n]}\varOmega^{jkl} + \varOmega^{ijp}\,G_{pp'nm}\,\varOmega^{p'kl}\Big)\,, \tag{3.11e}$$

$$R^{ij,kl,n} = \frac{1}{2}\,\hat{\partial}^{i[j}\,\varOmega^{kln]} - \frac{1}{2}\,\hat{\partial}^{j[i}\,\varOmega^{kln]} - \frac{1}{2}\,\hat{\partial}^{k[l}\,\varOmega^{ijn]} + \frac{1}{2}\,\hat{\partial}^{l[k}\,\varOmega^{ijn]} - \frac{1}{8}\,\varOmega^{ijm}\,\varOmega^{klp}\,\varOmega^{rsn}\,G_{mprs}\,, \tag{3.11f}$$

where we defined $\hat{\partial}^{ij} := \varOmega^{ijk}\,\partial_k$. These expressions were derived without using the restriction $d = 4$. One may directly observe that the fluxes $\widetilde{F}$ and $\widetilde{Q}$ are not independent from F and Q, as there are the trace relations

$$\widetilde{F}_{ijl}{}^{lk} = F_{ij}{}^k\,, \tag{3.12}$$

$$\widetilde{Q}_{im}{}^{jk,lm} = \frac{d}{4}\,Q_i{}^{jkl} + \frac{d-4}{16}\,\varOmega^{jkp}\,G_{ipqr}\,\varOmega^{qrl} - \frac{1}{4}\,\delta_i^{[j}\,\partial_n\varOmega^{k]ln} + \frac{1}{8}\,\delta_i^l\,\partial_n\varOmega^{jkn}\ . \tag{3.13}$$

For general d, the flux $Q_i{}^{jkl}$ is not completely antisymmetric in its three vector indices.

Let us now specialize to the four-dimensional case. First, it is then inevitable that only the contracted parts of $\widetilde{F}$ and $\widetilde{Q}$ play a role, since they both have more than four indices in total. Specifically, the expression (3.13) for $\widetilde{Q}$ results in[3]

$$\widetilde{Q}_{im}{}^{jk,lm} = Q_i{}^{jkl} - \frac{1}{2}\,\delta_i^{[j}\,Q_n{}^{k]ln} + \frac{1}{4}\,\delta_i^l\,Q_n{}^{jkn}\ . \tag{3.14}$$

Second, in four dimensions, the flux $Q_i{}^{jkl}$ is necessarily completely antisymmetric in its three vector indices, in agreement with expectations. Another important observation is that the last term in (3.11f) is identically zero in four dimensions for any antisymmetric 3-vector Ω. Thus it is useful to define the $(1,4)$ mixed-symmetry combination

$$\mathcal{R}^{i,jklm} = \frac{1}{2}\,\hat{\partial}^{i[j}\,\Omega^{klm]}\,, \tag{3.15}$$

which allows us to write the R-flux obtained by the higher Courant bracket in terms of the $\mathcal{R}$-flux as

$$R^{ij,kl,n} = \mathcal{R}^{i,jkln} - \mathcal{R}^{j,ikln} - \mathcal{R}^{k,lijn} + \mathcal{R}^{l,kijn}\ . \tag{3.16}$$

Thus unlike the string theory case where all indices of the trivector R-flux participate in the antisymmetrization, here the right-hand side contains terms with derivatives of the trivector Ω where one index is outside the antisymmetrization. Studying all possibilities for index assignments, it turns out that in all cases we necessarily end up with only one term[4] and the R-flux is a $(1,4)$ mixed-symmetry tensor, in agreement with expectations. We discuss this point further in the extended case of section 6, where we shall compare with previous results in the literature. For the time being, we can already conclude that by studying the higher Courant bracket, the types of geometric and non-geometric fluxes which arise in the four-dimensional case are

$$G_{ijkl}\,, \quad F_{ij}{}^k\,, \quad Q_i{}^{jkl} \quad \text{and} \quad \mathcal{R}^{i,jklm}\ . \tag{3.17}$$

Finally, the expressions above are written in the holonomic frame. The corresponding expressions in a non-holonomic frame may be obtained similarly, by considering a vielbein e_a and $e^{ab} = \frac{1}{2}\,e^a \wedge e^b$. We present them only for the relevant fluxes in the four-dimensional case, which are

$$G_{abcd} = 4\,\nabla_{[a}C_{bcd]}\,, \tag{3.18a}$$

$$F_{ab}{}^c = f_{ab}{}^c - \frac{1}{2}\,G_{abde}\,\Omega^{dec}\,, \tag{3.18b}$$

$$Q_a{}^{bcd} = \frac{1}{2}\left(\partial_a\Omega^{bcd} + 3\,\Omega^{e[bc}\,f_{ae}{}^{d]} - \frac{1}{2}\,\Omega^{def}\,\delta_a^{[b}\,f_{ef}{}^{c]} - \frac{1}{2}\,\Omega^{e[bc}\,G_{aefg}\,\Omega^{d]fg}\right), \tag{3.18c}$$

$$R^{ab,cd,e} = \frac{1}{2}\,\widehat{\nabla}^{a[b}\,\Omega^{cde]} - \frac{1}{2}\,\widehat{\nabla}^{b[a}\,\Omega^{cde]} - \frac{1}{2}\,\widehat{\nabla}^{c[d}\,\Omega^{abe]} + \frac{1}{2}\,\widehat{\nabla}^{d[c}\,\Omega^{abe]}\,, \tag{3.18d}$$

[3]Note that the trace of the second term in $\widetilde{Q}$, namely $\Omega^{jkp}\,G_{pqrs}\,\Omega^{qrs}$, vanishes in four dimensions.

[4]The argument works as follows. Fix $i = i_0 \in \{1,2,3,4\}$. If $i_0 = j$ then obviously the R-flux vanishes; if $i_0 = k$, then $R^{ij,kl,n} = \hat{\partial}^{i_0[j}\,\Omega^{i_0 ln]}$, and similarly if $i_0 = l$; if $i_0 = n$ then $R^{ij,kl,n} = \frac{1}{2}\,\hat{\partial}^{i_0[j}\,\Omega^{kli_0]}$. If $i_0 \neq j,k,l,n$, then fix $j = j_0$ and repeat the argument.

JHEP05(2019)055

where $\widehat{\nabla}^{ab} = \Omega^{abc}\,\nabla_c$, and we used the definitions and facts explained before. Once more, one should keep in mind that the R-flux effectively contains only one term.[5]

3.2 Bianchi identities

Bianchi identities for the above fluxes can be obtained upon calculating the Jacobiator for the higher Courant bracket using the basic property

$$[[A,B],C] + \text{cyclic}(A,B,C) = \frac{1}{3}\,\mathrm{d}\big(\langle [A,B],C\rangle + \text{cyclic}(A,B,C)\big)\ . \tag{3.19}$$

To calculate the Jacobiator it is useful to write down the pairing $\langle A,B\rangle$ for the basis elements (3.10a) and (3.10b), which reads as

$$\langle e_i, e_j\rangle = 0\,, \tag{3.20a}$$

$$\langle e_i, e^{jm}\rangle = \frac{1}{2}\,\delta_i^{[j}\,\mathrm{d}x^{m]}\,, \tag{3.20b}$$

$$\langle e^{ij}, e^{mn}\rangle = \frac{1}{2}\,\Omega^{[ijm}\,\delta_l^{n]}\,\mathrm{d}x^l\ . \tag{3.20c}$$

Using these expressions, the modified Jacobi identity containing only the basis elements e_i,

$$[[e_i,e_j],e_m] + \text{cyclic}(i,j,m) - \frac{1}{3}\,\mathrm{d}\big(\langle [e_i,e_j],e_m\rangle + \text{cyclic}(i,j,m)\big) = 0\,, \tag{3.21}$$

gives directly the Bianchi identities

$$\partial_{[m} G_{ijkl]} = -\frac{3}{5}\,G_{np[ij}\,\widetilde{F}_{m]kl}{}^{np} - \frac{3}{5}\,F_{[ij}{}^n\,G_{m]nkl}\,, \tag{3.22a}$$

$$\partial_{[m} F_{ij]}{}^l - \frac{1}{3}\,\hat{\partial}^{lk} G_{ijmk} = -G_{nk[ij}\,Q_{m]}{}^{nkl} - F_{[ij}{}^k\,F_{m]k}{}^l\ . \tag{3.22b}$$

As in section 3.1, these formulas hold in any dimension d. However, for the physically interesting case $d=4$, the first identity becomes algebraic, since the left-hand side is identically zero (five antisymmetrized indices in four dimensions), and taking into account the trace relation (3.12) between $\widetilde{F}$ and F, the result is

$$G_{n[lij}\,F_{mk]}{}^n = 0\ . \tag{3.23}$$

The rest of the Jacobi identities, involving different combinations of the basis elements e_i and e^{ij}, give six additional Bianchi identities for all fluxes which read as

$$\begin{aligned} 3\,\partial_{[i}\widetilde{F}_{jp]r}{}^{mn} - \delta_{[r}^{[n}\,\partial_{p]}F_{ij}{}^{m]} + \frac{1}{2}\,\hat{\partial}^{mn}G_{ijpr} + \Omega^{ks[m}\,\delta_{[p}^{n]}\,\partial_{r]}G_{ijks} \\ = G_{ijkl}\,\widetilde{Q}_{pr}{}^{kl,mn} + F_{ij}{}^k\,\widetilde{F}_{kpr}{}^{mn} + 2\,\widetilde{F}_{kl[i}{}^{mn}\,\widetilde{F}_{j]pr}{}^{kl} + 2\,Q_{[i}{}^{mnk}\,G_{j]kpr}\,, \end{aligned} \tag{3.24a}$$

[5] Upon dimensional reduction over the M-theory circle, the four-dimensional fluxes (3.18a)–(3.18d) reduce to the NS-NS fluxes (3.4a)–(3.4d) in three dimensions. In higher dimensions this is not generally true, as the G-flux can reduce to either the NS-NS H-flux or the 4-form RR flux depending on whether or not the 4-form G has a leg along the M-theory circle.

$$2\,\partial_{[i}Q_{j]}{}^{mnp}+\frac{1}{2}\,\hat\partial^{mn}F_{ij}{}^{p}-\frac{1}{2}\,\hat\partial^{p[n}F_{ij}{}^{m]}-\frac{1}{2}\,\hat\partial^{lp}\widetilde F_{ijl}{}^{mn}+\frac{1}{2}\,\Omega^{kl[m}\,\hat\partial^{n]p}G_{ijkl}$$
$$=G_{ijkl}\,R^{kl,mn,p}+3\,F_{[ij}{}^{k}\,Q_{k]}{}^{mnp}+2\,\widetilde F_{kl[i}{}^{mn}\,Q_{j]}{}^{klp}\,, \tag{3.24b}$$

$$3\,\partial_{[i}\widetilde Q_{st]}{}^{jk,mn}-\hat\partial^{jk}\widetilde F_{ist}{}^{mn}-(\{jk\}\to\{mn\})$$
$$=2\,\widetilde F_{ipr}{}^{jk}\,\widetilde Q_{st}{}^{pr,mn}-\widetilde F_{ist}{}^{pr}\,\widetilde Q_{pr}{}^{jk,mn}+2\,Q_{i}{}^{jkp}\,\widetilde F_{pst}{}^{mn}+R^{jk,mn,p}\,G_{pist}$$
$$-(\{jk\}\to\{mn\})\,, \tag{3.24c}$$

$$\partial_i R^{jk,mn,s}-\hat\partial^{jk}Q_i{}^{mns}-\hat\partial^{ps}\widetilde Q_{ip}{}^{jk,mn}-(\{jk\}\to\{mn\})$$
$$=2\,\widetilde F_{ipr}{}^{jk}\,R^{pr,mn,s}+F_{pi}{}^{s}\,R^{jk,mn,p}+2\,Q_i{}^{jkp}\,Q_p{}^{mns}-\widetilde Q_{pr}{}^{jk,mn}\,Q_i{}^{prs}$$
$$-(\{jk\}\to\{mn\})\,, \tag{3.24d}$$

$$\partial_{[t}R^{ij,kl,[m}\,\delta^{n]}_{s]}+\frac{3}{4}\,\hat\partial^{mn}\widetilde Q_{st}{}^{ij,kl}+\Omega^{[prm}\,\delta^{n]}_{[s}\,\partial_{t]}\widetilde Q_{pr}{}^{ij,kl}+\mathrm{cyclic}(ij,kl,mn)$$
$$=R^{ij,kl,p}\,\widetilde F_{pst}{}^{mn}+\widetilde Q_{pr}{}^{ij,kl}\,\widetilde Q_{st}{}^{pr,mn}+\widetilde Q_{pr}{}^{ij,kl}\,\delta^{[n}_{[s}\left(Q_{t]}{}^{prm]}+\frac{1}{4}\,\Omega^{pr\underline{a}}\,G_{t]abc}\,\Omega^{m]bc}\right)$$
$$+\,\mathrm{cyclic}(ij,kl,mn)\,, \tag{3.24e}$$

$$\hat\partial^{mn}R^{ij,kl,q}+\frac{2}{3}\,\hat\partial^{pq}R^{ij,kl,[m}\,\delta^{n]}_{p}+\frac{2}{3}\,\Omega^{[prm}\,\hat\partial^{n]q}\widetilde Q_{pr}{}^{ij,kl}+\mathrm{cyclic}(ij,kl,mn)$$
$$=2\,R^{ij,kl,p}\,Q_p{}^{mnq}+2\,\widetilde Q_{pr}{}^{ij,kl}\,R^{pr,mn,q}$$
$$+\frac{2}{3}\,\widetilde Q_{pr}{}^{ij,kl}\,\Omega^{dq[n}\left(Q_d{}^{prm]}+\frac{1}{4}\,\Omega^{pr\underline{a}}\,G_{dabc}\,\Omega^{m]bc}\right)+\mathrm{cyclic}(ij,kl,mn)\,. \tag{3.24f}$$

Again these expressions are given in the holonomic frame. The Bianchi identities in a non-holonomic frame may be found in the same way.

4 Threebrane sigma-models and homotopy algebroids

4.1 Overview

Let us begin with an explanation of which precise question we aim at answering in the following. To this end let us go back to the string theory case, and the expressions for fluxes and Bianchi identities there. As already mentioned, they may be determined with the same approach as the one we used in section 3 for M-theory. The essential point we would like to recall is as follows. First we note that the fluxes and Bianchi identities in the holonomic frame for generalized geometry may be written in the compact form

$$\rho^i{}_I\,\partial_i\rho^j{}_J-\rho^i{}_J\,\partial_i\rho^j{}_I-\eta^{KL}\,\rho^j{}_K\,T_{LIJ}=0\,, \tag{4.1}$$

$$4\,\rho^i{}_{[L}\,\partial_iT_{IJK]}+3\,\eta^{MN}\,T_{M[IJ}\,T_{KL]N}=0\,, \tag{4.2}$$

where indices $i,j,\dots$ run over $1,\dots,d$ while $I,J,\dots$ run through $1,\dots,2d$. Here $\rho^i{}_I=(\delta^i{}_j,\beta^{ij})$, η_{IJ} is the $O(d,d)$-invariant metric, and the 3-form T_{IJK} corresponds to the four fluxes H_{ijk}, $F_{ij}{}^{k}$, $Q_i{}^{jk}$ and R^{ijk} depending on the index position. We supplement these equations with a third one,

$$\eta^{IJ}\,\rho^i{}_I\,\rho^j{}_J=0\,, \tag{4.3}$$

which is identically satisfied as long as $\beta^{(ij)} = 0$, namely β^{ij} are components of an antisymmetric 2-vector (not necessarily a Poisson bivector). These three equations may be interpreted in three different but related ways:

- As the fluxes and Bianchi identities in the NS-NS sector of general string theory compactifications.
- As the local form of the axioms of a Courant algebroid on E_1 [40].
- As the conditions arising from the classical master equation in the BV-BRST quantization of generalized Wess-Zumino terms, or equivalently as the conditions for gauge invariance and on-shell closure of gauge transformations for the Courant sigma-model [25].

Read differently, the above statements say the following: given a Courant algebroid, one can uniquely write down a membrane sigma-model which gives the BV-BRST action for Wess-Zumino terms which appear as fluxes in string theory compactifications. This statement is based on [25–29]. In particular, in [29] the precise relation with topological sigma-models of AKSZ-type [32] is demonstrated.

Moving one worldvolume dimension higher, from the closed string to the closed M2-brane, we have already found the expressions for the fluxes and their Bianchi identities in section 3. This allows us to pose the following question: given a higher analogue of an exact Courant algebroid, can one write down uniquely a threebrane sigma-model which gives the BV-BRST action for generalized Wess-Zumino terms that appear as fluxes in M-theory compactifications?

In other words, we would like to have a set of expressions much like (4.1)–(4.3), which can be interpreted again in three different ways: as M-theory fluxes and Bianchi identities, as the local form of the properties of the higher Courant bracket, and as conditions that guarantee the classical master equation for some topological threebrane sigma-model (or, equivalently, its gauge invariance and on-shell closure of gauge transformations). We address this question in section 5 below. In the present section we first discuss the construction of threebrane sigma-models as the higher analogue of Courant sigma-models.

4.2 Topological threebrane sigma-models

The threebrane sigma-model that corresponds to the topological AKSZ theory for open threebranes, or in other words the BV-BRST action for a 4-form Wess-Zumino term, was contructed explicitly in [39]. We shall not review the full construction here,[6] but instead we consider just the zero-ghost topological action for the AKSZ topological sigma-model in

[6]Along the usual lines of the BV-BRST formalism this consists in considering the BRST symmetry by replacing gauge parameters by ghosts, higher gauge parameters by ghosts-for-ghosts, and introducing the corresponding antifields required of the BV formalism. In the full space of fields, ghosts and antifields, an antibracket is defined and a BV-BRST action in constructed in terms of superfields on a graded manifold, which is required to satisfy the classical master equation imposing BRST-invariance.

JHEP05(2019)055

four dimensions, which will be enough to make our main point. Its general form reads as

$$S[X,\alpha,A,F]=\int_{\Sigma_4}\Big(F_i\wedge \mathrm{d}X^i-\alpha_I\wedge \mathrm{d}A^I+\rho^i{}_I(X)\,F_i\wedge A^I+\frac12\,S^{IJ}(X)\,\alpha_I\wedge\alpha_J \\ +\frac12\,T^I{}_{JK}(X)\,\alpha_I\wedge A^J\wedge A^K+\frac1{4!}\,G_{IJKL}(X)\,A^I\wedge A^J\wedge A^K\wedge A^L\Big)\,. \tag{4.4}$$

Let us spell out the ingredients in this sigma-model action. We have a theory of maps from a threebrane worldvolume Σ_4 to a d-dimensional target space M (we shall mostly consider $d=4$ later, but the discussion here holds for any d),

$$X=(X^i):\Sigma_4\longrightarrow M \qquad \text{with} \quad i=1,\dots,d\,, \tag{4.5}$$

and $F\in\Omega^3(\Sigma_4,X^*T^*M)$ is an auxiliary worldvolume 3-form taking values in the pullback of the cotangent bundle of M by the map X. In addition, there is a worldvolume 1-form $A\in\Omega^1(\Sigma_4,X^*E)$ and a worldvolume 2-form $\alpha\in\Omega^2(\Sigma_4,X^*E^*)$. They take values in the pullback of a vector bundle $E\to M$ and its dual $E^*\to M$ respectively. For example this could be the tangent bundle, but not necessarily so (see below). Here I is a bundle index when a basis of local sections $\{e_I\}$ of E is chosen, with corresponding dual basis $\{e^I\}$ for E^*, while $\rho^i{}_I$ are the components of an anchor map $\rho:E\to TM$, and S^{IJ} is symmetric in its two bundle indices, which defines a symmetric bilinear pairing on sections of E^* (possibly degenerate). Finally, $T^I{}_{JK}$ are structure constants of an antisymmetric bracket on sections of E and G_{IJKL} is a generalized 4-form on E. The quantities S, T and G are all functions of $X(\sigma)$, where σ^α are the local coordinates of the threebrane worldvolume, and they will be understood as generalized Wess-Zumino couplings in the present setting.

There is a hierarchy of structures and fields exhibited in the following table:

$\dim\Sigma$	AKSZ σ-model	0-forms	1-forms	2-forms	3-forms
2	Poisson	X^i	$F_i\in\Gamma(X^*T^*M)$	—	—
3	Courant	X^i	$A^I\in\Gamma(X^*E)$	$F_i\in\Gamma(X^*T^*M)$	—
4	Threebrane	X^i	$A^I\in\Gamma(X^*E)$	$\alpha_I\in\Gamma(X^*E^*)$	$F_i\in\Gamma(X^*T^*M)$

Specifically, the AKSZ theory in two dimensions corresponds to the BV-BRST action for the Poisson sigma-model (the first order formulation of the topological bosonic string B-field amplitude), in three dimensions to the Courant sigma-model and in four dimensions to the threebrane sigma-model that we discuss here. This is of course part of a semi-infinite staircase of (higher) geometric structures and topological sigma-models. More details may be found for example in the review [40].

The action (4.4) comes with a host of conditions stemming from the classical master equation — or, equivalently, from gauge invariance. Recall that in the two-dimensional case these conditions are equivalent to the vanishing of the Schouten-Nijenhuis bracket for a bivector field, which is the condition for a Poisson structure on M or the Lie algebroid axioms for the cotangent bundle T^*M equipped with the Koszul-Schouten bracket, while in the three-dimensional case they are equivalent to the axioms of a Courant algebroid. In the four-dimensional case the conditions found in [39, 41] define a higher algebroid structure, called a Lie algebroid up to homotopy in [39] or more generally an H-twisted Lie algebroid in [41] (see also refs. [42, 43]). Instead of providing the geometric axioms defining such a structure, we find it more illuminating at this stage to reside on the equivalent local coordinate conditions imposed by the classical master equation; the two approaches anyway reflect the same structure. The action (4.4) is invariant under the gauge transformations parametrized by scalar, 1-form and 2-form gauge parameters ϵ, ζ and t:

$$\delta X^i = -\rho^i{}_I\, \epsilon^I\,, \tag{4.6a}$$

$$\delta A^I = \mathrm{d}\epsilon^I + S^{IJ}\,\zeta_J - T^I{}_{JK}\, A^J\, \epsilon^K\,, \tag{4.6b}$$

$$\delta\alpha_I = \mathrm{d}\zeta_I + \rho^i{}_I\, t_i + T^J{}_{IK}\,\zeta_J\wedge A^K + T^J{}_{IK}\,\alpha_J\,\epsilon^K + \frac{1}{2}\,G_{IJKL}\,\epsilon^J\, A^K\wedge A^L\,, \tag{4.6c}$$

$$\begin{aligned}\delta F_i = -\mathrm{d}t_i &+ \partial_i\rho^j{}_I\left(\epsilon^I\, F_j + t_j\wedge A^I\right) - \partial_i T^J{}_{LI}\,\epsilon^I\,\alpha_J\wedge A^L\\ &- \frac{1}{6}\,\partial_i G_{IJKL}\,\epsilon^I\, A^J\wedge A^K\wedge A^L + \frac{1}{2}\,\partial_i T^I{}_{JK}\,\zeta_I\wedge A^J\wedge A^K + \partial_i S^{IJ}\,\zeta_I\wedge\alpha_J\,,\end{aligned} \tag{4.6d}$$

provided the following conditions are met:

$$\rho^i{}_I\, S^{IJ} = 0\,, \tag{4.7a}$$

$$\rho^i{}_I\,\partial_i S^{JK} + S^{LJ}\, T^K{}_{IL} + S^{LK}\, T^J_{IL} = 0\,, \tag{4.7b}$$

$$\rho^i{}_I\,\partial_i\rho^j{}_J - \rho^i{}_J\,\partial_i\rho^j{}_I - \rho^j{}_K\, T^K{}_{IJ} = 0\,, \tag{4.7c}$$

$$3\,\rho^i{}_{[I}\,\partial_i T^J{}_{KL]} + S^{JM}\, G_{KLIM} - 3\, T^J{}_{M[K}\, T^M{}_{LI]} = 0\,, \tag{4.7d}$$

$$\rho^i{}_{[I}\,\partial_i G_{JKLM]} + T^N{}_{[IJ}\, G_{KLM]N} = 0\;. \tag{4.7e}$$

Therefore, given a set of structure functions that solve these conditions, the corresponding topological threebrane sigma-model is uniquely determined. One can then reconstruct the anchor, bracket and 4-form on the vector bundle E, as well as the pairing on its dual E^*, as derived brackets [39].

These relations define a homotopy deformation of a Lie algebroid on E: setting $S = G = 0$ reduces the conditions (4.7a)–(4.7e) to the usual axioms of a Lie algebroid, with the remaining non-trivial identities (4.7c) and (4.7d) corresponding to the homomorphism property, Leibniz rule and Jacobi identity for the anchor map and bracket on E. Generally, the bracket $[\,\cdot\,,\,\cdot\,]$ on E extends to give a higher analog of the (twisted) Courant bracket on sections of $E\oplus\bigwedge^2E^*$, which can again be computed as a derived bracket and for $S=0$ is given by [39]

$$[s_1+\gamma_1, s_2+\gamma_2]_G = [s_1,s_2] + \mathcal{L}_{s_1}\gamma_2 - \mathcal{L}_{s_2}\gamma_1 - \frac{1}{2}\,\mathrm{d}_E(\iota_{s_1}\gamma_2 - \iota_{s_2}\gamma_1) + \iota_{s_1}\iota_{s_2}G\,, \tag{4.8}$$

where $s_1, s_2\in\Gamma(E)$ and $\gamma_1,\gamma_2\in\Gamma(\bigwedge^2E^*)$. Here $\mathcal{L}_{s_i} = \mathrm{d}_E\,\iota_{s_i} + \iota_{s_i}\,\mathrm{d}_E$ and $\mathrm{d}_E:\Gamma(\bigwedge^pE^*)\to\Gamma(\bigwedge^{p+1}E^*)$ is the usual Lie algebroid differential defined by

$$(\mathrm{d}_E\,\omega)_{I_0I_1\cdots I_p} = \rho^i{}_{[I_0}\,\partial_i\,\omega_{I_1\cdots I_p]} + T^J{}_{[I_0I_1}\,\omega_{I_2\cdots I_{p-1}]J} \tag{4.9}$$

for $\omega = \frac{1}{p!}\,\omega_{I_1\cdots I_p}\, e^{I_1}\wedge\cdots\wedge e^{I_p}$. Then (4.7d) (with $S=0$) is the nilpotency condition $\mathrm{d}_E^2=0$, while (4.7e) is just the closure condition $\mathrm{d}_E G=0$ which states that the twisting 4-form G represents a class in the degree 4 Lie algebroid cohomology of M; this classifies the Lie algebroids up to homotopy over M with $S=0$ [39].

The relation of the threebrane sigma-model to the higher Courant bracket may also be established as follows. First recall that the generalized Wess-Zumino term for Courant sigma-models is

$$\int_{\Sigma_3}\frac{1}{3!}\,X^*(T_{IJK})\,A^I\wedge A^J\wedge A^K=\int_{\Sigma_3}\frac{1}{3}\,X^*\big(\langle e_I,[e_J,e_K]\rangle\big)\,A^I\wedge A^J\wedge A^K\,, \tag{4.10}$$

where the components T_{IJK} are directly related to the twisted Courant-Roytenberg bracket in a local basis $\{e_I\}$ of $E_1=TM\oplus T^*M$ as $T_{IJK}=2\,\langle e_I,[e_J,e_K]\rangle$ [29].[7] In a similar fashion, the last two terms in the action (4.4) may be written as

$$\frac{1}{2}\,T^I{}_{JK}(X)\,\alpha_I\wedge A^J\wedge A^K=X^*\big(\langle e^I,[e_J,e_K]\rangle\big)\,\alpha_I\wedge A^J\wedge A^K\,, \tag{4.11a}$$

$$\frac{1}{4}\,G_{IJKL}(X)\,A^I\wedge A^J\wedge A^K\wedge A^L=X^*\big(\langle e_I,\langle e_J,[e_K,e_L]\rangle\rangle\big)\,A^I\wedge A^J\wedge A^K\wedge A^L\,, \tag{4.11b}$$

where as before $\{e_I\}$ and $\{e^I\}$ are local bases of sections of E and E^* respectively, the bracket is the higher Courant bracket on $E\oplus\bigwedge^2E^*$ and the bilinear form corresponds to symmetric contraction as in (2.5):[8]

$$\langle s_1+\gamma_1,s_2+\gamma_2\rangle=\frac{1}{2}\,(\iota_{s_1}\gamma_2+\iota_{s_2}\gamma_1)\ . \tag{4.12}$$

Thus the higher geometric operations introduced above and in section 2.1 directly dictate the generalized Wess-Zumino terms in the threebrane sigma-model. This will be further exemplified in a number of examples below.

Therefore, the question we posed in section 4.1 may be rephrased as follows: what is the relation of the conditions (4.7a)–(4.7e) to the fluxes and Bianchi identities that we found in section 3? Before we delve into the answer, we first discuss some (known and new) characteristic examples for this structure.

4.3 Examples

Homotopy tangent algebroids. The simplest possibility is to choose $E=TM$ with the usual Lie bracket of vector fields. Then the bundle index I is identified with the coordinate index i, in a local basis $\{e_i\}$ of the tangent bundle. The worldvolume 1-form $A^i=A^i_\alpha\,\mathrm{d}\sigma^\alpha$

[7] We indicated explicitly the pullback by X here, in order to avoid confusion, in contrast to the customary short-hand notation $\frac{1}{3}\,\langle A,[A,A]\rangle$ for the integrand in (4.10) used e.g. in [36]; the latter notation should be treated with caution, since the fields A live in the pullback bundle X^*E_1, which unlike E_1 itself is not endowed naturally with a Courant algebroid structure. Thus the notation $[A,A]$ can be misleading, since no such bracket is defined on X^*E_1, and we refrain from using it here.

[8] In (4.11b) the two bilinear forms are in principle different: the first ("inner") bilinear form is defined on $E\oplus\bigwedge^2E^*$, while the second ("outer") bilinear form is the canonical pairing between the vector bundle E and its dual E^*. Since in both cases the corresponding contractions are understood from the context, we refrain from establishing a separate notation for these two operations.

is valued in the pullback of the tangent bundle X^*TM over Σ_4 and the worldvolume 2-form $\alpha_i = \frac{1}{2}\,\alpha_{i\,\alpha\beta}\,\mathrm{d}\sigma^\alpha \wedge \mathrm{d}\sigma^\beta$ is valued in the pullback of the cotangent bundle X^*T^*M.

We begin with an analysis of the conditions (4.7a)–(4.7e) that define a Lie algebroid up to homotopy. The condition (4.7a) reads as

$$\rho^i{}_k\, S^{kj} = 0 \ . \tag{4.13}$$

If the pairing S^{ij} is non-degenerate, this implies $\rho^i{}_j = 0$. This is a legitimate option, especially in the case that the base manifold M is a point and the algebroid structure is reduced to an algebra. Such cases were examined in [39]. For our purposes, it is more interesting to consider instead the case that the anchor is non-degenerate, in which case one concludes that

$$S^{ij} = 0 \ . \tag{4.14}$$

Then (4.7a) and also (4.7b) are satisfied automatically. Since ρ is non-degenerate, with inverse $\rho_i{}^j$, the condition (4.7c) requires that

$$T^i{}_{jk} = 2\,\rho_l{}^i\,\rho^m{}_{[j}\,\partial_{\underline{m}}\,\rho^l{}_{k]} \ . \tag{4.15}$$

Finally, the relations (4.7d) and (4.7e) resemble Bianchi identities and they explicitly read as

$$\rho^l{}_{[i}\,\partial_{\underline{l}} T^j{}_{mn]} - T^j{}_{l[m}\,T^l{}_{ni]} = 0\,, \tag{4.16a}$$

$$\rho^n{}_{[i}\,\partial_{\underline{n}} G_{jklm]} + T^n{}_{[ij}\,G_{klm]n} = 0 \ . \tag{4.16b}$$

We distinguish two particularly interesting cases below.

***G*-flux.** Choose the anchor to be the projection to the tangent bundle, namely $\rho = \mathrm{id}$, or more explicitly $\rho^i{}_j = \delta^i{}_j$. Then (4.7c) implies immediately that $T^i{}_{jk} = 0$, which automatically satisfies (4.7d); in this case the derived bracket (4.8) reproduces (for $G = 0$) the higher Courant bracket on $E_2 = TM \oplus \bigwedge^2 T^*M$ from (2.7). Finally (4.7e) is the Bianchi identity which simply states that the 4-form G is closed,

$$\partial_{[i} G_{jklm]} = 0\,, \tag{4.17}$$

or $\mathrm{d}G = 0$, implying that G defines a class in the degree 4 de Rham cohomology of M.

The corresponding threebrane sigma-model becomes

$$S[X,\alpha,A,F] = \int_{\Sigma_4} \Big(F_i \wedge (\mathrm{d}X^i + A^i) - \alpha_i \wedge \mathrm{d}A^i \\ + \frac{1}{4!}\, G_{ijkl}(X)\, A^i \wedge A^j \wedge A^k \wedge A^l \Big) \ . \tag{4.18}$$

In accord with the discussion at the end of section 4.2, the Wess-Zumino term is associated to the higher Courant bracket on $E_2 = TM \oplus \bigwedge^2 T^*M$ by means of the relation

$$\frac{1}{4}\, G_{ijkl}(X) = X^*\big(\langle e_i, \langle e_j, [e_k, e_l]\rangle\rangle\big)\,, \tag{4.19}$$

where the bracket and the bilinear form are given by (3.9a) and (3.20b) respectively. It is useful to add a boundary term

$$S_\partial[X,A] = \oint_{\partial\Sigma_4} \frac{1}{2}\, g_{ij}(X)\, A^i \wedge * A^j\,, \tag{4.20}$$

where the Hodge duality operation $*$ is in three dimensions, sending 1-forms to 2-forms. A topological boundary term of the form

$$S_{\partial,\mathrm{top}}[X,A,\alpha] = \oint_{\partial\Sigma_4} h^i_j(X)\, \alpha_i \wedge A^j \tag{4.21}$$

is also possible, as well as boundary terms of type $\alpha \wedge *\alpha$ and $A \wedge A \wedge A$, however we do not include them in this example. The field equation for the auxiliary 3-form F_i gives

$$A^i = -\mathrm{d}X^i\,, \tag{4.22}$$

which with $G = \mathrm{d}C$ leads locally to the M2-brane action

$$S_\partial[X] = \oint_{\partial\Sigma_4} \left(\frac{1}{2}\, g_{ij}\, \mathrm{d}X^i \wedge * \mathrm{d}X^j + \frac{1}{3!}\, C_{ijk}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k\right)\,. \tag{4.23}$$

This is recognized as the action for a closed M2-brane coupled to a 3-form C-field, whose field strength is the 4-form G-flux. This example, without the metric term, was considered in [39], and recently in more detail in [44] where the double dimensional reduction of the Wess-Zumino term for wrapped threebranes along the M-theory circle is shown to reproduce the standard Wess-Zumino membrane coupling to an NS-NS H-flux.

M-theory on twisted tori. Motivated by Scherk-Schwarz reductions of M-theory on twisted tori, studied in detail in [37], we can choose $\rho^i{}_j$ to be equal to the components of a globally defined coframe $\mathsf{E}^i{}_j(X)$ for a twisted torus.[9] The simplest example, but by no means the only one, is to take a twisted 4-torus which is a trivial circle bundle over the three-dimensional Heisenberg nilmanifold, see for example [4, 6–10]. Then $T^i{}_{jk}$ is simply given by (4.15) and it is constant, corresponding to the structure constants of the associated nilpotent Lie algebra. Due to the Jacobi identity, the condition (4.7d) is identically satisfied. One may further choose $G = 0$, in which case (4.7e) is also an identity and all conditions are solved.

The sigma-model is

$$S[X,\alpha,A,F] = \int_{\Sigma_4} \left(F_i \wedge (\mathrm{d}X^i + \mathsf{E}^i{}_j\, A^j) - \alpha_i \wedge \mathrm{d}A^i + \frac{1}{2}\, T^i{}_{jk}\, \alpha_i \wedge A^j \wedge A^k\right)\,, \tag{4.24}$$

where the last term corresponds to

$$\frac{1}{2}\, T^i{}_{jk} = X^*\big(\langle e^i, [e_j, e_k]\rangle\big)\,, \tag{4.25}$$

[9] As explained in e.g. [36], one has to choose an isomorphism $\mathsf{E} : TN \to TM$ between the tangent bundles of M and the twisted torus N with corresponding components $\mathsf{E}^m{}_i$, whose inverse $\mathsf{E}^{-1} : TM \to TN$ has components $\mathsf{E}_m{}^i$, and identify the anchor with those. We present a short-cut version of this construction here, hoping that no confusion is caused.

JHEP05(2019)055

and we add the same boundary term as before. The equation of motion for F_i yields

$$A^i = -\mathsf{E}^i = -\mathsf{E}_j{}^i\,\mathrm{d}X^j \qquad \text{with} \quad \mathrm{d}\mathsf{E}^i = -\frac{1}{2}\,T^i{}_{jk}\,\mathsf{E}^j\wedge\mathsf{E}^k\,, \tag{4.26}$$

where we used the Maurer-Cartan equations. Inserting this in the action, the bulk terms cancel completely and the boundary term becomes

$$S_\partial[X] = \oint_{\partial\Sigma_4} \frac{1}{2}\,g_{ij}\,\mathsf{E}^i\wedge *\,\mathsf{E}^j\,, \tag{4.27}$$

which is the correct M2-brane action for an M-theory background with purely geometric torsion flux $T^i{}_{jk}$.

One may already consider a more general situation, by allowing a non-vanishing constant 4-form G. Then one would simply obtain the same model decorated with an additional G-flux, subject to the algebraic identity

$$T^n{}_{[ij}\,G_{klm]n} = 0\,, \tag{4.28}$$

due to (4.7e). This is a special case of the algebraic Bianchi identity (3.23), and it is welcoming to see that the very same condition was considered in the context of Scherk-Schwarz reductions with both fluxes turned on, see [37, eq. (2.5)], where it ensures that the flux $\mathsf{G} = \frac{1}{4!}\,G_{ijkl}\,\mathsf{E}^i\wedge\mathsf{E}^j\wedge\mathsf{E}^k\wedge\mathsf{E}^l$ is closed. In the present context this identity is imposed by the gauge invariance of the threebrane sigma-model via the classical master equation, and it ensures closure of the Wess-Zumino term $X^*(\mathsf{G})$.

Homotopy cotangent algebroids. An option that has not been explored so far is the choice of vector bundle $E = T^*M$. As we shall see, this is the analogue of the R-flux with Poisson structure in the case of (contravariant) Courant algebroids [52], which leads to a *geometric* R-flux. The difference is that in the Courant algebroid, the analogues of the fields α and A are treated symmetrically because they are both degree 1 fields. Here this "symmetry" breaks down.

We choose a local coframe $\{e^i\}$ for the cotangent bundle, with dual local frame $\{e_i\}$ of the tangent bundle. According to the explanations of section 4.2, the worldvolume 1-form $A = A_i\,e^i$ with $A_i = A_{i\,\alpha}\,\mathrm{d}\sigma^\alpha$ is now valued in X^*T^*M, while the worldvolume 2-form $\alpha = \alpha^i\,e_i$ with $\alpha^i = \frac{1}{2}\,\alpha^i_{\alpha\beta}\,\mathrm{d}\sigma^\alpha\wedge\mathrm{d}\sigma^\beta$ is valued in X^*TM. In the case of exact Courant algebroids, where $E = E_1 = TM\oplus T^*M$, taking instead the dual $E_1^* = T^*M\oplus TM$ does not lead to any difference but a renaming of the fields; however, this is not the case here, since the fields have different degrees.

Let us proceed with the analysis of the conditions for a Lie algebroid up to homotopy. First, the anchor components $\rho^i{}_I$ now become ρ^{ij} and they correspond to a bivector. The condition (4.7a) becomes

$$\rho^{ij}\,S_{jk} = 0\,, \tag{4.29}$$

which implies $S_{ij} = 0$ if as before we assume that the anchor is a non-degenerate map. Once more, the condition (4.7b) is then automatically satisfied. On the other hand, the relation (4.7c) becomes

$$\rho^{li}\,\partial_l\rho^{jk} - \rho^{lk}\,\partial_l\rho^{ji} - \rho^{jl}\,T_l^{ik} = 0\,. \tag{4.30}$$

This is solved by identifying $\rho^{ij} = \Pi^{ij}$ with the components of a Poisson bivector Π, satisfying $[\Pi, \Pi]_{\rm SN} = 0$ with respect to the Schouten-Nijenhuis bracket on multivector fields, and

$$T_i^{jk} = -\,Q_i{}^{jk} := -\,\partial_i \Pi^{jk} \ . \tag{4.31}$$

This Q-flux satisfies the Bianchi identity (4.7d),

$$\Pi^{l[i}\,\partial_l Q_j{}^{km]} = Q_j{}^{l[k}\,Q_l{}^{mi]} \ . \tag{4.32}$$

If we allow a non-vanishing generalized 4-form G, which in the present case is a tetravector, it has to satisfy the Bianchi identity (4.7e) which reads as

$$\Pi^{l[i}\,\partial_l G^{jkmn]} + Q_l{}^{[ij}\,G^{kmn]l} = 0\,, \tag{4.33}$$

which may be written in the suggestive form

$$\mathrm{d}_\Pi G := [\Pi, G]_{\rm SN} = 0 \ . \tag{4.34}$$

This is reminiscent of the Bianchi identity $[\Pi, R]_{\rm SN} = 0$ in the case of the Poisson Courant algebroid. In this case the higher Courant bracket (4.8) on $E_2^* = T^*M \oplus \bigwedge^2 TM$ is written using the Lichnerowicz differential $\mathrm{d}_E = \mathrm{d}_\Pi = [\Pi, \cdot\,]_{\rm SN}$ defined on multivector fields, together with the Koszul-Schouten bracket on $E = T^*M$ which for 1-forms $\tau_1, \tau_2 \in \Gamma(T^*M)$ reads as

$$[\tau_1, \tau_2]_\Pi = \mathcal{L}_{\iota_{\tau_1}\Pi}\,\tau_2 - \mathcal{L}_{\iota_{\tau_2}\Pi}\,\tau_1 - \mathrm{d}\,\iota_{\tau_1}\iota_{\tau_2}\Pi \ . \tag{4.35}$$

Then the twisting 4-vector G defines a class in the degree 4 Poisson cohomology of M.

Now that we have satisfied all the conditions for a Lie algebroid up to homotopy, we are all set to write down the threebrane sigma-model for the above data. It is given by the action

$$\begin{aligned} S[X,\alpha,A,F] = \int_{\Sigma_4} \Big(& F_i \wedge \mathrm{d}X^i - \alpha^i \wedge \mathrm{d}A_i + \Pi^{ij}\,F_i \wedge A_j \\ & - \frac{1}{2}\,Q_i{}^{jk}\,\alpha^i \wedge A_j \wedge A_k + \frac{1}{4!}\,G^{ijkl}\,A_i \wedge A_j \wedge A_k \wedge A_l \Big) \ . \end{aligned} \tag{4.36}$$

The equation of motion for F_i gives

$$\mathrm{d}X^i = -\Pi^{ij}\,A_j\,, \tag{4.37}$$

which can be inverted due to the non-degeneracy of the Poisson bivector to get

$$A_i = -\Pi^{-1}_{ij}\,\mathrm{d}X^j \ . \tag{4.38}$$

Choosing local Darboux coordinates in which both G^{ijkl} and Π^{ij} are constant, and adding a suitable boundary term, one obtains the M2-brane sigma model

$$\begin{aligned} S_\partial[X] = & \oint_{\partial\Sigma_4} \frac{1}{2}\,\big(g_{ij} - \Pi^{-1}_{ik}\,g^{kl}\,\Pi^{-1}_{lj}\big)\,\mathrm{d}X^i \wedge *\,\mathrm{d}X^j \\ & + \oint_{\partial\Sigma_4} \frac{1}{4!}\,G^{pqrs}\,\Pi^{-1}_{ip}\,\Pi^{-1}_{jq}\,\Pi^{-1}_{kr}\,\Pi^{-1}_{ls}\,X^i\,\mathrm{d}X^j \wedge \mathrm{d}X^k \wedge \mathrm{d}X^l \ . \end{aligned} \tag{4.39}$$

This is a non-trivial example of a Lie algebroid up to homotopy that generalizes to open threebranes the Poisson R-flux model for the open membrane. In the present case, the 4-form flux is controlled by a 4-vector G that satisfies $[\Pi, G]_{\rm SN} = 0$.

We stress that this is not the analogue of the *non-geometric* R-flux in M-theory, as it does not correspond to the lift of the nonassociative closed string R-flux deformation along the M-theory circle. It is simply the analogue of the Poisson Courant algebroid [34, 36, 52] at one level higher in the geometric staircase, and it is a geometric model. The analogue of the non-geometric R-flux will be discussed below.

5 Threebrane sigma-models and M-theory fluxes

We would now like to address the question posed in section 4.1, and relate the general M-theory fluxes and their Bianchi identities computed in section 3 to the threebrane sigma-models presented in section 4.2. This question can now be stated in more precise terms as follows: do the conditions (4.7a)–(4.7e) generate all the fluxes and Bianchi identities? We will answer this question in the affirmative and thus enable ourselves to write down the corresponding threebrane sigma-model.

The answer is that in the context of the threebrane sigma-model we should consider the vector bundle $E = E_2 = TM \oplus \bigwedge^2 T^*M$. Although this choice seems very reasonable in view of our previous discussion, one should appreciate that it is not the most natural choice, which is perhaps the reason that it has not been considered before. This may be explained by invoking the analogy with the Courant sigma-model. In that case, one has two different worldvolume 1-forms, say q^i and p_i, taking values in dual bundles, say L and L^* respectively. However, since they are of the same degree, they can be combined in a single 1-form A^I taking values in $E_1 = L \oplus L^*$, so that L is a maximally isotropic subbundle of E_1 with respect to the symmetric contraction pairing. Then the natural choice would be $L = TM$, which gives rise to the generalized tangent bundle. (The second choice $L = T^*M$ leads to the same bundle, as we already mentioned before.) On the contrary, in the present case, the two fields taking values in dual bundles E and E^* are of different degree and they cannot be combined. The most natural choice for E, the direct analogue of $L = TM$ above, would be either the tangent or cotangent bundle, the two choices being now inequivalent. But this is exactly what we have already done in section 4.3. There we saw that this can account for the G-flux and for the geometric torsion flux $f^i{}_{jk}$, but not for the rest of the SL(5) fluxes. We also saw that there is a consistent case with a 4-vector flux, which however is not one of the SL(5) fluxes.

Here our sole purpose is to find under which conditions the general theory yields instead the full set of SL(5) fluxes and nothing more. In contrast to the case of Courant algebroids, where the $O(d, d)$-structure is intrinsic, the structure group for a general Lie algebroid up to homotopy is not naturally tailored for the U-duality group SL(5) in four dimensions. Thus additional projections to SL(5) tensors are needed to make contact with the SL(5) fluxes. This is actually a strength of the present formalism, as there is some room for the same expressions below to also give a subset of the fluxes in higher dimensions for other exceptional U-duality groups.

JHEP05(2019)055

As in the examples of section 4.3, based on $E = TM$ and $E = T^*M$ respectively, we directly set $S^{IJ} = 0$, since this pairing does not play any further role in the identifications. Once more, this assumption takes care of the conditions (4.7a) and (4.7b). As we now show, this means that the relevant SL(5) fluxes are determined by the structure constants of a suitable Lie algebroid on $E = E_2$, whose bracket is specified by these structure constants.

Considering the bundle E_2 over a four-dimensional target space M means that its local basis index I takes 10 values, split into sets of 4 and 6 as before, namely a lower (upper) I becomes either a lower (upper) i or a set of upper (lower) $[ij]$ indices, with $i, j = 1, 2, 3, 4$. Based on this, we write the components of the anchor ρ as

$$(\rho^i{}_I) = (\rho^i{}_j, \rho^{ijk}), \tag{5.1}$$

and for our purposes here we further identify

$$\rho^i{}_j = \delta^i{}_j \qquad \text{and} \qquad \rho^{ijk} = \frac{1}{2}\,\Omega^{ijk}, \tag{5.2}$$

where we assume that ρ^{ijk} is a completely antisymmetric 3-vector, but not necessarily a Nambu-Poisson tensor. Then the condition (4.7c) yields three different equations. For both I, J being i, j the equation is

$$T^m{}_{ij} + \frac{1}{2}\,\Omega^{mkl}\,T_{klij} = 0, \tag{5.3}$$

which is precisely the expression (3.11b), provided we make the identifications $T^k{}_{ij} = F^k{}_{ij}$ and $T_{ijkl} = G_{ijkl}$. The latter identification is non-trivial, since T_{ijkl} is not completely antisymmetric *a priori*, but instead it is a *reducible* mixed-symmetry tensor of type $(2, 2)$. This means that in order to make contact with the G-flux, we consider only the irreducible fully antisymmetric component to be non-vanishing. This is a legitimate assumption, as long as the consistency conditions are satisfied, which is the case here. In a certain sense, it corresponds to a projection to SL(5) representations.

Second, for $I = i$ and J being (upper) $[jk]$ (or vice-versa), the equation we obtain is

$$T_i{}^{jkl} = \frac{1}{2}\,\partial_i \Omega^{jkl} - \frac{1}{2}\,\Omega^{lmp}\,T_{mpi}{}^{jk}. \tag{5.4}$$

This instructs us to identify $T_i{}^{jkl} = Q_i{}^{jkl}$ and $T_{mpi}{}^{jk} = \widetilde{F}_{mpi}{}^{jk}$. This looks like (3.11d), provided that we manage to fix $\widetilde{F}$ properly with the remaining equations. Third, taking I to be (upper) $[ij]$ and J to be (upper) $[kl]$, we obtain

$$T^{mijkl} + \frac{1}{2}\,\Omega^{mpq}\,T_{pq}{}^{ijkl} = \frac{1}{4}\,\Omega^{nkl}\,\partial_n \Omega^{mij} - \frac{1}{4}\,\Omega^{nij}\,\partial_n \Omega^{mkl}. \tag{5.5}$$

This expression indicates the identifications $T^{ijklm} = R^{jk,lm,i}$ and $T_{pq}{}^{ijkl} = \widetilde{Q}_{pq}{}^{ij,kl}$. Whether or not it is identical to (3.11f) remains to be shown, provided we are able to fix $\widetilde{Q}$ from what follows.

We now move on to the condition (4.7d), whose middle term is zero by assumption. Taking I, K, L as single indices and J as a doubled index, we obtain directly the Bianchi identity (3.22a) with the same identifications as above. This is then solved by taking $\widetilde{F}$ to be as in (3.11c), in which case indeed (5.4) becomes identical to (3.11d). Similarly, from (4.7d) we also identify $\widetilde{Q}$ as in (3.11e), and thus (5.5) is identical to (3.11f).

Finally, in the present context (4.7e) is redundant and we can take a vanishing twist $G_{IJKL} = 0$. By (4.11b), the vanishing locus of G defines an isotropic subbundle of $E = TM \oplus \bigwedge^2 T^*M$. It is precisely on this isotropic subbundle that the Bianchi identities coming from the higher Courant bracket in section 3.2 give the second condition (4.7d) coming from gauge invariance of the threebrane action with our projection.

Thus we conclude that indeed the sought-for equations for the fluxes and Bianchi identities are the conditions (4.7a)–(4.7e) under the above identifications. This provides a correspondence between the higher Courant bracket and this special case of the general AKSZ threebrane sigma-model. Let us therefore use this correspondence to write down the sigma-model explicitly. First, the 1-form A and the 2-form α have components

$$A^I = (A^i, A_{ij}) =: (q^i, p_{ij})\,, \tag{5.6a}$$

$$\alpha_I = (\alpha_i, \alpha^{ij}) =: (p_i, q^{ij})\,, \tag{5.6b}$$

involving 1-forms taking values in (the pullback bundles of) TM and $\bigwedge^2 T^*M$, and 2-forms taking values in T^*M and $\bigwedge^2 TM$ respectively. It would therefore appear that we have introduced an overabundance of worldvolume fields. However, our assumption that only the fully antisymmetric component of the reducible tensor T_{ijkl} survives and is identical to the G-flux indicates that the 2-form q^{ij} is decomposable, namely $q^{ij} = c\, q^i \wedge q^j$, where we choose a convenient scaling factor $c \in \mathbb{R}^\times$. What is more, the fact that we restrict the target space dimension to be four, and that the fluxes $\widetilde{F}$ and $\widetilde{Q}$ obey the relations (3.12) and (3.13), dictates that $q^i \wedge p_{ij} = \frac{1}{\tilde{c}}\, p_j$, where $\tilde{c} \in \mathbb{R}^\times$. Thus we are not dealing with an arbitrary threebrane sigma-model of the type discussed in section 4, but rather with a special case of it dictated by the relation to the M-theory fluxes. In the quantum theory, this would mean projecting the domain of the path integral to worldvolume field configurations constrained in this way.

Therefore, putting the above ingredients together, the sought-for threebrane sigma-model reads explicitly as

$$\begin{aligned} S = \int_{\Sigma_4} \Big(& F_i \wedge \mathrm{d}X^i - \tilde{c}\, q^i \wedge p_{ij} \wedge \mathrm{d}q^j - c\, q^i \wedge q^j \wedge \mathrm{d}p_{ij} + F_i \wedge q^i + \frac{1}{2}\, \Omega^{ijk}\, F_i \wedge p_{jk} \\ & + \Big(3\,c + \frac{\tilde{c}}{2}\Big)\, F^m{}_{jk}\, q^i \wedge q^j \wedge q^k \wedge p_{im} + \frac{c}{2}\, G_{ijkl}\, q^i \wedge q^j \wedge q^k \wedge q^l \\ & + (\tilde{c} + 2\,c)\, Q_l{}^{ijk}\, q^l \wedge q^m \wedge p_{mi} \wedge p_{jk} + \frac{c}{2}\, R^{jk,lm,i}\, q^n \wedge p_{ni} \wedge p_{jk} \wedge p_{lm} \Big)\,, \end{aligned} \tag{5.7}$$

up to boundary terms. This is indeed a sigma-model that contains all types of fluxes G, F, Q and R. Due to the conditions we have imposed, it depends only on two first order fields q^i and p_{ij}, which are both 1-forms on Σ_4. One could think of them as the first order variables for spacetime coordinates X^i and their dual wrapping coordinates $\widetilde{X}_{ij}$, although a truly $\widetilde{X}$-inclusive model requires extension of the base manifold M as in exceptional field theory, which is not the case here. With our specific projections in the threebrane sigma-model, it is straightforward if a bit lengthy to check that the projected threebrane action (5.7) is invariant under the corresponding restriction of the gauge transformations (4.6a)–(4.6d).

We now observe that the G-flux model may be obtained in an alternative way using (5.7). One simply takes $\Omega = 0$, i.e. the anchor to be the projection to the tangent bundle of M, and the only non-vanishing flux to be G. The field equation for F_i leads to $q^i = -\mathrm{d}X^i$ and the threebrane sigma-model reduces on-shell to

$$S_G[X] = \oint_{\partial\Sigma_4} \frac{1}{2}\, g_{ij}\, \mathrm{d}X^i \wedge * \,\mathrm{d}X^j + \int_{\Sigma_4} \frac{1}{4!}\, G_{ijkl}\, \mathrm{d}X^i \wedge \mathrm{d}X^j \wedge \mathrm{d}X^k \wedge \mathrm{d}X^l \,, \tag{5.8}$$

where we consider the case that the threebrane has a non-empty closed M2-brane boundary and, using $\tilde{c}\, q^i \wedge p_{ij} = p_j$ and $c = \frac{1}{12}$, we imposed the boundary condition

$$p_i = -\,6\,\tilde{c}\, g_{ij}\, * \mathrm{d}X^j \qquad \text{on} \quad \partial\Sigma_4 \,. \tag{5.9}$$

This is the same as the sigma-model with G-flux that we obtained in a different way in section 4.3. Here the symmetric term corresponds to the upper-left entry of the generalized metric (2.18) for $C = 0$.

Alternatively, one may consider the completely dual situation, where the anchor maps to the tangent bundle only through the trivector Ω, i.e. taking $\rho^i{}_j = 0$. For simplicity, let us suppose that Ω is a constant non-degenerate trivector, and that the only non-vanishing (constant) flux is R. Then the field equation for F_i is

$$\mathrm{d}X^i = -\frac{1}{2}\, \Omega^{ijk}\, p_{jk} \,. \tag{5.10}$$

This implies $\Omega^{ijk}\, \mathrm{d}p_{jk} = 0$ for all i, and hence, by our assumption that Ω^{ijk} is non-degenerate, that $\mathrm{d}p_{jk} = 0$. This means that locally we can define functions $\widetilde{X}_{ij}$ on Σ_4 through

$$p_{ij} = \mathrm{d}\widetilde{X}_{ij} \,. \tag{5.11}$$

This is not meant to be a wrapping coordinate, however the notation is indicative of what one would expect in the case of an extended base manifold. Then on-shell the threebrane sigma-model reduces to

$$S_R[\widetilde{X}] = \oint_{\partial\Sigma_4} \frac{1}{2}\, g^{ijkl}\, \mathrm{d}\widetilde{X}_{ij} \wedge * \,\mathrm{d}\widetilde{X}_{kl} + \int_{\Sigma_4} \frac{1}{4!}\, R^{jk,lm,i}\, q^n \wedge \mathrm{d}\widetilde{X}_{ni} \wedge \mathrm{d}\widetilde{X}_{jk} \wedge \mathrm{d}\widetilde{X}_{lm} \,, \tag{5.12}$$

where, taking into account $q^{ij} = \frac{1}{12}\, q^i \wedge q^j$, we imposed the boundary condition

$$q^{ij} = \frac{1}{12\,\tilde{c}}\, g^{ijkl}\, * \mathrm{d}\widetilde{X}_{kl} \qquad \text{on} \quad \partial\Sigma_4 \,, \tag{5.13}$$

with

$$g^{ijkl} = g^{ik}\, g^{jl} - g^{il}\, g^{jk} \,. \tag{5.14}$$

This is indeed the metric naturally appearing in M2-brane duality rotations [53], and it also corresponds to the lower-right entry of the generalized metric (2.18).

An integrability problem in realizing the full SL(5) U-duality group in the worldvolume theory for a closed M2-brane was identified in [54], based on previous work of [53]. It would be interesting to see whether the simple constructions we presented here can be generalized to include the M2-brane wrapping modes and shed some light on the problem of defining a manifestly U-duality invariant sigma-model.

6 SL(5) exceptional field theory fluxes

Going one step further, we would now like to determine in a systematic way the fluxes in SL(5) exceptional field theory, as first discussed in [4]. In that case, the base manifold M is extended to include coordinates conjugate to both the momentum and wrapping modes of closed M2-branes. In the present case the extended space $\mathcal{M}$ is 10-dimensional with local coordinates

$$x^I = (x^i, \tilde{x}_{ij}) \ . \tag{6.1}$$

Explicitly, $x^I = x^{\bar{a}\bar{b}} = -x^{\bar{b}\bar{a}}$ are coordinates in the antisymmetric representation **10** of SL(5) with $\bar{a}, \bar{b} = 1,2,3,4,5$. The spacetime coordinates are given by $x^{i5} = -x^{5i} = x^i$ with $i = 1,2,3,4$ and the wrapping coordinates are given by dualization $\tilde{x}_{ij} = \frac{1}{2}\,\epsilon_{ijkl}\,x^{kl}$ in four dimensions. A field in the antisymmetric representation of SL(5) is a section of the generalized tangent bundle $E_2 = TM \oplus \bigwedge^2 T^*M$, so a local model for the extended space may be taken to be the total space $\mathcal{M} = \bigwedge^2 T^*M$ of the bundle of 2-forms on M, with x^i local coordinates on the base M and $\tilde{x}_{ij} = -\tilde{x}_{ji}$ local fibre coordinates. Correspondingly, dual derivatives are defined in addition to the standard ones,

$$\partial_I = (\partial_i, \tilde{\partial}^{ij}) \ . \tag{6.2}$$

It is instructive at this point to recall that the local symmetries of exceptional field theory are generated by a generalized Lie derivative

$$\mathscr{L}_\xi A^I = \xi^J\,\partial_J A^I - A^J\,\partial_J \xi^I + Y^{IJ}_{KL}\,A^K\,\partial_J \xi^L \ , \tag{6.3}$$

where ξ^I is a gauge generator for generalized diffeomorphisms and Y^{IJ}_{KL} is an invariant tensor of the U-duality group, here SL(5). It reads as

$$Y^{IJ}_{KL} = \epsilon^{\bar{a}IJ}\,\epsilon_{\bar{a}KL} \ . \tag{6.4}$$

The role of this tensor becomes evident when closure of the algebra of gauge transformations is imposed. This happens when a section condition is satisfied, which reads as

$$Y^{IJ}_{KL}\,\partial_I \otimes \partial_J = 0 \ . \tag{6.5}$$

This operator equation expresses the fact that when it acts on any field or product of fields the result should vanish. Restricting to fields satisfying this section condition, one finds

$$[\mathscr{L}_{\xi_1}, \mathscr{L}_{\xi_2}] = \mathscr{L}_{[\![\xi_1,\xi_2]\!]} \ , \tag{6.6}$$

JHEP05(2019)055

where the bracket appearing on the right-hand side is defined as

$$[\![\xi_1, \xi_2]\!] = \frac{1}{2}\left(\mathscr{L}_{\xi_1}\xi_2 - \mathscr{L}_{\xi_2}\xi_1\right) . \tag{6.7}$$

This bracket is the SL(5) covariantization of the higher Courant bracket (2.7), in the same way that the $O(d,d)$ covariantization of the standard Courant bracket is the C-bracket of double field theory [55]. Indeed, when solving the section condition (6.5) by setting the dual derivatives $\tilde{\partial}^{ij}$ to zero, the bracket of (6.7) becomes precisely the higher Courant bracket (2.7), and moreover $\mathscr{L}_\xi \cdot \ = \xi \circ \cdot$ is given by the higher Dorfman bracket (2.6). This is exactly how this bracket was originally constructed in [56] and used in [6] to relate the geometric formulation of SL(5) exceptional field theory to the embedding tensor formalism of seven-dimensional gauged supergravity, where the embedding tensor is related to the fluxes and in turn to the structure constants of the algebra of generalized diffeomorphisms.

For our purposes, we employ the procedure given in [57] for double field theory fluxes. A representation of the algebra (3.9a)–(3.9c) can be given by the Lie bracket of certain vector fields which are sections of the tangent bundle $T\mathcal{M}$ of the extended manifold of exceptional field theory. We introduce two such fields in the holonomic basis, given by

$$D_i = \partial_i + \frac{1}{2}\, C_{ijk}\, \tilde{\partial}^{jk} \,, \tag{6.8a}$$

$$\tilde{D}^{jk} = \frac{1}{2}\, \tilde{\partial}^{jk} + \frac{1}{2}\, \Omega^{jkl}\, D_l \,, \tag{6.8b}$$

and calculate their Lie brackets. Here the components of the 3-form $C = \frac{1}{6}\, C_{ijk}\, \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$ and the 3-vector $\Omega = \frac{1}{6}\, \Omega^{ijk}\, \partial_i \wedge \partial_j \wedge \partial_k$ can depend on both physical and wrapping coordinates. The result is

$$[D_i, D_j] = G_{ijkl}\, \tilde{D}^{kl} + F_{ij}{}^m\, D_m \,, \tag{6.9a}$$

$$[D_i, \tilde{D}^{jk}] = \widetilde{F}_{ilm}{}^{jk}\, \tilde{D}^{lm} + Q_i{}^{jkm}\, D_m \,, \tag{6.9b}$$

$$[\tilde{D}^{ij}, \tilde{D}^{kl}] = R^{ij,kl,n}\, D_n + \widetilde{Q}_{mn}{}^{ij,kl}\, \tilde{D}^{mn} \,, \tag{6.9c}$$

where we defined the exceptional field theory fluxes

$$G_{ijkl} = 4\, \partial_{[i} C_{jkl]} + 2\, C_{mn[i}\, \tilde{\partial}^{mn} C_{jkl]} \,, \tag{6.10a}$$

$$F_{ij}{}^m = -\frac{1}{2}\, G_{ijkl}\, \Omega^{klm} + \tilde{\partial}^{mk} C_{ijk} \,, \tag{6.10b}$$

$$\widetilde{F}_{ilm}{}^{jk} = \frac{1}{2}\, G_{ilmn}\, \Omega^{njk} - \frac{1}{2}\, \tilde{\partial}^{jk} C_{ilm} \,, \tag{6.10c}$$

$$Q_i{}^{jkm} = \frac{1}{2}\Big(\partial_i \Omega^{jkm} + \frac{1}{2}\, C_{iln}\, \tilde{\partial}^{ln} \Omega^{jkm} + \frac{1}{2}\, \Omega^{lnm}\, \tilde{\partial}^{jk} C_{iln} + \Omega^{ljk}\, \tilde{\partial}^{mn} C_{iln} - \frac{1}{2}\, \Omega^{jkn}\, G_{inps}\, \Omega^{psm} \Big) \,, \tag{6.10d}$$

$$\widetilde{Q}_{mn}{}^{ij,kl} = \frac{1}{4}\left(\Omega^{ijp}\, G_{pp'mn}\, \Omega^{p'kl} + \Omega^{klr}\, \tilde{\partial}^{ij} C_{rmn} - \Omega^{ijr}\, \tilde{\partial}^{kl} C_{rmn} \right)$$

JHEP05(2019)055

$$
\begin{aligned}
&- \frac{1}{4}\left(\delta^l_{[m}\,\partial_{n]}\Omega^{ijk} - \delta^k_{[m}\,\partial_{n]}\Omega^{ijl} - \delta^j_{[m}\,\partial_{n]}\Omega^{ikl} + \delta^i_{[m}\,\partial_{n]}\Omega^{jkl}\right) \\
&- \frac{1}{8}\Big(\delta^l_{[m}\,C_{n]sp}\,\tilde\partial^{sp}\Omega^{ijk} - \delta^k_{[m}\,C_{n]sp}\,\tilde\partial^{sp}\Omega^{ijl} \\
&\qquad\qquad - \delta^j_{[m}\,C_{n]sp}\,\tilde\partial^{sp}\Omega^{ikl} + \delta^i_{[m}\,C_{n]sp}\,\tilde\partial^{sp}\Omega^{jkl}\Big)\,,
\end{aligned}
\tag{6.10e}
$$

$$
\begin{aligned}
R^{ij,kl,n} &= \frac{1}{2}\,\hat\partial^{i[j}\,\Omega^{kln]} - \frac{1}{2}\,\hat\partial^{j[i}\,\Omega^{kln]} - \frac{1}{2}\,\hat\partial^{k[l}\,\Omega^{ijn]} + \frac{1}{2}\,\hat\partial^{l[k}\,\Omega^{ijn]} - \frac{1}{8}\,\Omega^{ijm}\,\Omega^{klp}\,\Omega^{rsn}\,G_{mprs} \\
&+ \frac{1}{4}\,C_{mpr}\left(\Omega^{mi[j}\,\tilde\partial^{\underline{pr}}\Omega^{kln]} - \Omega^{mj[i}\,\tilde\partial^{\underline{pr}}\Omega^{kln]} - \Omega^{mk[l}\,\tilde\partial^{\underline{pr}}\Omega^{ijn]} + \Omega^{ml[k}\,\tilde\partial^{\underline{pr}}\Omega^{ijn]}\right) \\
&+ \frac{1}{8}\left(\Omega^{stn}\,\Omega^{ijr}\,\tilde\partial^{kl} - \Omega^{stn}\,\Omega^{klr}\,\tilde\partial^{ij} + 2\Omega^{ijt}\,\Omega^{klr}\,\tilde\partial^{ns}\right)C_{rst}\,,
\end{aligned}
\tag{6.10f}
$$

and $\hat\partial^{ij} = \tilde\partial^{ij} + \Omega^{ijk}\,\partial_k$. These expressions rely on the section condition, as also happens in the case of double field theory. As in the case of exceptional generalized geometry, these expressions are valid for any d, in which case the dimension of the extended space is $d + \frac{d\,(d-1)}{2}$, however they simplify for the physically relevant case of $d = 4$, where $\widetilde F$ and $\widetilde Q$ can be related to F and Q respectively.

To write these expressions in a non-holonomic frame, we introduce a vielbein $e_a = e_a{}^i\,\partial_i$ whose components $e_a{}^i$ can depend on both physical and wrapping coordinates, together with the dual vector fields $\tilde e^{ab} = e^{[a}{}_i\,e^{b]}{}_j\,\tilde\partial^{ij}$. Then the fluxes acquire additional terms and in the four-dimensional case they read as

$$
G_{abcd} = 4\,\nabla_{[a}C_{bcd]} + 2\,C_{ef[a}\,\widetilde\nabla^{ef}C_{bcd]}\,,
\tag{6.11a}
$$

$$
F_{ab}{}^c = f_{ab}{}^c + C_{de[a}\,\tilde\Gamma^{de}{}_{b]}{}^c - \frac{1}{2}\,\Omega^{dec}\,G_{abde} + \widetilde\nabla^{cd}C_{dab}\,,
\tag{6.11b}
$$

$$
\begin{aligned}
Q_a{}^{bcd} &= \frac{1}{2}\,\partial_a\Omega^{bcd} - \frac{3}{2}\,\tilde\Gamma^{[bc}{}_a{}^{d]} + \frac{3}{2}\,\Omega^{e[bc}\,f_{ae}{}^{d]} + \frac{1}{4}\,C_{aef}\,\widetilde\nabla^{ef}\,\Omega^{bcd} - \frac{3}{4}\,C_{efg}\,\tilde\Gamma^{ef}{}_a{}^{[d}\,\Omega^{bc]g} \\
&+ \frac{1}{2}\,\tilde\Gamma^{de}{}_e{}^{[c}\,\delta_a^{b]} + \frac{1}{4}\,\Omega^{def}\,C_{fgh}\,\tilde\Gamma^{gh}{}_e{}^{[c}\,\delta_a^{b]} - \frac{1}{4}\,\Omega^{def}\,f_{ef}{}^{[c}\,\delta_a^{b]} \\
&+ \frac{1}{4}\,\Omega^{efd}\,\widetilde\nabla^{bc}C_{aef} + \frac{1}{2}\,\Omega^{bce}\,\widetilde\nabla^{df}\,C_{aef} - \frac{1}{4}\,\Omega^{e[bc}\,\Omega^{d]fg}\,G_{aefg}\,,
\end{aligned}
\tag{6.11c}
$$

$$
\begin{aligned}
R^{ab,cd,e} &= \frac{1}{2}\,\widehat\nabla^{a[b}\,\Omega^{cde]} - \frac{1}{2}\,\widehat\nabla^{b[a}\,\Omega^{cde]} - \frac{1}{2}\,\widehat\nabla^{c[d}\,\Omega^{abe]} + \frac{1}{2}\,\widehat\nabla^{d[c}\,\Omega^{abe]} \\
&+ \frac{1}{4}\,C_{fgh}\left(\Omega^{fa[b}\,\widetilde\nabla^{\underline{gh}}\Omega^{cde]} - \Omega^{fb[a}\,\widetilde\nabla^{\underline{gh}}\Omega^{cde]} - \Omega^{fc[d}\,\widetilde\nabla^{\underline{gh}}\Omega^{abe]} + \Omega^{fd[c}\,\widetilde\nabla^{\underline{gh}}\Omega^{abe]}\right) \\
&+ \frac{1}{8}\left(\Omega^{fge}\,\Omega^{abh}\,\widetilde\nabla^{cd} - \Omega^{fge}\,\Omega^{cdh}\,\widetilde\nabla^{ab} + 2\Omega^{abg}\,\Omega^{cdh}\,\widetilde\nabla^{ef}\right)C_{hfg}\,,
\end{aligned}
\tag{6.11d}
$$

where we defined the dual connection

$$
\tilde\Gamma^{ab}{}_c{}^d = e^d{}_k\,e^{[a}{}_i\,e^{b]}{}_j\,\tilde\partial^{ij}e^k{}_c\,,
\tag{6.12}
$$

and

$$
\widetilde\nabla^{ab}C_{cde} = \tilde\partial^{ab}C_{cde} - \tilde\Gamma^{ab}{}_c{}^f\,C_{fde} - \tilde\Gamma^{ab}{}_d{}^f\,C_{cfe} - \tilde\Gamma^{ab}{}_e{}^f\,C_{cdf}\,,
\tag{6.13a}
$$

$$
\widehat\nabla^{ab} = \widetilde\nabla^{ab} + \Omega^{abc}\,\nabla_c\,,
\tag{6.13b}
$$

are the dual covariant derivatives.

Finally, let us compare our results with the SL(5) fluxes described in [4, 10], where a group theoretical derivation in terms of SL(5) representation theory was used. Based on the embedding tensor formalism for gaugings of seven-dimensional maximal supergravity [58], the relevant representations of SL(5) are $\overline{\mathbf{15}} \oplus \overline{\mathbf{40}} \oplus \overline{\mathbf{10}}$, and therefore the fluxes should exhaust these representations. This may be confirmed by using their branching decompositions under the embedding $\mathrm{SL}(5) \supset \mathrm{SL}(4) \times \mathbb{R}^+$, which read as[10]

$$\overline{\mathbf{10}}\,\big|_{\mathrm{SL}(4)} = \overline{\mathbf{4}} \oplus \mathbf{6}\,, \tag{6.14a}$$

$$\overline{\mathbf{15}}\,\big|_{\mathrm{SL}(4)} = \overline{\mathbf{10}} \oplus \overline{\mathbf{4}} \oplus \mathbf{1}\,, \tag{6.14b}$$

$$\overline{\mathbf{40}}\,\big|_{\mathrm{SL}(4)} = \overline{\mathbf{20}} \oplus \mathbf{10} \oplus \mathbf{6} \oplus \overline{\mathbf{4}}\;. \tag{6.14c}$$

The available fluxes that should be matched with these representations are the 4-form G-flux G_{abcd}, the geometric torsion flux $f_{ab}{}^{c}$, the Q-flux $Q_a{}^{bcd}$ and the $\mathcal{R}$-flux $\mathcal{R}^{a,bcde}$. Clearly, the only singlet in these decompositions corresponds to the 4-form G-flux. Moreover, the $\mathcal{R}$-flux, as a mixed-symmetry $(1,4)$ tensor, lives in one of the three $\overline{\mathbf{4}}$ representations of SL(4) — the one in the decomposition of $\overline{\mathbf{40}}$. The torsion flux $f_{ab}{}^{c}$ has 24 components and lives in the representations $\overline{\mathbf{20}} \oplus \overline{\mathbf{4}}$, corresponding to its trace and traceless parts. The flux $Q_a{}^{bcd}$, antisymmetric in its upper three indices, has 16 components, corresponding to the representations $\overline{\mathbf{10}} \oplus \mathbf{6}$, again containing a trace and a traceless part. What remains is the trace part of the dual torsion flux $\tilde{\Gamma}^{ab}{}_{b}{}^{c}$, which contains a symmetric and an antisymmetric part with a total of 16 components living in the representations $\mathbf{10} \oplus \mathbf{6}$. The last $\overline{\mathbf{4}}$ representation corresponds to the determinant of the seven-dimensional metric, and this exhausts all representations of SL(4) appearing above. Therefore, the geometric approach based on the higher Courant bracket reproduces all the fluxes obtained using the group theoretical approach.

7 Conclusions and outlook

Motivated by the relation between T-duality and non-geometric fluxes in closed string theory, the properties of the Courant bracket in generalized geometry, and the gauge structure of AKSZ-type topological membrane sigma-models, in this paper we have investigated whether such relations extend to the case of U-duality and fluxes in M-theory, the higher Courant bracket in exceptional generalized geometry, and AKSZ-type topological threebrane sigma-models. We established that upon a certain projection to SL(5) tensors, the local coordinate form of the axioms for a specific Lie algebroid up to homotopy based on the extended bundle $TM \oplus \bigwedge^2 T^*M$ coincides with the general expressions for geometric and non-geometric fluxes in $(7+4)$-dimensional compactifications of M-theory, together with their Bianchi identities. The same expressions are also interpreted as the conditions for gauge invariance and closure of gauge transformations for a topological threebrane sigma-model, where the fluxes appear as generalized Wess-Zumino terms. It would be interesting to understand better the geometric features of the algebroid structure defined by our SL(5)

[10]We refrain from presenting the additional $\mathbb{R}^+$ charges here.

projection of the Lie algebroid up to homotopy on $TM \oplus \bigwedge^2 T^*M$. Given that the higher Courant algebroid structure on this extended bundle has a well-known realization as an L_∞-algebra, see for example [18, 19, 59], it is an interesting problem to see how our specific Lie algebroid up to homotopy fits into recent discussions of the L_∞-algebra structure of gauge symmetries underlying the tensor hierarchy in exceptional field theory, see for instance [60–62].

This observation can serve as a first step toward a new geometric understanding of the worldvolume approach to M-theory. This requires extension of the analysis presented here in several directions. In this paper we focused on the case of the exceptional group SL(5), which is the (continuous) U-duality group for M-theory compactified on a four-dimensional torus. Obviously, a more complete treatment would require studying the fluxes for lower (higher) number of external (internal) dimensions, where the U-duality group is larger. A standard complication in that case is that M5-brane charges emerge, and the extended bundles are larger and not of the type $TM \oplus \bigwedge^p T^*M$, while the corresponding topological sigma-model that encodes the fluxes as Wess-Zumino terms is expected to be more complicated as U-duality now exchanges charged objects of different dimensionalities. Some discussions of these issues can be found in [44, 61, 63].

Another open problem is to construct a threebrane sigma-model with an extended base space, similarly to double field theory where the coordinates are doubled, whose generalized Wess-Zumino terms would accomodate the exceptional field theory fluxes, derived for the U-duality group SL(5) in section 6. In the SL(5) case this would require a ten-dimensional extended space, and even higher-dimensional for larger U-duality groups. Target space exceptional field theories with manifest U-duality invariance were constructed in recent years in [64–68], however the corresponding worldvolume problem remains open [54]. Applying the strategy developed in [36] for the construction of T-duality invariant membrane sigma-models could be helpful in that case, although its precise implementation is not straightforward. In addition, one should then deal with the section condition and understand its geometric origin in the context of weaker algebroid structures, similar to the (pre-)DFT algebroids defined in [36] and formulated in [69] in terms of an AKSZ-type construction.

Finally, another issue that we did not discuss in the present paper is the nonassociativity of the M2-brane phase space, which is deformed by the presence of the locally non-geometric M-theory R-flux [7, 8]. This problem requires the extension of the base manifold M and falls in the same line of discussion as previously. In other words, one would have to include the wrapping coordinates of closed M2-branes in order to see how nonassociativity manifests itself in the worldvolume approach. Similarly, it would be interesting to find a geometric interpretation for the fact that the phase space of M-theory, when R-flux is turned on, is dimensionally reduced from eight to seven dimensions due to the absence of momentum modes along the M-theory circle [7–10].

Acknowledgments

We thank Ralph Blumenhagen, Mark Bugden, Fech Scen Khoo, Zoltán Kökényesi, Emanuel Malek, Erik Plauschinn, Felix Rudolph and Marc Syväri for helpful discussions. We acknowledge support by COST (European Cooperation in Science and Technology) in the framework of the Action MP1405 QSPACE. The work of A.Ch. is supported by the Croatian Science Foundation Project "New Geometries for Gravity and Spacetime" (IP-2018-01-7615). A.Ch. and L.J. are partially supported by the European Union through the European Regional Development Fund — The Competitiveness and Cohesion Operational Programme (KK.01.1.1.06) and by the H2020 CSA Twinning Project No. 692194 "RBI-T-WINNING". The work of D.L. was partially supported by the ERC Advanced Grant "Strings and Gravity" (Grant No. 320045). The work of R.J.S. was supported by the Consolidated Grant ST/P000363/1 from the U.K. Science and Technology Facilities Council.

References

[1] M. Graña, *Flux compactifications in string theory: a comprehensive review*, *Phys. Rept.* **423** (2006) 91 [hep-th/0509003] [INSPIRE].

[2] E. Plauschinn, *Non-geometric backgrounds in string theory*, *Phys. Rept.* **798** (2019) 1 [arXiv:1811.11203] [INSPIRE].

[3] R.J. Szabo, *Higher quantum geometry and non-geometric string theory*, PoS(CORFU2017)151 [arXiv:1803.08861] [INSPIRE].

[4] C.D.A. Blair and E. Malek, *Geometry and fluxes of* SL(5) *exceptional field theory*, *JHEP* **03** (2015) 144 [arXiv:1412.0635] [INSPIRE].

[5] D.S. Berman and M.J. Perry, *Generalized geometry and M-theory*, *JHEP* **06** (2011) 074 [arXiv:1008.1763] [INSPIRE].

[6] P. du Bosque, F. Hassler, D. Lüst and E. Malek, *A geometric formulation of exceptional field theory*, *JHEP* **03** (2017) 004 [arXiv:1605.00385] [INSPIRE].

[7] M. Günaydin, D. Lüst and E. Malek, *Non-associativity in non-geometric string and M-theory backgrounds, the algebra of octonions and missing momentum modes*, *JHEP* **11** (2016) 027 [arXiv:1607.06474] [INSPIRE].

[8] V.G. Kupriyanov and R.J. Szabo, *G_2-structures and quantization of non-geometric M-theory backgrounds*, *JHEP* **02** (2017) 099 [arXiv:1701.02574] [INSPIRE].

[9] D. Lüst, E. Malek and R.J. Szabo, *Non-geometric Kaluza-Klein monopoles and magnetic duals of M-theory R-flux backgrounds*, *JHEP* **10** (2017) 144 [arXiv:1705.09639] [INSPIRE].

[10] D. Lüst, E. Malek and M. Syvari, *Locally non-geometric fluxes and missing momenta in M-theory*, *JHEP* **01** (2018) 050 [arXiv:1710.05919] [INSPIRE].

[11] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, *Phys. Rev.* **D 47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[12] W. Siegel, *Superspace duality in low-energy superstrings*, *Phys. Rev.* **D 48** (1993) 2826 [hep-th/9305073] [INSPIRE].

[13] C. Hull and B. Zwiebach, *Double field theory*, *JHEP* **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[14] C.M. Hull, *Generalised geometry for M-theory*, *JHEP* **07** (2007) 079 [hep-th/0701203] [INSPIRE].

[15] P. Pires Pacheco and D. Waldram, *M-theory, exceptional generalised geometry and superpotentials*, *JHEP* **09** (2008) 123 [arXiv:0804.1362] [INSPIRE].

[16] Y. Hagiwara, *Nambu-Dirac manifolds*, *J. Phys.* **A 35** (2002) 1263.

[17] G. Bonelli and M. Zabzine, *From current algebras for p-branes to topological M-theory*, *JHEP* **09** (2005) 015 [hep-th/0507051] [INSPIRE].

[18] M. Zambon, *L_∞ algebras and higher analogues of Dirac structures and Courant algebroids*, *J. Symplectic Geom.* **10** (2012) 563 [arXiv:1003.1004] [INSPIRE].

[19] Y. Bi and Y. Sheng, *On higher analogues of Courant algebroids*, *Sci. China Math.* **A 54** (2011) 437 [arXiv:1003.1350].

[20] P. Bouwknegt and B. Jurčo, *AKSZ construction of topological open p-brane action and Nambu brackets*, *Rev. Math. Phys.* **25** (2013) 1330004 [arXiv:1110.0134] [INSPIRE].

[21] T.J. Courant, *Dirac manifolds*, *Trans. Amer. Math. Soc.* **319** (1990) 631.

[22] Z.-J. Liu, A. WEinstein and P. Xu, *Manin triples for Lie bialgebroids*, *J. Diff. Geom.* **45** (1997) 547 [dg-ga/9508013] [INSPIRE].

[23] N. Halmagyi, *Non-geometric backgrounds and the first order string σ-model*, arXiv:0906.2891 [INSPIRE].

[24] R. Blumenhagen, A. Deser, E. Plauschinn and F. Rennecke, *Bianchi identities for non-geometric fluxes — From quasi-Poisson structures to Courant algebroids*, *Fortsch. Phys.* **60** (2012) 1217 [arXiv:1205.1522] [INSPIRE].

[25] N. Ikeda, *Chern-Simons gauge theory coupled with BF theory*, *Int. J. Mod. Phys.* **A 18** (2003) 2689 [hep-th/0203043] [INSPIRE].

[26] J.-S. Park, *Topological open p-branes*, in the proceedings of the *Symplectic geometry and mirror symmetry*, 4th *KIAS Annual International Conference*, August 14–18, Seoul, South Korea (2000), hep-th/0012141 [INSPIRE].

[27] C. Hofman and J.-S. Park, *BV quantization of topological open membranes*, *Commun. Math. Phys.* **249** (2004) 249 [hep-th/0209214] [INSPIRE].

[28] D. Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, math/0203110 [INSPIRE].

[29] D. Roytenberg, *AKSZ-BV formalism and Courant algebroid-induced topological field theories*, *Lett. Math. Phys.* **79** (2007) 143 [hep-th/0608150] [INSPIRE].

[30] M. Henneaux, *Lectures on the antifield-BRST formalism for gauge theories*, *Nucl. Phys. Proc. Suppl.* **18A** (1990) 47 [INSPIRE].

[31] J. Gomis, J. Paris and S. Samuel, *Antibracket, antifields and gauge theory quantization*, *Phys. Rept.* **259** (1995) 1 [hep-th/9412228] [INSPIRE].

[32] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, *Int. J. Mod. Phys.* **A 12** (1997) 1405 [hep-th/9502010] [INSPIRE].

[33] D. Mylonas, P. Schupp and R.J. Szabo, *Membrane σ-models and quantization of non-geometric flux backgrounds*, *JHEP* **09** (2012) 012 [arXiv:1207.0926] [INSPIRE].

JHEP05(2019)055

[34] A. Chatzistavrakidis, L. Jonke and O. Lechtenfeld, *σ-models for genuinely non-geometric backgrounds*, *JHEP* **11** (2015) 182 [arXiv:1505.05457] [INSPIRE].

[35] T. Bessho, M.A. Heller, N. Ikeda and S. Watamura, *Topological membranes, current algebras and H-flux–R-flux duality based on Courant algebroids*, *JHEP* **04** (2016) 170 [arXiv:1511.03425] [INSPIRE].

[36] A. Chatzistavrakidis, L. Jonke, F.S. Khoo and R.J. Szabo, *Double field theory and membrane σ-models*, *JHEP* **07** (2018) 015 [arXiv:1802.07003] [INSPIRE].

[37] C.M. Hull and R.A. Reid-Edwards, *Flux compactifications of M-theory on twisted tori*, *JHEP* **10** (2006) 086 [hep-th/0603094] [INSPIRE].

[38] I. Bakhmatov et al., *Exotic branes in exceptional field theory: the* SL(5) *duality group*, *JHEP* **08** (2018) 021 [arXiv:1710.09740] [INSPIRE].

[39] N. Ikeda and K. Uchino, *QP-structures of degree* 3 *and* 4*D topological field theory*, *Commun. Math. Phys.* **303** (2011) 317 [arXiv:1004.0601] [INSPIRE].

[40] N. Ikeda, *Lectures on AKSZ σ-models for Physicists*, in the proceedings of the *Workshop on Strings, Membranes and Topological Field Theory (WSPC)*, arXiv:1204.3714 [INSPIRE].

[41] M. Grützmann, *H-twisted Lie algebroids*, *J. Geom. Phys.* **61** (2011) 476 [arXiv:1005.5680].

[42] M. Grützmann and T. Strobl, *General Yang-Mills type gauge theories for p-form gauge fields: From physics-based ideas to a mathematical framework or From Bianchi identities to twisted Courant algebroids*, *Int. J. Geom. Meth. Mod. Phys.* **12** (2014) 1550009 [arXiv:1407.6759].

[43] U. Carow-Watamura et al., *Off-shell covariantization of algebroid gauge theories*, *PTEP* **2017** (2017) 083B01 [arXiv:1612.02612].

[44] Z. Kökényesi, A. Sinkovics and R.J. Szabo, *AKSZ constructions for topological membranes on G_2-manifolds*, *Fortsch. Phys.* **66** (2018) 1800018 [arXiv:1802.04581] [INSPIRE].

[45] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math.* **54** (2003) 281 [math/0209099] [INSPIRE].

[46] M. Gualtieri, *Generalized complex geometry*, Ph.D. thesis, Oxford University, Oxford, U.K. (2003), math/0401221 [INSPIRE].

[47] D. Roytenberg, *Courant algebroids, derived brackets and even symplectic supermanifolds*, Ph.D. Thesis, University of California at Berkeley, Berkeley, U.S.A. (1999), math/9910078.

[48] Y. Kosmann-Schwarzbach, *Quasi, twisted, and all that... in Poisson geometry and Lie algebroid theory*, *Progr. Math.* **232** (2005) 363 [math/0310359].

[49] D. Andriot, M. Larfors, D. Lüst and P. Patalong, *A ten-dimensional action for non-geometric fluxes*, *JHEP* **09** (2011) 134 [arXiv:1106.4015] [INSPIRE].

[50] A. Chatzistavrakidis, L. Jonke and O. Lechtenfeld, *Dirac structures on nilmanifolds and coexistence of fluxes*, *Nucl. Phys.* **B 883** (2014) 59 [arXiv:1311.4878] [INSPIRE].

[51] C.M. Hull and R.A. Reid-Edwards, *Flux compactifications of string theory on twisted tori*, *Fortsch. Phys.* **57** (2009) 862 [hep-th/0503114] [INSPIRE].

[52] T. Asakawa, H. Muraki, S. Sasa and S. Watamura, *Poisson-generalized geometry and R-flux*, *Int. J. Mod. Phys.* **A 30** (2015) 1550097 [arXiv:1408.2649] [INSPIRE].

[53] M.J. Duff and J.X. Lu, *Duality rotations in membrane theory*, *Nucl. Phys.* **B 347** (1990) 394 [INSPIRE].

JHEP05(2019)055

[54] M.J. Duff et al., *Membrane duality revisited*, *Nucl. Phys.* **B 901** (2015) 1 [arXiv:1509.02915] [INSPIRE].

[55] C. Hull and B. Zwiebach, *The gauge algebra of double field theory and Courant brackets*, *JHEP* **09** (2009) 090 [arXiv:0908.1792] [INSPIRE].

[56] D.S. Berman, H. Godazgar, M. Godazgar and M.J. Perry, *The local symmetries of M-theory and their formulation in generalised geometry*, *JHEP* **01** (2012) 012 [arXiv:1110.3930] [INSPIRE].

[57] R. Blumenhagen, X. Gao, D. Herschmann and P. Shukla, *Dimensional oxidation of non-geometric fluxes in type II orientifolds*, *JHEP* **10** (2013) 201 [arXiv:1306.2761] [INSPIRE].

[58] H. Samtleben and M. Weidner, *The maximal $D = 7$ supergravities*, *Nucl. Phys.* **B 725** (2005) 383 [hep-th/0506237] [INSPIRE].

[59] D. Baraglia, *Leibniz algebroids, twistings and exceptional generalized geometry*, *J. Geom. Phys.* **62** (2012) 903 [arXiv:1101.0856] [INSPIRE].

[60] M. Cederwall and J. Palmkvist, *L_∞ algebras for extended geometry from Borcherds superalgebras*, arXiv:1804.04377 [INSPIRE].

[61] A.S. Arvanitakis, *Brane Wess-Zumino terms from AKSZ and exceptional generalised geometry as an L_∞-algebroid*, arXiv:1804.07303 [INSPIRE].

[62] Y. Cagnacci, T. Codina and D. Marques, *L_∞ algebras and tensor hierarchies in exceptional field theory and gauged supergravity*, *JHEP* **01** (2019) 117 [arXiv:1807.06028] [INSPIRE].

[63] D.S. Berman, E.T. Musaev and R. Otsuki, *Exotic branes in exceptional field theory: $E_{7(7)}$ and beyond*, *JHEP* **12** (2018) 053 [arXiv:1806.00430] [INSPIRE].

[64] D.S. Berman, H. Godazgar, M.J. Perry and P. West, *Duality invariant actions and generalised geometry*, *JHEP* **02** (2012) 108 [arXiv:1111.0459] [INSPIRE].

[65] O. Hohm and H. Samtleben, *Exceptional form of $D = 11$ supergravity*, *Phys. Rev. Lett.* **111** (2013) 231601 [arXiv:1308.1673] [INSPIRE].

[66] O. Hohm and H. Samtleben, *Exceptional field theory I: $E_{6(6)}$ covariant Form of M-theory and type IIB*, *Phys. Rev.* **D 89** (2014) 066016 [arXiv:1312.0614] [INSPIRE].

[67] O. Hohm and H. Samtleben, *Exceptional field theory. II. $E_{7(7)}$*, *Phys. Rev.* **D 89** (2014) 066017 [arXiv:1312.4542] [INSPIRE].

[68] O. Hohm and H. Samtleben, *Exceptional field theory. III. $E_{8(8)}$*, *Phys. Rev.* **D 90** (2014) 066002 [arXiv:1406.3348] [INSPIRE].

[69] Z. Kökényesi, A. Sinkovics and R.J. Szabo, *Double field theory for the A/B-models and topological S-duality in generalized geometry*, *Fortsch. Phys.* **66** (2018) 1800069 [arXiv:1805.11485] [INSPIRE].