Quantum twist-deformed $D = 4$ phase spaces with spin sector and Hopf algebroid structures

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A B S T R A C T

We consider the generalized $(10 + 10)$-dimensional $D = 4$ quantum phase spaces containing translational and Lorentz spin sectors associated with the dual pair of twist-quantized Poincare Hopf algebra $\mathbb{H}$ and quantum Poincare Hopf group $\widehat{\mathbb{G}}$. Two Hopf algebroid structures of generalized phase spaces with spin sector will be investigated: first one $\mathcal{H}^{(10,10)}$ describing dynamics on quantum group algebra $\widehat{\mathbb{G}}$ provides by the Heisenberg double algebra $\mathcal{HD} = \mathbb{H} \rtimes \widehat{\mathbb{G}}$, and second, denoted by $\tilde{\mathcal{H}}^{(10,10)}$, describing twisted Hopf algebroid with base space containing twisted noncommutative Minkowski space $\hat{x}_\mu$. We obtain the first explicit example of Hopf algebroid structure of relativistic quantum phase space which contains quantum-deformed Lorentz spin sector.

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1. Introduction

Following recent description of noncommutative quantum phase spaces as bialgebroids and Hopf algebroids (see [1–10]) we shall study in this paper such algebraic structures in quantum phase spaces derived from twisted quantum Poincare symmetries. In comparison with recent studies [5–10] the novelty in our approach is the appearance of Hopf algebroid description of $D = 4$ generalized relativistic quantum-deformed phase spaces with additional coordinates and momenta describing Lorentz spin sector.

The quantum-deformed relativistic phase spaces $\mathcal{H}^{(4,4)}$, spanned by the degrees of freedom $(\hat{x}_\mu, \hat{p}_\mu)$, are described by quantum deformations of canonical relativistic Heisenberg algebra[1]

$$\left[p_\mu, x_\nu \right] = -i\eta_{\mu\nu}, \qquad \left[x_\mu, x_\nu \right] = \left[p_\mu, p_\nu \right] = 0. \tag{1}$$

Such choice of phase space is suitable only for the description of standard spinless dynamics. In this paper we shall consider the generalized $D = 4$ quantum phase spaces $\mathcal{H}^{(10,10)}$ and $\tilde{\mathcal{H}}^{(10,10)}$. The additional generators $\hat{\Lambda}_{\mu\nu}$ will be given by quantum counterpart of six Lorentzian angles $(\hat{\Lambda}\hat{\Lambda}^T = \hat{\Lambda}^T\hat{\Lambda} = I)$ dual to the generators $M_{\mu\nu}$ describing quantum-deformed Lorentz algebra.

The phase space description of spin dynamics as Heisenberg double has its roots in the half century old idea of Souriau [11] and Kostant [12] who described symplectic dynamics of point-like objects by the geometry of cosets $\mathcal{K} = G/S$, where G is the space–time symmetry group, and S its so-called stability group. If G is the Poincare group $\mathcal{P}^4$, for providing the dynamics of spin degrees of freedom the coset $\mathcal{K}$ should include besides translations as well some Lorentz group parameters (see e.g. [13–17]). In such a way one can describe e.g. the infinite-dimensional spin multiplets by adding to coordinate sector the spinorial Weyl spinor coordinates $\eta_\alpha, \eta_{\dot{\alpha}} = \bar{\eta}_\alpha (\alpha = 1, 2)$ defined by fourdimensional coset of $SL(2; \mathbb{C})$. In this way one can introduce the generalized wave functions (classical fields) $\Psi(x; \eta_\alpha, \eta_{\dot{\alpha}})$ obtained if S is a suitable two-dimensional subgroup of Lorentz group.[2]

The Souriau–Kostant approach as well as its generalization to Poisson manifolds [23], [24] and Poisson–Lie theory [25–27] can be also extended to models based on noncommutative (NC) geometry. The construction of quantum-deformed NC space–times extended

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[1] We denote the undeformed canonical phase space variables and canonical Poincare algebra generators by letters without hats.

[2] It can be shown (see e.g. [18], [15]) that for $D = 4$ the generalized field $\psi(x; \eta, \bar{\eta})$ can be defined as introduced on 8-dimensional coset $\mathcal{P}^4/H_2$, where H_2 is generated by Lorentz generators $(M_{01} + M_{31}, M_{02} + M_{32})$ and $SL(2; \mathbb{C})/H_2$ is parametrized by a pair $(\eta_\alpha, \bar{\eta}_{\dot{\alpha}})$ of complex-conjugated $D = 4$ Weyl spinors. The idea of studying dynamics on 10-dimensional Poincare group manifold was early advocated by Lurcat [19], [14]. Other way of introducing additional variables describing spin is to supplement space–time by two-parameter auxiliary manifold described by the sphere S^2 [20–22].

by the additional quantum Lorentz group parameters has been recently also proposed in the case of quantum-deformed Poincare group (see e.g. [28,29] where the case $D = 2 + 1$ was considered).

In this paper we shall study firstly the generalized quantum-deformed phase space $\mathcal{H}^{(10,10)} = (\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}; p_\mu, M_{\mu\nu})$ with non-commutative generalized coordinates described by the algebraic manifold of full quantum $D = 4$ Poincare group (i.e. $S = 1$), and supplemented with dual algebra of generalized momenta defined by ten generators of quantum-deformed Poincare algebra. The quantization will be obtained by the twisting procedure considered recently in [10] (see also [30]), which generate the Lie-algebraic deformation of space–time translations algebra and the quadratic algebra describing the commutators of all ten quantum Poincare group generators $(\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu})$. Subsequently, following [1] we anticipate the Hopf algebroid structure describing the dynamics on algebraic quantum Poincare group manifold.

Further, following [2], [9] we introduce $(10 + 10)$-dimensional quantum phase space $\tilde{\mathcal{H}}^{(10,10)} = (\hat{x}_\mu, \tilde{\Lambda}_{\mu\nu}; p_\mu, M_{\mu\nu})$ with Hopf algebroid structure generated by twist-deformed $D = 4$ Poincare–Hopf algebra $\mathbb{H}$, with base space described by $\mathbb{H}$-module $\hat{X}_A = (\hat{x}_\mu, \tilde{\Lambda}_{\mu\nu})$, $(A = 1 \ldots 10)$, its noncommutativity structure determined by twist-deformed star product multiplication

$$\hat{X}_A \cdot \hat{X}_B \simeq X_A \star_{\mathcal{F}} X_B = m[\mathcal{F}^{-1} \circ (X_A \otimes X_B)]$$
$$= (\mathcal{F}^{-1}_{(1)} \triangleright X_A)(\mathcal{F}^{-1}_{(2)} \triangleright X_B) \qquad (2)$$

where the Drinfeld twist factor $\mathcal{F} = \sum \mathcal{F}_{(1)} \otimes \mathcal{F}_{(2)} \in \mathbb{H} \otimes \mathbb{H}$ satisfies the two-cocycle condition [2], [31]. In nondeformed relativistic theory one can identify the translations fourvectors $\hat{\xi}_\mu$ and Minkowski space–time coordinates $\hat{x}_\mu$; in the presence of quantum deformations the algebraic properties of base spaces $(\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu})$ and $(\hat{x}_\mu, \tilde{\Lambda}_{\mu\nu})$ are usually different.[3]

We shall consider therefore two possible types of Hopf algebroids describing quantum phase spaces which are defined by twist-quantized Hopf algebra $\mathbb{H}$ (quantum deformed Poincare algebra) defining generalized momenta and generalized coordinates, given by two choices of the $\mathbb{H}$-module:

i) described by Heisenberg double $\mathcal{HD} = \mathbb{H} \rtimes \widehat{\mathbb{G}}$ defined by smash product[4] of dual Hopf algebras $\mathbb{H}, \widehat{\mathbb{G}}$, with built-in Hopf action $h \blacktriangleright \hat{g} = \hat{g}_{(1)}\langle h, \hat{g}_{(2)} \rangle$ and $h \blacktriangleright (\hat{g}\hat{g}') = (h_{(1)} \blacktriangleright \hat{g})(h_{(2)} \blacktriangleright \hat{g}')$ $(h \in \mathbb{H}; \hat{g}, \hat{g}' \in \widehat{\mathbb{G}}; \Delta(h) = h_{(1)} \otimes h_{(2)})$. The associative multiplication formula in $\widehat{\mathbb{G}} \otimes \mathbb{H}$ algebra is given by

$$(\hat{g} \otimes h)(\hat{g}' \otimes h') = \hat{g}(h_{(1)} \blacktriangleright \hat{g}') \otimes h_{(2)}h' \qquad (3)$$

where $\widehat{\mathbb{G}}$ describes generalized coordinates.

Heisenberg double data are specified by the pair of dual Hopf algebras $(\mathbb{H}, \widehat{\mathbb{G}})$ with the pairing $\langle \cdot, \cdot \rangle$ (see also footnote 4), which expresses the duality of $\mathbb{H}$ and $\widehat{\mathbb{G}}$ by the formulae

$$\langle h, \hat{g}\hat{g}' \rangle = \langle \Delta(h), \hat{g} \otimes \hat{g}' \rangle \qquad (4)$$
$$\langle hh', \hat{g} \rangle = \langle h \otimes h', \Delta(\hat{g}) \rangle . \qquad (5)$$

Heisenberg doubles provide the associative algebras of quantum phase spaces endowed with quantum-deformed NC symplectic structure (see e.g. [27]), however without the Hopf-algebraic coalgebra sector.[5]

ii) We introduce the twist-deformed coordinate sector described by $\hat{X}_A \in \widehat{\mathbb{X}}$, where $\widehat{\mathbb{X}}$ is the $\mathbb{H}$-module algebra, however not obtained from $\mathbb{H}$ via Hopf-algebraic duality.[6] We shall attach to the quantum phase space algebra $\tilde{\mathcal{H}}^{(10,10)} = (\hat{x}_\mu, \tilde{\Lambda}_{\mu\nu}; p_\mu, M_{\mu\nu})$ the coalgebraic sector and antipodes, which define the twisted Hopf algebroid structure (see [2], [9]). For twisted Hopf algebroid $\tilde{\mathcal{H}}^{(10,10)}$ the base algebra $(\widehat{\mathbb{X}}, \star_{\mathcal{F}})$ is endowed with the multiplication defined by the star product (2), which describes the generalized NC coordinates $\hat{X}_A = (\hat{x}_\mu, \tilde{\Lambda}_{\mu\nu})$, which will describe the Lie-algebraic type of noncommutativity.

The plan of our paper is the following: In Sect. 2 following [10] and [30] we shall describe the pair of dual twist-deformed Poincare–Hopf algebras $\mathbb{H}, \widehat{\mathbb{G}}$. In Sect. 3 we shall introduce the generalized $D = 4$ quantum phase spaces $\mathcal{H}^{(10,10)}$ containing space–time translations generators $\hat{\xi}_\mu$ as well as $\tilde{\mathcal{H}}^{(10,10)}$ with the quantum Minkowski space–time coordinates $\hat{x}_\mu$. In Sect. 4 by following Xu twisted bialgebroids framework [2] we will show how to get for the quantum phase space $\tilde{\mathcal{H}}^{(10,10)}$ the explicit Hopf algebroid structure. In Sect. 5 we present final remarks and comments on possible extensions of presented results.

2. Dual pair of twisted Hopf algebras

2.1. Twist-deformed Poincare algebra $\mathbb{H}$

The classical $D = 4$ Poincare–Hopf algebra looks as follows

$$[p_\mu, p_\nu] = 0 \qquad (6)$$
$$[M_{\mu\nu}, p_\rho] = \eta_{\nu\rho} p_\mu - \eta_{\mu\rho} p_\nu \qquad (7)$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\sigma} - \eta_{\mu\rho} M_{\nu\sigma} - \eta_{\nu\sigma} M_{\mu\rho} + \eta_{\mu\sigma} M_{\nu\rho} \qquad (8)$$

where $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$ and

$$\Delta_0(p_\mu) = p_\mu \otimes 1 + 1 \otimes p_\mu \quad \Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} \qquad (9)$$
$$S_0(p_\mu) = -p_\mu \quad S_0(M_{\mu\nu}) = -M_{\mu\nu} \qquad (10)$$
$$\epsilon_0(p_\mu) = 0 \quad \epsilon_0(M_{\mu\nu}) = 0. \qquad (11)$$

We define twist $\mathcal{F}$ as an element of $\mathbb{H} \otimes \mathbb{H}$ $(\mathbb{H} = \mathcal{U}(\mathcal{P}))$ which has an inverse, satisfies the cocycle condition

$$\mathcal{F}_{12} (\Delta_0 \otimes 1) \mathcal{F} = \mathcal{F}_{23} (1 \otimes \Delta_0) \mathcal{F} \qquad (12)$$

and the normalization condition

$$(\epsilon \otimes 1)\mathcal{F} = (1 \otimes \epsilon)\mathcal{F} = 1 \qquad (13)$$

where $\mathcal{F}_{12} = \mathcal{F} \otimes 1$ and $\mathcal{F}_{23} = 1 \otimes \mathcal{F}$. It is known, that $\mathcal{F}$ does not modify the algebraic part and the counit, but changes the coproducts and the antipodes in the following way

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[3] In the canonical case of $\theta_{\mu\nu}$-deformation this difference is exposed e.g. in [30], [32].

[4] Smash product algebra (see e.g. [33]) is a special kind of cross-product algebra $\mathbb{H} \rtimes \mathbb{V}$ when the left $\mathbb{H}$-module $\mathbb{V}$ is provided by dual Hopf algebra $\widehat{\mathbb{G}}$, with the action $\mathbb{H} \otimes \widehat{\mathbb{G}} \to \widehat{\mathbb{G}}$ defined with the help of bilinear pairing $\langle \cdot, \cdot \rangle : \mathbb{H} \otimes \widehat{\mathbb{G}} \to \mathbb{C}$. In the derivation of quantum-mechanical Heisenberg algebra (see (1)) as Heisenberg double the $\mathbb{C}$-number pairing is assumed to be proportional to $\hbar$; in this paper we shall put $\hbar = 1$.

[5] Partial coalgebraic structure can be introduced only separately in generalized coordinate and generalized momenta sectors, described respectively by dual Hopf algebras. If in quantum phase space we introduce the bialgebroid coproducts (see Sect. 4), the group coproducts are changed, but the partial coalgebraic structure in generalized momenta sector can be preserved.

[6] In distinction to the action $h \blacktriangleright g$, the action $h \triangleright X_A$ is described by a white triangle.

$$\Delta_{\mathcal{F}}(h) = \mathcal{F} \circ \Delta_0(h) \circ \mathcal{F}^{-1} \tag{14}$$

$$S_{\mathcal{F}}(h) = U \, S_0(h) \, U^{-1} \tag{15}$$

where due to (12) the coproduct $\Delta_{\mathcal{F}}$ is coassociative, $U = \sum f_{(1)} S(f_{(2)})$ (we use Sweedler's notation $\mathcal{F} = \sum f_{(1)} \otimes f_{(2)}$). In such a way we obtain for any choice of the twist $\mathcal{F}$ the quasitriangular Hopf algebra.

Our family of Abelian twists is[7]

$$\mathcal{F}_u := \exp(f_u)$$
$$= \exp\left((1-u)\frac{a \cdot p}{2\kappa} \otimes \theta^{\alpha\beta} M_{\alpha\beta} - u\theta^{\alpha\beta} M_{\alpha\beta} \otimes \frac{a \cdot p}{2\kappa}\right)$$
$$= \exp\left(\frac{1-u}{2}\mathcal{K}^{\alpha\beta} \otimes M_{\alpha\beta} - \frac{u}{2} M_{\alpha\beta} \otimes \mathcal{K}^{\alpha\beta}\right) \tag{16}$$

where $a \cdot p = a^\mu \eta_{\mu\nu} p^\nu$. The parameter $u \in [0, 1]$, κ is the deformation parameter with dimension of mass, $a^2 \equiv a_\mu a^\mu \in \{-1, 0, 1\}$,

$$\mathcal{K}^{\mu\nu} = \frac{a \cdot p}{\kappa} \theta^{\mu\nu} \tag{17}$$

and further

$$\theta^{\mu\nu} = -\theta^{\nu\mu}, \qquad a_\mu \theta^{\mu\nu} = 0. \tag{18}$$

From twists (16) one gets the coproducts

$$\Delta_{\mathcal{F}}(p_\mu) = p_\alpha \otimes (e^{-u\mathcal{K}})^\alpha{}_\mu + (e^{(1-u)\mathcal{K}})^\alpha{}_\mu \otimes p_\alpha \tag{19}$$

$$\Delta_{\mathcal{F}}(M_{\mu\nu}) = M_{\alpha\beta} \otimes (e^{-u\mathcal{K}})^\alpha{}_\mu (e^{-u\mathcal{K}})^\beta{}_\nu$$
$$+ (e^{(1-u)\mathcal{K}})^\alpha{}_\mu (e^{(1-u)\mathcal{K}})^\beta{}_\nu \otimes M_{\alpha\beta}$$
$$+ \frac{\theta^{\alpha\beta}}{2\kappa}(a_\mu \delta^\gamma_\nu - a_\nu \delta^\gamma_\mu)\Big[(1-u)p_\delta \otimes M_{\alpha\beta}(e^{-u\mathcal{K}})^\delta{}_\gamma$$
$$- u M_{\alpha\beta}(e^{(1-u)\mathcal{K}})^\delta{}_\gamma \otimes p_\delta\Big] \tag{20}$$

where $\mathcal{K}_{\mu\nu}$ is given by equation (17), and corresponding antipodes are:

$$S_{\mathcal{F}}(p_\mu) = -(e^{-(1-2u)\mathcal{K}})^\alpha{}_\mu p_\alpha \tag{21}$$

$$S_{\mathcal{F}}(M_{\mu\nu}) = -M_{\alpha\beta}(e^{-(1-2u)\mathcal{K}})^\alpha{}_\mu (e^{-(1-2u)\mathcal{K}})^\beta{}_\nu$$
$$+ \frac{1}{\kappa}(a_\mu \delta^\alpha_\nu - a_\nu \delta^\alpha_\mu)\big[S(p_\alpha) + (1-2u)\theta_{\alpha\beta}S(p^\beta)\big] \tag{22}$$

The counit is trivial:

$$\epsilon(p_\mu) = 0, \qquad \epsilon(M_{\mu\nu}) = 0. \tag{23}$$

2.2. RTT matrix quantum Poincare group $\widehat{\mathbb{G}}$

The universal $\mathcal{R}$-matrix $((a \wedge b = a \otimes b - b \otimes a))$

$$\mathcal{R} = \mathcal{F}_u^T \mathcal{F}_u^{-1} = \exp[\frac{1}{2}(M_{\alpha\beta} \wedge \mathcal{K}^{\alpha\beta})] \qquad (a \otimes b)^T = b \otimes a \tag{24}$$

can be used for the description of 10-generator deformed $D = 4$ Poincaré group. Using the 5×5 – matrix realization of the Poincaré generators

$$(M_{\mu\nu})^A{}_B = \delta^A_\mu \eta_{\nu B} - \delta^A_\nu \eta_{\mu B} \qquad (P_\mu)^A{}_B = -i\delta^A_\mu \delta^4_B \tag{25}$$

7 See [30] and [10]; the twists corresponding to special cases $u = 0$ and $u = \frac{1}{2}$ were considered also respectively in [35] and [36]. The twists (16) with different u can be related by coboundary twist (see e.g. [34], Sect. 3.1) which does not modify the universal $\mathcal{R}$-matrix.

we can show that in (24) only the linear term is non-vanishing

$$\mathcal{R} = 1 \otimes 1 + \frac{1}{2}(M_{\alpha\beta} \wedge \mathcal{K}^{\alpha\beta}). \tag{26}$$

To find the matrix quantum group which is dual to our Hopf algebra $\mathbb{H}$ we introduce the following 5×5 – matrices

$$\hat{\mathcal{T}}_{AB} = \begin{pmatrix} \hat{\Lambda}_{\mu\nu} & \hat{\xi}_\mu \\ 0 & 1 \end{pmatrix} \tag{27}$$

where $\hat{\Lambda}_{\mu\nu}$ parametrizes the quantum Lorentz rotation and $\hat{\xi}_\mu$ denotes quantum translations. In the framework of the FRT quantization procedure [37], the algebraic relations defining such a quantum group $\widehat{\mathbb{G}}$ is described by the following relation

$$\mathcal{R}\hat{\mathcal{T}}_1\hat{\mathcal{T}}_2 = \hat{\mathcal{T}}_2\hat{\mathcal{T}}_1\mathcal{R} \tag{28}$$

while the composition law for the coproduct remains classical

$$\Delta(\hat{\mathcal{T}}_{AB}) = \hat{\mathcal{T}}_{AC} \otimes \hat{\mathcal{T}}^C_B \tag{29}$$

with $\hat{\mathcal{T}}_1 = \hat{\mathcal{T}} \otimes 1$, $\hat{\mathcal{T}}_2 = 1 \otimes \hat{\mathcal{T}}$ and quantum $\mathcal{R}$-matrix given in the representation (25).

In terms of the operator basis $(\hat{\Lambda}_{\mu\nu}, \hat{\xi}_\mu)$ the algebraic relations (28) describing quantum group algebra can be written as follows

$$\left[\hat{\xi}_\mu, \hat{\xi}_\nu\right] = \frac{i}{\kappa}(a_\nu \theta_\mu{}^\alpha - a_\mu \theta_\nu{}^\alpha)\hat{\xi}_\alpha \tag{30}$$

$$[\hat{\xi}_\mu, \hat{\Lambda}_{\rho\sigma}] = \frac{i}{\kappa}\left(\hat{\Lambda}_{\rho\alpha}\theta^\alpha_\sigma a^\gamma \hat{\Lambda}_{\mu\gamma} - a_\mu \theta_\rho{}^\alpha \hat{\Lambda}_{\alpha\sigma}\right) \tag{31}$$

$$[\hat{\Lambda}_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = 0 \tag{32}$$

while the coproduct (29) takes the well known primitive forms

$$\Delta(\hat{\Lambda}_{\mu\nu}) = \hat{\Lambda}_{\mu\rho} \otimes \hat{\Lambda}^\rho{}_\nu \qquad \Delta(\hat{\xi}_\mu) = \hat{\Lambda}_{\mu\nu} \otimes \hat{\xi}^\nu + \hat{\xi}_\mu \otimes 1. \tag{33}$$

One can check that coproducts (33) are consistent with the commutators (30)–(32).

2.3. Duality between quantum algebra $\mathbb{H}$ and quantum group $\widehat{\mathbb{G}}$

Two Hopf algebras $\mathbb{H}, \widehat{\mathbb{G}}$ are said to be dual if there exists a nondegenerate bilinear form

$$\langle\,,\,\rangle : \mathbb{H} \times \widehat{\mathbb{G}} \longrightarrow \mathbb{C} \qquad (h, \hat{g}) \longrightarrow \langle h, \hat{g} \rangle \tag{34}$$

such that the duality relations (4)–(5) are satisfied. One can check, using the following pairing relations

$$< p_\mu, \hat{\xi}_\nu > = -i\eta_{\mu\nu} \quad < M_{\mu\nu}, \hat{\Lambda}_{\alpha\beta} > = -(\eta_{\mu\alpha}\eta_{\nu\beta} - \eta_{\nu\alpha}\eta_{\mu\beta})$$
$$< 1, \hat{\Lambda}_{\mu\nu} > = \eta_{\mu\nu} \tag{35}$$

that we have in our case $(h, h' = \{M_{\mu\nu}, p_\mu\}; \hat{g}, \hat{g}' = \{\hat{\Lambda}_{\mu\nu}, \hat{\xi}_\mu\})$

$$\langle h, [\hat{g}, \hat{g}'] \rangle = \langle \Delta(h), \hat{g} \otimes \hat{g}' - \hat{g}' \otimes \hat{g} \rangle \tag{36}$$

$$\langle [h, h'], \hat{g} \rangle = \langle h \otimes h' - h' \otimes h, \Delta(\hat{g}) \rangle. \tag{37}$$

The basic action of $\mathbb{H}$ on $\widehat{\mathbb{G}}$ promoting $\widehat{\mathbb{G}}$ to the $\mathbb{H}$-module is given by the relation

$$h \blacktriangleright \hat{g} = \hat{g}_{(1)} \langle h, \hat{g}_{(2)} \rangle. \tag{38}$$

After using (4) one gets the relation

$$h \blacktriangleright (\hat{g}\hat{g}') = \hat{g}\hat{g}'_{(1)} \langle \Delta h, \hat{g}_{(2)} \otimes \hat{g}'_{(2)} \rangle = \hat{g}\hat{g}'_{(1)} \langle h_{(1)}, \hat{g}_{(2)} \rangle \langle h_{(1)}, \hat{g}'_{(2)} \rangle$$
$$= (h_{(1)} \blacktriangleright \hat{g})(h_{(2)} \blacktriangleright \hat{g}') \tag{39}$$

what establishes that algebra $\widehat{\mathbb{G}}$ is the $\mathbb{H}$-module.

3. Heisenberg double $\mathcal{HD}$ and generalized $D = 4$ quantum phase spaces

Using the formula (3), in Heisenberg double framework we can obtain cross commutators between the algebra $\mathbb{H}$ and group $\widehat{\mathbb{G}}$ by the following relation

$$[h, \hat{g}] = \hat{g}_{(2)} \langle h_{(1)}, \hat{g}_{(1)} \rangle h_{(2)} - \hat{g} h$$
$$h = \{M_{\mu\nu}, p_\mu\}; \ \hat{g} = \{\hat{\Lambda}_{\mu\nu}, \hat{\xi}_\mu\}. \tag{40}$$

Using pairing (35) and coproducts (19), (20) and (33) we get

$$[p_\mu, \hat{\xi}_\nu] = -i(e^{-u\mathcal{K}})_{\nu\mu} + i(1-u)\frac{a_\nu}{\kappa}\theta_\mu{}^\alpha p_\alpha \tag{41}$$

$$[p_\mu, \hat{\Lambda}_{\rho\sigma}] = 0 \tag{42}$$

$$[M_{\mu\nu}, \hat{\Lambda}_{\rho\sigma}] = \frac{u}{\kappa}\theta_\rho{}^\alpha \hat{\Lambda}_{\alpha\sigma}(a_\mu p_\nu - a_\nu p_\mu) \tag{43}$$
$$+ \hat{\Lambda}_{\alpha\sigma}(e^{-u\mathcal{K}})^\alpha{}_{[\mu}(e^{-u\mathcal{K}})_{\rho\nu]}$$

$$[M_{\mu\nu}, \hat{\xi}_\rho] = \hat{\xi}_\gamma(e^{u\mathcal{K}})^\gamma_{[\mu}(e^{u\mathcal{K}})_{\nu]\rho} + \frac{u}{\kappa}\theta_\rho{}^\alpha \hat{\xi}_\alpha(a_\mu p_\nu - a_\nu p_\mu) \tag{44}$$
$$+ \frac{i(u-1)}{2\kappa}\theta^{\alpha\beta}M_{\alpha\beta}a_{[\mu}(e^{-u\mathcal{K}})_{\rho\nu]}$$
$$+ \frac{i(u-1)}{\kappa}a_\rho M_{[\mu\alpha}\theta^\alpha_{\nu]}$$

One can show that the generators of quantum Poincare group algebra, satisfying relations (30)–(32), (41)–(44) can be expressed in terms of classical group parameters $(\xi_\mu, \Lambda_{\rho\sigma})$ as follows

$$\hat{\xi}_\mu = \xi_\alpha(e^{-u\mathcal{K}})_\mu{}^\alpha + (1-u)\frac{ia_\mu}{2\kappa}\theta M \tag{45}$$

$$\hat{\Lambda}_{\rho\sigma} = (e^{-u\mathcal{K}})_\rho{}^\alpha \Lambda_{\alpha\beta}(e^{\frac{1}{\kappa}(p\wedge a)\theta})^\beta{}_\sigma. \tag{46}$$

The group manifold $\widehat{\mathbb{G}}$ as $\mathbb{H}$-module algebra can be promoted to the base algebra of Hopf algebroid. We get in such a way the application of Lu construction [1], which provides for Heisenberg double $\mathcal{HD} = \mathbb{H} \rtimes \widehat{\mathbb{G}}$, with generators $(\hat{\xi}_\mu, \hat{\Lambda}_{\mu\nu}; p_\mu, M_{\mu\nu})$ the Hopf algebroid structure.[8]

Let us introduce in place of relations (31) the following set of parameter-dependent algebraic relations (s-real parameter)

$$[\hat{\xi}_\mu^{(s)}, \hat{\Lambda}_{\rho\sigma}^{(s)}] = \frac{i}{\kappa}\left(s\hat{\Lambda}_{\rho\alpha}^{(s)}\theta^\alpha_\sigma a^\gamma \hat{\Lambda}_{\mu\gamma}^{(s)} - a_\mu\theta_\rho{}^\alpha \hat{\Lambda}_{\alpha\sigma}^{(s)}\right). \tag{47}$$

One can show that the relation (47) forms together with relations (30), (32) the consistent algebra of associative generalized quantum Poincare phase spaces, satisfying Jacobi relations. However, for $\xi = 0$ and $\xi = 1$ one can supplement the Hopf algebroid structure, namely

i) if $s = 1$ ($\hat{\xi}_\mu^{(1)} \equiv \hat{\xi}_\mu, \hat{\Lambda}_{\rho\sigma}^{(1)} \equiv \hat{\Lambda}_{\rho\sigma}$) one can construct the Hopf algebroid $\mathcal{H}^{(10,10)}$ with base algebra $\widehat{\mathbb{G}} = (\hat{\xi}_\mu, \hat{\Lambda}_{\rho\sigma})$ described by algebraic quantum Poincare group manifold.

ii) if $s = 0$ ($\hat{\xi}_\mu^{(0)} \equiv \hat{x}_\mu, \hat{\Lambda}_{\rho\sigma}^{(0)} \equiv \tilde{\Lambda}_{\rho\sigma}$) one gets alternative Hopf algebroid, denoted by $\tilde{\mathcal{H}}^{(10,10)}$, with algebra $\hat{\tilde{\mathbb{X}}} = (\hat{x}_\mu, \tilde{\Lambda}_{\mu\nu})$ as base algebra, with the multiplication given by the star product formula (2). The formula (2) can be also written as follows

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$$f(X) \star_\mathcal{F} k(X') = \widehat{f(X)} \triangleright k(X') \tag{48}$$

where $\widehat{f(X)}$ denotes the noncommutative representation of $f(\hat{X})$ defined as follows

$$f(\hat{X}) \simeq \widehat{f(X)} = m[\mathcal{F}^{-1}(\triangleright \otimes 1)(f(X) \otimes 1)] \tag{49}$$

where in formulae (48), (49) we use (see also (2))

$$h \triangleright X_A = [h, X_A], \qquad h = \{p_\mu, M_{\mu\nu}\}, X_A = \{x_\mu, \Lambda_{\mu\nu}\} \tag{50}$$

and for Lorentz sector we obtain

$$[M_{\mu\nu}, \Lambda_{\rho\sigma}] = \eta_{\rho\nu}\Lambda_{\mu\sigma} - \eta_{\rho\mu}\Lambda_{\nu\sigma}. \tag{51}$$

Substituting in (49) the twist (16) we get the following explicit formulas describing base algebra $\hat{X}_A = \{\hat{x}_\mu, \tilde{\Lambda}_{\mu\nu}\}$

$$\hat{x}_\mu = m[\mathcal{F}^{-1}(\triangleright \otimes 1)(x_\mu \otimes 1)] \tag{52}$$
$$= x_\alpha(e^{-u\mathcal{K}})_\mu{}^\alpha + (1-u)\frac{ia_\mu}{2\kappa}\theta M$$

$$\tilde{\Lambda}_{\rho\sigma} = m[\mathcal{F}^{-1}(\triangleright \otimes 1)(\Lambda_{\rho\sigma} \otimes 1)] \tag{53}$$
$$= (e^{-u\mathcal{K}})_\rho{}^\alpha \Lambda_{\alpha\sigma}$$

satisfying the following algebraic relations

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{\kappa}(a_\nu\theta_\mu{}^\alpha - a_\mu\theta_\nu{}^\alpha)\hat{x}_\alpha \tag{54}$$

$$[\hat{x}_\mu, \tilde{\Lambda}_{\rho\sigma}] = -\frac{i}{\kappa}a_\mu\theta_\rho{}^\alpha \tilde{\Lambda}_{\alpha\sigma} \tag{55}$$

$$[\tilde{\Lambda}_{\mu\nu}, \tilde{\Lambda}_{\rho\sigma}] = 0. \tag{56}$$

The generators $\Lambda_{\rho\sigma}$ and $M_{\mu\nu}$ satisfying the relation (51) describe the pair of undeformed dual variables in Lorentz sector, defined by the limit $\kappa \to \infty$ of the formula (43).

4. Hopf algebroid structures

The Hopf algebroid is described by the data $\mathcal{B}(H, A; s, t; \tilde{\Delta}, S, \epsilon)$ where H is total algebra, A the base algebra, and bialgebroidal coproducts $\tilde{\Delta}$ are described by the maps $H \to H \otimes_A H$ into (A, A) bimodules $H \otimes_A H$, which do not inherit the H algebra structure. The bimodule (A, A) property follows from the assumed existence of two maps:

i) source map $s(a) : A \to H$ (homomorphism),
ii) target maps $t(a) : A \to H$ (antihomomorphism),

where

$$[s(a), t(b)] = 0 \qquad a, b \in A \qquad s(a), t(b) \in H. \tag{57}$$

The relation (57) permits to introduce the basic (A, A) bimodule formula, namely $ahb = ht(a)s(b)$. Further, the algebra $H \otimes_A H$ can be defined as the quotient of $H \otimes H$ by left ideal $\mathcal{I}_L$ generated by the following elements[9]

$$\mathcal{I}_L = s(a) \otimes 1 - 1 \otimes t(a) \qquad a \in A \tag{58}$$

The canonical choice $s(a) = a$ of the source map leads to canonical form of (58)

$$\mathcal{I}_L^c = a \otimes 1 - 1 \otimes t^c(a) \qquad a \in A \tag{59}$$

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with the target map $t^c(a)$ determining the explicit form of coalgebra gauge freedom (see e.g. [7]). Further the canonical coproducts of the elements of base algebra A are chosen to be half-primitive (see e.g. [1,2])

$$\tilde{\Delta}(\hat{a}) = \hat{a} \otimes 1 \qquad \hat{a} \in A \tag{60}$$

i.e. the coproducts (60) are homomorphic in trivial way to the algebra structure of A.

In this paper we have two choices of algebra A: given by quantum group algebra $\widehat{\mathbb{G}}$, with quantum space–times translations $\hat{\xi}_\mu$, and by generalized coordinates sector $\widehat{\mathbb{X}}$ with quantum-deformed Minkowski space–time coordinates introduced as the module algebra of twisted Poincare algebra. Below we shall consider in some detail the second choice.

If the quantum deformation is generated by twist, the bialgebroid source and target map are described by the formulae analogous to relations (49) (see [2], [9]). For the choice (16) of twist factor one gets:

- source map

$$s_{\mathcal{F}}(\hat{x}_\mu) = m[\mathcal{F}^{-1}(\triangleright \otimes 1)(s_0(x_\mu) \otimes 1)] \tag{61}$$
$$= x_\alpha (e^{-u\mathcal{K}})_\mu{}^\alpha + (1-u)\frac{ia_\mu}{2\kappa}\theta M \equiv \hat{x}_\mu$$

$$s_{\mathcal{F}}(\tilde{\Lambda}_{\rho\sigma}) = m[\mathcal{F}^{-1}(\triangleright \otimes 1)(s_0(\Lambda_{\rho\sigma}) \otimes 1)] \tag{62}$$
$$= (e^{-u\mathcal{K}})_\rho{}^\alpha \Lambda_{\alpha\sigma} \equiv \tilde{\Lambda}_{\rho\sigma}$$

- target map

$$t(\hat{x}_\mu) = m[(\mathcal{F}^{-1})^\tau (\triangleright \otimes 1)(t_0(x_\mu) \otimes 1)] \tag{63}$$
$$= x_\alpha (e^{(1-u)\mathcal{K}})_\mu{}^\alpha - u\frac{ia_\mu}{2\kappa}\theta M = \hat{x}_\alpha (e^{\mathcal{K}})_\mu{}^\alpha - i\frac{a_\mu}{2\kappa}\theta M$$

$$t(\tilde{\Lambda}_{\rho\sigma}) = m[(\mathcal{F}^{-1})^\tau (\triangleright \otimes 1)(t_0(\Lambda_{\rho\sigma}) \otimes 1)] \tag{64}$$
$$= (e^{(1-u)\mathcal{K}})_\rho{}^\alpha \Lambda_{\alpha\sigma} = (e^{\mathcal{K}})_\rho{}^\alpha \tilde{\Lambda}_{\alpha\sigma}.$$

One can check that images of source and target maps commute as follows

$$[s(\hat{X}_A), s(\hat{X}_B)] = \mathcal{C}_{AB}{}^C s(\hat{X}_C) \tag{65}$$

$$[t(\hat{X}_A), t(\hat{X}_B)] = -\mathcal{C}_{AB}{}^C t(\hat{X}_C) \tag{66}$$

$$[s(\hat{X}_A), t(\hat{X}_B)] = 0 \tag{67}$$

where constant structures $\mathcal{C}_{AB}{}^C$ describe the Lie algebraic structure of commutation relations (54)–(56). The coproducts of base algebra elements are given by relations (60), and the coalgebra of generalized momentum sector H is described by Hopf-algebraic formulae (14) and (19)–(20).

In the case of our twisted Hopf algebroid the canonical ideal (59) is the following

$$\mathcal{I}_L^c(\hat{x}_\mu) = \hat{x}_\mu \otimes 1 - 1 \otimes (\hat{x}_\alpha (e^{\mathcal{K}})_\mu{}^\alpha - i\frac{a_\mu}{2\kappa}\theta M) \tag{68}$$

$$\mathcal{I}_L^c(\tilde{\Lambda}_{\rho\sigma}) = \tilde{\Lambda}_{\rho\sigma} \otimes 1 - 1 \otimes (e^{\mathcal{K}})_\rho{}^\alpha \tilde{\Lambda}_{\alpha\sigma} \tag{69}$$

with counits given by the canonical formula

$$\epsilon(\hat{X}_A) = m[\mathcal{F}^{-1}(\triangleright \otimes 1)(\epsilon_0(X_A) \otimes 1)] = \hat{X}_A \tag{70}$$

and $\epsilon(h) = 0$ and $\epsilon(1) = 1$. The antipodes should satisfy the following relations

$$S(t(\hat{X}_A)) = s(\hat{X}_A) = \hat{X}_A \tag{71}$$

$$m[(1 \otimes S) \circ \tilde{\Delta}] = s\epsilon = \epsilon \tag{72}$$

$$m[(S \otimes 1) \circ \tilde{\Delta}] = t\epsilon S. \tag{73}$$

Using (71) one gets explicit formulae for the antipodes

$$S(\hat{x}_\mu) = (e^{\mathcal{K}})_\mu{}^\alpha \hat{x}_\alpha - i\frac{a_\mu}{2\kappa}\theta^{\alpha\beta}M_{\alpha\beta} = t(\hat{x}_\mu) \tag{74}$$

$$S(\tilde{\Lambda}_{\mu\nu}) = (e^{\mathcal{K}})_\mu{}^\alpha \tilde{\Lambda}_{\alpha\nu} = t(\tilde{\Lambda}_{\mu\nu}). \tag{75}$$

In our case we have $S^2 = 1$ and the anchor map is not needed (see [1–3]).

5. Final remarks

The Hopf algebras, which we define over commutative ring, can be generalized to Hopf-algebraic structures over noncommutative ring A, what leads to the notion of Hopf algebroids. If the algebra sector of Hopf algebroid $\mathcal{B}$ is endowed with (pre)symplectic structure, as it occurs in Heisenberg double construction presented here, we obtain various models of quantum-deformed phase space algebras. We presented in this paper two types of quantum-deformed phase spaces with Hopf algebroid structure: with coordinates described by quantum-deformed Poincare group algebra and with NC space–time coordinates generated by twist-dependent star product formula (48).

Our considerations have been illustrated by explicit example of Drinfeld twist, but our considerations can be generalized to arbitrary Drinfeld twist case. It has been shown (see [3], Sect. 4) that there exists a generic construction of bialgebroid associated with FRT quantization method of quantum matrix groups, which also in general case provides the braid-commutative choice of base algebras defining the Yetter–Drinfeld module [3].

Hopf algebroids are defined in their coalgebraic sectors by the coproducts $\tilde{\Delta} : H \to H \otimes H$ defined modulo the coproduct gauge freedom (see e.g. [7]; we stress that $\otimes$ denotes standard tensor product). It is an interesting problem to characterize the algebraic manifold $\mathcal{C}$ parametrizing the coproduct gauge transformations, and subsequently define the coproduct gauge-invariant quantum two-particle phase space as the factor algebra $H \otimes H/\mathcal{C}$ which does not depend on the choice of coproduct gauge.

We add that $\mathcal{HD}$ construction and corresponding Hopf algebroid structures can be introduced also for infinite-dimensional affine algebras, which are linked with integrable many-body systems of Calogero type (e.g. Calogero–Moser [38,39] and Ruijsenaars–Schneider [40,41] integrable hierarchies) as well as applied for dynamical quantum groups, described by parameter-dependent (dynamical) Yang–Baxter equations (see e.g. [42]). Other extension, which has as well the link with algebroid structures, is related with recent development in string models with applications of generalized geometries and T-duality covariance. Such generalized phase space dynamics formulated in the framework of string theories, related with double field theories, has been recently investigated under the name of metastring theories (see e.g. [43], [44]).

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