ЧЕБЫШЕВСКИЙ СБОРНИК
Том 20. Выпуск 2.

УДК 511.3 DOI 10.22405/2226-8383-2019-20-2-161-169

Гиперметрический конус и многогранник на графах

М. Дьютор

Дютур-Сикирич Мэтью — Старший Научный Сотрудник Института Рудьер Боскович Биеница, Загреб, Хорватия
e-mail: mathieu.dutour@gmail.com

Аннотация

Гиперметрический конус был определен в [9] и был широко изучен Мишелем Дезой и его сотрудниками. Еще одним ключевым его интересом был отрезной и метрический многогранник, который он рассматривал в своих последних работах в случае графов.

Здесь мы объединяем оба интереса, рассматривая гиперметрию на графах. Мы определяем их для любого графа и даем алгоритм вычисления экстремальных лучей и граней гиперметрического конуса на графах. Мы вычисляем гиперметрический конус для первого нетривиального случая $K_7 - \{e\}$. Мы также вычисляем гиперметрический конус в случае графов без K_5 минора.

Ключевые слова: алгебраические решётки, алгебраические сетки, тригонометрические суммы алгебраических сеток с весами, весовые функции.

Библиография: 21 название.

Для цитирования:
M. Dutour. The hypermetric cone and polytope on graphs // Чебышевский сборник, 2019, т. 20, вып. 2, с. 161–169.

CHEBYSHEVSKII SBORNIK
Vol. 20. No. 2.

UDC 511.3 DOI 10.22405/2226-8383-2019-20-2-161-169

The hypermetric cone and polytope on graphs

M. Dutour

M. Dutour — Senior Associate Researcher, Institut Rudjer Boskovic Bijenicka, Zagreb, Croatia
e-mail: mathieu.dutour@gmail.com

Abstract

The hypermetric cone was defined in [9] and was extensively studied by Michel Deza and his collaborators. Another key interest of him was cut and metric polytope which he considered in his last works in the case of graphs.

Here we combine both interest by considering the hypermetric on graphs. We define them for any graph and give an algorithm for computing the extreme rays and facets of hypermetric cone on graphs. We compute the hypermetric cone for the first non-trivial case of $K_7 - \{e\}$. We also compute the hypermetric cone in the case of graphs with no K_5 minor.

Keywords: algebraic lattices, algebraic net, trigonometric sums of algebraic net with weights, weight functions.

Bibliography: 21 titles.

For citation:
M. Dutour, 2019, "The hypermetric cone and polytope on graphs *Chebyshevskii sbornik*, vol. 20, no. 2, pp. 161–169.

1. Introduction

Given an integer n and a vector $b \in \mathbb{Z}^n$ such that $\sum_i b_i = 1 + 2s$ with $s \in \mathbb{Z}$ the *hypermetric inequality* is defined as

$$H(b,d) = \sum_{1 \le i < j \le n} b_i b_j d(i,j) \le s(s+1)$$

The *hypermetric cone* $\mathrm{HYP}(K_n)$ is the set of functions $d : \{1,\dots,n\}^2 \to \mathbb{R}$, such that $H(b,d) \le 0$ is satisfied for all $b \in \mathbb{Z}^n$ with $\sum_i b_i = 1$. The elements of $\mathrm{HYP}(K_n)$ are named *hypermetric.*

A priori the hypermetric cone is not polyhedral since it is defined by an infinity of inequalities. However, the polyhedrality of the hypermetric cone was proved in [19, 9, 10]. The hypermetric cone is interpreted in term of parameter space of Delaunay polytopes and this viewpoint was introduced first in [3]. A complete description of the facets of the hypermetric cone was achieved in [4] for $n = 6$, [5, 20, 7] for $n = 7$ and [8].

Another viewpoint for the parameter space of Delaunay polytopes is the *Erdahl cone*. It is the set of quadratic functions on $\mathbb{R}^n$ such that $f(x) \ge 0$ for $x \in \mathbb{Z}^n$. This viewpoint is used and developed in [18, 17, 14].

The *hypermetric polytope* $\mathrm{HYPP}(K_n)$ is the set of functions $d : \{1,\dots,n\}^2 \to \mathbb{R}$, such that $H(b,d) \le s(s+1)$ is satisfied for all $b \in \mathbb{Z}^n$ with $\sum_i b_i = 1 + 2s$ and $s \in \mathbb{Z}$. It was defined in [8, 15] and it is related to centrally symmetric Delaunay polytopes (see 2.2 for some summary).

In this work we define the hypermetric cone $\mathrm{HYP}(G)$ and polytope $\mathrm{HYPP}(G)$ of a graph G. This extends the construction of cut and metric cone and polytopes of graphs (see [19, 11]). We provide algorithms for checking if an hypermetric belong to those.

In Section 2 we provide needed preliminaries on cut, metric cones and polytopes of graphs G. We also define the hypermetric cone and polytopes for the complete graph K_n.

In Section 4 we define the hypermetric cone and polytope for a graph G. In Theorem 3 we give algorithms to test if a distance function on the edge set of a graph is an hypermetric or not.

In Section 4 we compute the facets and extreme rays of the first non-trivial case $K_7 - \{e\}$. We also prove that for graphs with k edges removed, the facet defining inequalities are obtained as sum of at most k hypermetric inequalities. We also characterize the facets of the hypermetric cone of graphs without K_5 minor.

Characterizing the facet inequalities of other graphs is an interesting problem. In particular one good question is characterize the graphs G for which $\mathrm{MET}(G) = \mathrm{HYP}(G)$ or $\mathrm{HYP}(G) = \mathrm{CUT}(G)$.

2. Preliminary definitions

2.1. Cut and metric cones and polytopes

Given a graph $G = (V,E)$ with $n = |V|$, for a vertex subset $S \subseteq V = \{1,\dots,n\}$, the *cut semimetric* $\delta_S(G)$ is a vector (actually, a symmetric $\{0,1\}$-matrix) defined as

$$\delta_S(x,y) = \begin{cases} 1 & \text{if } \{x,y\} \in E \quad \text{and} \quad |S \cap \{x,y\}| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

If G is connected, which will be the case in our work, there are exactly 2^{n-1} distinct cut semimetrics. The *cut polytope* $\mathrm{CUTP}(G)$ and the *cut cone* $\mathrm{CUT}(G)$ are defined as the convex hull of all such semimetrics and the positive span of all non-zero ones among them, respectively. Their number of vertices, respectively extreme rays is 2^{n-1}, respectively $2^{n-1}-1$ and their dimension is $|E|$, i.e., the number of edges of G.

The *metric cone* $\mathrm{MET}(K_n)$ is the set of all *semimetrics* on n points, i.e., the functions $d : \{1, \dots, n\}^2 \to \mathbb{R}_{\geqslant 0}$ satisfying $d(i,i) = 0$, $d(i,j) = d(j,i)$ for $1 \le i,j \le n$ and the *triangle inequalities*

$$d(i,j) \le d(i,k) + d(j,k) \text{ for } 1 \le i,j,k \le n. \tag{1}$$

The *metric polytope* $\mathrm{METP}(K_n)$ is defined as the elements of $\mathrm{MET}(K_n)$ satisfying the *perimeter inequalities*

$$d(i,j) + d(j,k) + d(k,i) \le 2 \text{ for } 1 \le i,j,k \le n. \tag{2}$$

For a graph $G = (V,E)$ of *order* $|V| = n$, let $\mathrm{MET}(G)$ and $\mathrm{METP}(G)$ denote the projections of $\mathrm{MET}(K_n)$ and $\mathrm{METP}(K_n)$, respectively, on the subspace $\mathbb{R}^E$ indexed by the edge set of G. Clearly, $\mathrm{CUT}(G)$ and $\mathrm{CUTP}(G)$ are the projections of, respectively, $\mathrm{CUT}(K_n)$ and $\mathrm{CUTP}(K_n)$ on $\mathbb{R}^E$. We have the relaxation property:

$$\mathrm{CUT}(G) \subseteq \mathrm{MET}(G) \text{ and } \mathrm{CUTP}(G) \subseteq \mathrm{METP}(G).$$

Definition 9. *Let $G = (V,E)$ be a graph.*
(i) Given an edge $e \in E$, the edge inequality *is*

$$x(e) \ge 0.$$

(ii) For a cycle C, and an odd size set $F \subseteq C$ the cycle inequality *is*

$$x(F) - x(C \setminus F) \le |F| - 1$$

where $x(U) = \sum_{u \in U} x(u)$.

$\mathrm{METP}(G)$ is defined by all edge, bounding inequality $x(e) \le 1$ and s-cycle inequalities, while $\mathrm{MET}(G)$ is defined by all edge inequalities and s-cycle inequalities with $|F| = 1$ (see [2] and [12, Section 27.3]).

2.2. Hypermetric cone and Delaunay polytope

By a *distance matrix* $D = (D_{i,j})_{0 \le i,j \le n}$ we mean a matrix with $D_{i,i} = 0$ and $D_{i,j} = D_{j,i}$. We define $e_0 = 0$, $e_1 = (1,0,\dots,0)$, $\dots$, $e_n = (0,\dots,0,1)$.

We can associate to this a $n \times n$ symmetric matrix Q, a vector $v \in \mathbb{R}^n$, a scalar $c \in \mathbb{R}$ and a function f defined on $\mathbb{R}^n$ by

$$f(x) = Q[x] + \langle v, x \rangle + c$$

and which satisfies $f(e_0) = f(e_1) = \cdots = f(e_n) = 0$. The matrix D satisfies $D_{i,j} = Q[e_i - e_j]$.

This correspondence relates the hypermetric cone $\mathrm{HYP}(K_n)$ with the Erdahl cone. See for example [14]. In other words we have that $D \in \mathrm{HYP}(K_{n+1})$ if and only if $f(x) \ge 0$ for all $x \in \mathbb{Z}^n$.

If we express the problem purely in term of geometry of numbers what we have is that a distance matrix $D \in \mathrm{HYP}(K_{n+1})$ if and only there exist a k-dimensional lattice L with $k \le n$, a Delaunay polytopes P of L and $D_{i,j} = \|v_i - v_j\|^2$ for some vertices v_i of P (see [12, Chapter 2]).

Given a $D \in \mathrm{HYP}(K_{n+1})$ which correspond to a Delaunay polytope P of dimension n. Then the set of vertices corresponds to the set of vector $b \in \mathbb{Z}^{n+1}$ with $H(b,D) = 0$. In this respect the set v_0, v_1, $\dots$, v_n form an *affine basis*. The set of vertices of P is then expressed as $\sum_{i=0}^{n} b_i v_i$ for $H(b,D) = 0$. This can be used to describe the Delaunay polytopes and this method was used in [13] for describing the Delaunay polytopes of dimension six.

Unfortunately, while powerful the method of hypermetric does not work out completely because there are Delaunay polytopes which are not basic (see [16]). We found out that the Erdahl cone provides a better replacement in many contexts (see [18, 17, 14]).

The symmetries of the hypermetric cone $\mathrm{HYP}(K_n)$ is $Sym(n)$ for $n \ne 4$ (see [6]).

2.3. Hypermetric polytope and centrally symmetric Delaunay polytope

We cite following [8, Theorem 6]:

THEOREM 1. *A distance function d belongs to* $\mathrm{HYPP}(K_n)$ *if and only if there exist a centrally symmetric n-dimensional Delaunay polytope of center c, circumradius* 1 *and vertices v_i, $2c - v_i$ for $1 \le i \le n$ with $\|v_i - v_j\|^2 = 4d_{ij}$.*

The correspondence can be made more precise. Consider the lattice

$$L = \left\{ x \in \mathbb{Z}^{n+1} \text{ s.t. } \sum_i x_i \equiv 0 \pmod 2 \right\}.$$

and the point $b^0 = (1, 0^{n-1})$. We define the matrix

$$A(d) = (A_{ij})_{1 \le i,j \le n} \text{ with } A_{ij} = \begin{cases} 1 & \text{if } i = j \\ 1 - 2d_{ij} & \text{otherwise.} \end{cases}$$

Then we have that $d \in \mathrm{HYPP}(K_n)$ if and only if we have (see [8]):

$$b^t A(d) b \ge 1 \text{ for all } b \in b^0 + L. \tag{3}$$

The set of $b \in b^0 + L$ such that $b^t A(d) b = 1$ correspond to the vertices of the centrally symmetric Delaunay polytope of center b^0 for the lattice L. The points $\pm e_i$ for $1 \le i \le n$ will always be among those.

It is important to point out that it is unlikely that all centrally symmetric Delaunay polytopes could be expressed in this way because of the negative result [16] but we do not know any counterexample. However, we could construct a variant of the Erdahl cone for this centrally symmetric setting

$$Erdahl_{Cent}(n) = \{Q \in S^n \text{ such that } Q[x - e_1/2] \ge 1\}$$

with S^n the set of $n \times n$ quadratic forms. The center of the Delaunay polytope will be $e_1/2$ and the circumradius 1. The vertices of the Delaunay polytope will be the set of $x \in \mathbb{Z}^n$ such that $Q[x - e_1/2] = 1$.

3. Definition of hypermetric cone on graphs

The hypermetric cone $\mathrm{HYP}(K_n)$ is defined as the set of metrics satisfying all hypermetric inequalities, that is

$$\mathrm{HYP}(K_n) = \left\{ d \in \mathbb{R}^{E(K_n)} \text{ with } H(b,d) \le 0 \text{ for } b \in \mathbb{Z}^n, \sum_i b_i = 1 \right\}.$$

DEFINITION 10. *Given a graph G on n vertices the hypermetric cone* $\mathrm{HYP}(G)$ *is defined as the projection of* $\mathrm{HYP}(K_n)$ *on* $\mathbb{R}^{E(G)}$.

Since we know that $\mathrm{HYP}(K_n) = \mathrm{CUT}(K_n)$ for $n \le 6$ we have $\mathrm{HYP}(G) = \mathrm{CUT}(G)$ for G on graph on at most 6 vertices.

Another elementary property is that $\mathrm{HYP}(G)$ is polyhedral since it is the projection of a polyhedral cone.

THEOREM 2. *Given a graph G on n vertices and a $d \in \mathrm{HYP}(G)$. The set of possible distances $D \in \mathrm{HYP}(K_n)$ such that $proj(D) = d$ is bounded if and only G is a connected graph.*

PROOF. Given a $d \in \mathrm{HYP}(G)$ with G being connected. Given two vertices v and w there exist a path $v = v_0, v_1, \ldots, v_m = w$ with v_i adjacent to v_j. By iterating the triangle inequality one obtains

$$D(v,w) \leq d(v_0, v_1) + \cdots + d(v_{m-1}, v_m) = C(d)$$

The value $C(d)$ does not depend on D and so the set of possible D is connected.

If $d \in \mathrm{HYP}(G)$ then there exist at least one D with $Proj(D) = d$. Since G is not connected, there exist a subset $S \subset \{1, \ldots, n\}$ with vertices in S and $\{1, \ldots, n\} - S$ not being adjacent. As a consequence we have $Proj(\delta_S) = 0$. Since $\delta_S \in \mathrm{HYP}(K_n)$ we have that for all $\alpha > 0$ the relation $Proj(D + \alpha\delta_S) = d$ with $D + \alpha\delta_S \in \mathrm{HYP}(K_n)$. ■

THEOREM 3. *Given a graph G on n vertices, testing if a given $d \in \mathbb{R}^{E(G)}$ belongs to* HYP(G) *can be done by iteratively solving linear programs.*

PROOF. We take a distance function $d \in \mathbb{R}^{E(G)}$ and we want to find a matrix D with $Proj(D) = d$. Thus we need to find the possible values D_{ij} with $\{i,j\} \notin E(G)$.

We need to solve following linear program:

$$\begin{array}{cc} \text{minimize} & \sum_{\{i,j\} \notin E(G)} D_{i,j} \\ \text{satisfying to} & H(b, D) \leq 0 \\ & \text{for } b \in \mathbb{Z}^n \text{ with } \sum_i b_i = 1 \end{array}$$

In other words what we have is an infinite linear program. What we can solve only is finite linear system.

The algorithm for solving that is to start from a finite set S of vectors b and then gradually expand it until a conclusion is reached. The finite starting point is the triangular inequalities of the metric cone. Then we iterate:

(i) We solve the program for a fixed set S.

(ii) If the program is unfeasible then this means that the elements d does not belong to HYP(K_n). The problem is resolved.

(iii) If the optimal solution D_0 of the linear program belongs to HYP(K_n) then the problem is resolved.

(iv) On the other hand if D_0 is not an hypermetric then there exist a d such that $H(b, D_0) > 0$. We add b in D and reiterate.

Since the hypermetric cone is polyhedral, after a finite set of addition one will eventually obtain a solution of the problem. ■

THEOREM 4. *If $f(x)$ is a linear function defined on $\mathbb{R}^{E(G)}$ then we can check whether f is valid on* HYP(G) *by a sequence of linear program.*

PROOF. Since f is defined on $\mathbb{R}^{E(G)}$ we can trivially extend it to $\mathbb{R}^{E(K_n)}$ by setting $f(e) = 0$ for all $e \notin E(G)$.

The idea is to consider the linear program

$$\begin{array}{cc} \text{minimize} & f(D) \\ \text{satisfying to} & H(b, D) \leq 0 \\ & \text{for } b \in \mathbb{Z}^n \text{ with } \sum_i b_i = 1 \\ & \text{and } \sum_{1 \leq i,j \leq n} D_{i,j} \leq 1. \end{array}$$

This infinite linear programming is very similar to the one of Theorem 3 and the same iterative strategy can be used. Let's denote D_0 the optimal solution which is an hypermetric. If $f(D_0) < 0$ then we have proved that f is not valid on HYP(G). If on the other hand $f(D_0) \geq 0$ then the inequality is valid. ■

In practice the implementation of the above algorithms can be fairly complex. The linear programs are large and hard to solve. In our implementation we use `cdd` which uses exact arithmetic and provides both primal and dual solution in exact rational arithmetic. However, `cdd` uses the simplex algorithm and is very small in some cases. The idea is then to use floating point arithmetic and the `glpk` program which has better algorithm and can approximately solve linear programs. From the approximate solution we can derive the incidence and from the incidence get an exact solution in most cases. If this approach fails, then we fall back to the more expensive in time `cdd`. We only accept an approximate solution if we can derive a primal and dual solution. In any case of failure we fall back to `cdd`.

If we have a distance matrix D checking if it belongs to HYP(K_n) is done in the following way. This defines a $n \times n$-rational quadratic form Q, a vector $v \in \mathbb{Q}^n$ and a scalar C such that

$$f(x) = Q[x] + \langle v, x\rangle + C$$

with $f(0) = f(e_1) = \cdots = f(e_n) = 0$. What we need is check if there is a $x \in \mathbb{Z}^n$ such that $f(x) < 0$. If Q is not positive semidefinite then we can find a negative eigenvalue and a corresponding eigenvector. By approximating the eigenvector with a rational vector and multiplying with some factor we can find a $x \in \mathbb{Z}^n$ with $f(x) < 0$. If Q is positive semidefinite then take the kernel $K = \{x \in \mathbb{Z}^n \text{ with } Q[x] = 0\}$ and L a subspace of $\mathbb{Z}^n$ such that $K \oplus_{\mathbb{Z}} L = \mathbb{Z}^n$. By restricting the problem to L we can restrict to the positive definite case. In the positive definite case, the decision problem of finding $x \in \mathbb{Z}^n$ with $f(x) < 0$ is a Closest Vector Problem and we can solve it by using a [19].

The interest of above two algorithms is that they give an algorithm for computing HYP(G). We can start with a list of hypermetric of full dimension in HYP(G). This is not difficult to obtain: We can for example take the cuts and project them on $\mathbb{R}^{E(G)}$.

Then we iterate the following:

(i) We compute the facets of the convex body defined by those hypermetrics.

(ii) For each facet we check if the corresponding facet defining inequality $f(x)$ is also valid on HYP(G).

(iii) If all inequalities are also valid on HYP(G) then we have computed the list of extreme rays and facets of HYP(G).
If not then we insert the hypermetrics that were found to be counterexample to the initial list of hypermetrics and reiterate.

Each insertion will increase the hypermetric cone until one has the complete description of HYP(G).

We haven't implemented the algorithms of this section and it would be hard to do so. The main measure of the complexity should be the number of edges of the graph because it is a direct measure of the complexity of the problem. A way to speed up the process is to use the symmetries of the graph for the computation. Based on that, the first interesting case would likely be HYP$(Petersen)$.

All of the above is for hypermetric cone. But we can just as well define the hypermetric polytope for a graph: Take HYPP(G) be the projection of HYPP(K_n) on $\mathbb{R}^{E(G)}$. The algorithms can be adapted just as well to this case.

4. Computing HYP(G) for some graphs

THEOREM 5. *The hypermetric cone* HYP($K_7 - \{e\}$) *has 8782 extreme rays and 7210 facets.*

PROOF. The hypermetric cone HYP(K_7) is defined by 3773 facet inequalities in 14 orbits and it has 37170 extreme rays in 29 orbits (see [7]). Let us denote $e = \{1, 2\}$. We consider the projection obtained by eliminating the component d_e. The normal to the equation $d_e = 0$ is the distance function determined by $d^e_{i,j} = 1$ if $\{i, j\} = e$ and 0 otherwise.

The facets F_i of the cone HYP(K_7) are defined by inequalities $f_i(d) \geq 0$. The facets of HYP($K_7 - \{e\}$) are obtained in two ways:

(i) The facets f_i of HYP(K_7) with $f_i(d^e) = 0$. The corresponding facet defining inequality of HYP($K_7 - \{e\}$) is $f_i(x) \geq 0$.

(ii) The ridges of HYP(K_7) obtained as intersection $F_i \cap F_j$ of two facets with $f_i(d^e) \times f_j(d^e) < 0$. We can find $\alpha > 0$ and $\beta > 0$ with $(\alpha f_i + \beta f_j)(d^e) = 0$. The corresponding facet defining inequality of HYP($K_7 - \{e\}$) is $\alpha f_i(x) + \beta f_j(x) \geq 0$.

By using this result and computing the ridges of HYP(K_7) we can get the facets of the projection in The symmetries of HYP($K_7 - \{e\}$) are induced by the symmetries of the graph $K_7 - \{e\}$ and the group is Sym($\{1, 2\}$) $\times$ Sym($\{3, 4, 5, 6, 7\}$) of order 240. The total number of orbits of facets is thus 7210 in 80 orbits.

The extreme rays of HYP($K_7 - \{e\}$)) are the projection of extreme rays of HYP(K_7). We check whether the projection are extreme rays or not by using the facets. This gives us 8782 extreme rays in 73 orbits. ■

THEOREM 6. *If G is a complete graph with k edges removed then the facets of* HYP(G) *are determined as a sum* $\sum_{i=1}^{m} \alpha_i H(b^i, d)$ *for* $m \leq k$, $\alpha_i > 0$ *and* $b^i \in \mathbb{Z}^n$ *with* $\sum_j b^i_j = 1$.

PROOF. The method of 5 can be generated to any n. It implies that the facets of HYP($K_n - \{e\}$) are induced by hypermetric inequalities and sum with positive coefficient of two hypermetric inequalities. The method extends to any number of edges and give us that the facets of HYP($K_n - \{e_1, \ldots, e_k\}$) are formed by hypermetric inequalities and sums with positive coefficients of at most k hypermetric inequalities. ■

Above theorem is in a sense a relatively negative result. For graphs with few edges it gives the facets as sums of too many hypermetric inequality to be practical.

THEOREM 7. *If G is a graph without K_5 minor then the facets of* HYP(G) *and* HYPP(G)) *are induced by the cycle inequalities and the non-negative inequalities.*

PROOF. By the result of [21, 1] we have that MET(G) = CUT(G) and METP(G) = CUTP(G) if G has no K_5 minor. Since we have

$$\text{CUT}(K_n) \subset \text{HYP}(K_n) \subset \text{MET}(K_n) \quad \text{and} \quad \text{CUTP}(K_n) \subset \text{HYPP}(K_n) \subset \text{METP}(K_n).$$

As a consequence we get that the facets of MET(G) = HYP(G) and METP(G) = HYPP(G). Thus the facets of MET(G) and METP(G) are induced by the cycle and facet inequalities. ■

Seymour's theorem [21] is even stronger and states that CUT(G) = MET(G) occurs if and only if G has no K_5 minor. Could it be that MET(G) = HYP(G) occurs for other graphs that have K_5 as minor?

Also interesting would be to characterize cases where HYP(G) = CUT(G). If G has at most 6 vertices then equality holds. The example of $K_7 - \{e\}$ shows that we cannot characterize the equality with K_6 minor.

REFERENCES

1. F. Barahona. The max-cut problem on graphs not contractible to K_5. *Oper. Res. Lett.*, 2(3):107–111, 1983.
2. F. Barahona. On cuts and matchings in planar graphs. *Math. Programming*, 60(1, Ser. A):53–68, 1993.
3. E. P. Baranovskiĭ. Volumes of L-simplexes of five-dimensional lattices. *Mat. Zametki*, 13:771–782, 1973.
4. E. P. Baranovskii. About l-simplexes of 6-dimensional lattices (in russian). In *Second International conference "Algebraic, Probabilistic, Geometrical, Combinatorial and Functional Methods in the theory of numbers.* 1995.
5. E. P. Baranovskii. The conditions for a simplex of 6-dimensional lattice to be l-simplex (in russian). *Nauch. Trud. Ivan.*, (2):18–24, 1999.
6. A. Deza, B. Goldengorin, and D. V. Pasechnik. The isometries of the cut, metric and hypermetric cones. *J. Algebraic Combin.*, 23(2):197–203, 2006.
7. M. Deza and M. Dutour. The hypermetric cone on seven vertices. *Experiment. Math.*, 12(4):433–440, 2003.
8. M. Deza and M. Dutour Sikirić. The hypermetric cone and polytope on eight vertices and some generalizations. *J. Symbolic Comput.*, 88:67–84, 2018.
9. M. Deza, V. P. Grishukhin, and M. Laurent. Extreme hypermetrics and L-polytopes. In *Sets, graphs and numbers (Budapest, 1991)*, volume 60 of *Colloq. Math. Soc. János Bolyai*, pages 157–209. North-Holland, Amsterdam, 1992.
10. M. Deza, V. P. Grishukhin, and M. Laurent. Hypermetrics in geometry of numbers. In *Combinatorial optimization (New Brunswick, NJ, 1992–1993)*, volume 20 of *DIMACS Ser. Discrete Math. Theoret. Comput. Sci.*, pages 1–109. Amer. Math. Soc., Providence, RI, 1995.
11. M. Deza and M. D. Sikirić. Enumeration of the facets of cut polytopes over some highly symmetric graphs. *Int. Trans. Oper. Res.*, 23(5):853–860, 2016.
12. M. M. Deza and M. Laurent. *Geometry of cuts and metrics*, volume 15 of *Algorithms and Combinatorics*. Springer, Heidelberg, 2010. First softcover printing of the 1997 original [MR1460488].
13. M. Dutour. The six-dimensional Delaunay polytopes. *European J. Combin.*, 25(4):535–548, 2004.
14. M. Dutour Sikirić. The seven dimensional perfect Delaunay polytopes and Delaunay simplices. *Canad. J. Math.*, 69(5):1143–1168, 2017.
15. M. Dutour Sikirić, M.-M. Deza, and E. I. Deza. Computations of metric/cut polyhedra and their relatives. In *Handbook of geometric constraint systems principles*, Discrete Math. Appl. (Boca Raton), pages 213–231. CRC Press, Boca Raton, FL, 2019.
16. M. Dutour Sikirić and V. Grishukhin. How to compute the rank of a Delaunay polytope. *European J. Combin.*, 28(3):762–773, 2007.

17. M. Dutour Sikirić, A. Schürmann, and F. Vallentin. Inhomogeneous extreme forms. *Ann. Inst. Fourier (Grenoble)*, 62(6):2227–2255 (2013), 2012.

18. R. Erdahl. A cone of inhomogeneous second-order polynomials. *Discrete Comput. Geom.*, 8(4):387–416, 1992.

19. U. Fincke and M. Pohst. Improved methods for calculating vectors of short length in a lattice, including a complexity analysis. *Math. Comp.*, 44(170):463–471, 1985.

20. S. S. Ryshkov and E. P. Baranovskii. Repartitioning complexes in n-dimensional lattices (with full description for $n \leq 6$). In *Proceedings of conference "Voronoi impact on modern science Book 2*, pages 115–124. 1998.

21. P. D. Seymour. Matroids and multicommodity flows. *European J. Combin.*, 2(3):257–290, 1981.

Получено 13.06.2019 г.
Принято в печать 12.07.2019 г.