Physics Letters B 792 (2019) 298–303

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

Decoherence in neutrino oscillations, neutrino nature and CPT violation

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ARTICLE INFO

Article history:
Received 29 September 2018
Received in revised form 10 March 2019
Accepted 29 March 2019
Available online 3 April 2019
Editor: A. Ringwald

ABSTRACT

We analyze many aspects of the phenomenon of the decoherence for neutrinos propagating in long baseline experiments. We show that, in the presence of an off-diagonal term in the dissipative matrix, the Majorana neutrino can violate the CPT symmetry, which, on the contrary, is preserved for Dirac neutrinos. We show that oscillation formulas for Majorana neutrinos depend on the choice of the mixing matrix U. Indeed, different choices of U lead to different oscillation formulas. Moreover, we study the possibility to reveal the differences between Dirac and Majorana neutrinos in the oscillations. We use the present values of the experimental parameters in order to relate our theoretical proposal with experiments.

1. Introduction

The phenomena of neutrino mixing and oscillation, induced by the non-zero neutrino mass, represent an hint of physics beyond the Standard Model of particles. It has been confirmed by many experiments [1–6]. At the present, the main issues of the neutrino physics are the determination of the absolute neutrino mass and its nature. As a matter of fact, since neutrino is electrically neutral, two possibilities exist, either neutrino is distinct from its antiparticle and hence is a Dirac fermion, or it is equal to its antiparticle and it is a Majorana fermion.

To reveal the neutrino nature, many experiments, based on the detection of the neutrinoless double beta decay, have been proposed [7]. Recently, it has been shown that quantities such as the Leggett-Garg K_3 quantity [8] and the geometric phase for neutrinos [9], can, in principle, discriminates between Dirac and Majorana neutrinos. Moreover, it has been shown that in the presence of decoherence, the neutrino oscillation formulas can depend on the Majorana phase [10]. However, at the moment the nature of the neutrino remains an open question.

On the other hand, particle mixing phenomenon, in particular the $B^0-\overline{B}^0$ mixing, is used to test the CPT symmetry. The CPT theorem, affirms that the simultaneous transformations of charge conjugation C, parity transformation P, and time reversal T, is an exact symmetry of nature at the fundamental level [11].

In this paper, we show that, if quantum decoherence appears in neutrino oscillations, then long baseline experiments might allow to investigate the nature of neutrinos and the CPT symmetry. Moreover, if the neutrinos are Majorana particles, the decoherence could allow the right choice of the matrix mixing U.

The phenomena of dissipation and decoherence are consequences of interaction with the environment, which, in neutrino case, could be originated by quantum gravity effects, or strings and branes. A significant research effort had been undertaken in the study of dissipation in neutrino oscillations [10,12,13]. It has been shown that such phenomena can modify the oscillation frequencies and the oscillation formulas. Moreover, it has been noted that the dissipation can generate oscillation formulas for Majorana neutrinos different with respect to the ones for Dirac neutrinos [10]. Still, other theoretical results can be obtained which are extremely relevant.

Here, by considering the neutrino as an open quantum system interacting with its environment, we analyze some features of the decoherence effect in flavor mixing. We study the time evolution of the density matrix representing the neutrino state in the flavor basis and we analyze the case in which the matrix describing the dissipator has off-diagonal terms. Specifically, we reveal the possible CPT symmetry breaking in the Majorana neutrino oscillation, and we study the differences between Majorana and Dirac neutrinos. We prove that, the presence of off-diagonal terms in the decoherence matrix leads to probability of transitions depending on the representation of the Majorana mixing matrix. Thus, if the decoherence exists in the neutrino propagation, the oscillation for-

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https://doi.org/10.1016/j.physletb.2019.03.062

mulas could provide a tool to determine the choice of the mixing matrix U. Moreover, by considering the parameters of DUNE experiment and the recent constraints on decoherence parameters [14,15], we show that long baseline experiments on neutrinos, could reveal the nature of neutrino and could allow to test the CPT symmetry.

We analyze the neutrinos propagation in the vacuum and through a medium. In the absence of decoherence, the matter effects are taken into account by replacing in the oscillation formulas in vacuum, Δm^2 with $\Delta m^2_m = \Delta m^2 R_\pm$, and $\sin 2\theta$ with $\sin 2\theta_m = \sin 2\theta / R_\pm$. The coefficients $R_\pm$ describing the Mikhaev-Smirnow-Wolfenstein (MSW) effect [16,17] are given by, $R_\pm = \sqrt{\left(\cos 2\theta \pm \frac{2\sqrt{2} G_F n_e E}{\Delta m^2}\right)^2 + \sin^2 2\theta}$, with $+$ for oscillation of antineutrinos and $-$ for oscillations of neutrinos. Here, n_e is the electron density in the matter, G_F is the Fermi constant, and E is the neutrino energy. In the presence of decoherence, one has to consider the formalism presented in ref. [18]. In the following we adopt such a formalism to describe the neutrino propagation through the matter. Notice that the effect of the matter on the neutrino can be relevant in the $\nu_e \leftrightarrow \nu_\mu$ oscillations, since the ν_e and ν_μ indices of refraction are different in media like the Earth and the Sun, $\kappa(\nu_e) \neq \kappa(\nu_\mu)$. In particular, $\kappa(\nu_e) - \kappa(\nu_\mu) = -\sqrt{2} G_F n_e / p$. On the contrary, in the case of the $\nu_\mu \leftrightarrow \nu_\tau$ mixing, the ν_μ and ν_τ indices of refraction are different only in very dense matter, like the core of supernovae, but they are almost identical in the matter of Earth and the Sun. Therefore, in such media, $R \sim 1$ and the $\nu_\mu \leftrightarrow \nu_\tau$ oscillations are almost identical to the ones in vacuum [19]. In the following we consider the propagation through Earth in which the matter effect is relevant only for $\nu_e \leftrightarrow \nu_\mu$ oscillations.

The paper is organized as follows. In Sec. 2 we report the main differences between Majorana and Dirac neutrinos. In Sec. 3 we analyze quantum decoherence in neutrino propagation and we show the effects induced on neutrino by an off-diagonal term in the dissipation matrix. Numerical analysis, for neutrino propagation in vacuum and through matter, are reported in Sec. 4. Sec. 5 is devoted to the conclusions.

2. Majorana and Dirac neutrinos

A crucial difference between Dirac and Majorana neutrinos is the following: while the Dirac Lagrangian is invariant under $U(1)$ global transformation, and hence the charges associated (electric, leptonic, etc.) with the transformations are conserved, the Majorana Lagrangian breaks the $U(1)$ symmetry.[1] Therefore, processes violating the lepton number, such as neutrinoless double beta decay, are allowed for Majorana neutrinos and forbidden for Dirac ones. In the case of neutrino mixing, the breaking of the $U(1)$ global symmetry of the Majorana Lagrangian implies that the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix with dimension $n \times n$, contains a total number of physical phases for Majorana neutrinos different with respect to that of Dirac neutrinos. Indeed, in the case of the mixing of n Dirac fields, one has N_D physical phases given by $N_D = \frac{(n-1)(n-2)}{2}$, and in the case of the mixing of n Majorana fields, one has N_M phases given by $N_M = \frac{n(n-1)}{2}$. The $n-1$ extra phases present in the Majorana neutrino mixing (called Majorana phases) represent another important distinction between Dirac and Majorana neutrinos. The detection of such phases can allow to fix the nature of neutrinos.

The mixing matrices for Majorana U_M and for Dirac neutrinos U_D can be related by the equation,

$$U_M = U_D \cdot diag(1, e^{i\phi_1}, e^{i\phi_2}, ..., e^{i\phi_{n-1}}) , \tag{1}$$

where ϕ_i, with $i = 1, ..., n-1$, are the Majorana phases. Other representations of Majorana mixing matrix can be obtained by the rephasing the lepton charge fields in the charged current weak-interaction Lagrangian, (for details see Ref. [20]). For example, for two mixed Majorana fields ν_σ and ν_ϱ, with σ and ϱ neutrino flavors, (in the following we will consider the $\nu_e \leftrightarrow \nu_\mu$ and $\nu_\mu \leftrightarrow \nu_\tau$ oscillations) one can consider the following mixing relations

$$\begin{pmatrix} \nu_\sigma \\ \nu_\varrho \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{i\phi} \\ -\sin\theta & \cos\theta\, e^{i\phi} \end{pmatrix} \begin{pmatrix} \nu_j \\ \nu_k \end{pmatrix} , \tag{2}$$

or

$$\begin{pmatrix} \nu_\sigma \\ \nu_\varrho \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ -\sin\theta\, e^{i\phi} & \cos\theta \end{pmatrix} \begin{pmatrix} \nu_j \\ \nu_k \end{pmatrix} , \tag{3}$$

where θ is the mixing angle, ν_j, ν_k are the fields with definite masses and ϕ is the Majorana phase. Such a phase can be removed for Dirac neutrinos by rephasing the mass term of the Dirac Lagrangian. For example, in Eq. (3), ϕ is eliminated by means of the replacements, $\nu_j \to \nu_j$ and $\nu_k \to \nu_k\, e^{i\phi}$.

Notice that, in absence of decoherence (i.e. in standard treatment of neutrino mixing) and in the case of dissipation with diagonal decoherence matrix, the probabilities of neutrino transitions are invariant under the rephasing $U_{\alpha l} \to e^{i\phi_l} U_{\alpha l}$ (with $\alpha = \sigma, \varrho$, and $l = j, k$). Thus, in such cases, the presence of the Majorana phases ϕ_l do not affect the oscillation formulas for neutrino propagating in the vacuum, through matter and through a magnetic field, being such formulas equivalent for Majorana and for Dirac neutrinos [21]. By contrast, in the case of decoherence matrix with off-diagonal terms, the oscillation formulas for Majorana neutrinos depend on the phases ϕ_l [10].

Moreover, we will reveal other two important aspects: a) Majorana neutrinos can violate CPT symmetry; b) different choices of the mixing matrix for such neutrinos can lead to different probability of transitions, (for example, the oscillation formulas obtained by using Eq. (2) can be different with respect to the ones obtained by means of Eq. (3), see below).

3. Decoherence and neutrino oscillations

The evolution of the neutrino considered as an open system, can be expressed by the Lidbland–Kossakowski master equation [22]

$$\frac{\partial \rho(t)}{\partial t} = -\frac{i}{\hbar}[H_{eff}, \rho(t)] + D[\rho(t)] . \tag{4}$$

Here, $H_{eff} = H^\dagger_{eff}$ is the effective Hamiltonian, and $D[\rho(t)]$ is the dissipator defined as

$$D[\rho(t)] = \frac{1}{2} \sum_{i,j=0}^{N^2-1} a_{ij} \left([F_i\, \rho(t)\, , F^\dagger_j] + [F_i\, , \rho(t)\, F^\dagger_j] \right) . \tag{5}$$

[1] Indeed, Dirac fields ψ are composed by the left-handed ψ_L and the right-handed ψ_R components. Then, in the free Lagrangian $L = \overline{\psi}(i\gamma^\mu \partial_\mu - m)\psi$, the mass term can be written as $L_m = -m\overline{\psi}\psi = -m(\overline{\psi_L + \psi_R})(\psi_L + \psi_R) = -m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L)$, (the $\overline{\psi_L}\psi_L$ and $\overline{\psi_R}\psi_R$ terms vanish identically). In such a case, L_m (and then L) is invariant under the $U(1)$ transformations $\psi \to e^{i\phi}\psi$ and $\overline{\psi} \to e^{-i\phi}\overline{\psi}$. Therefore the charges (electric, leptonic, etc.) associated with the transformations are conserved. On the other hand, for Majorana fields the right-handed component of ψ is the charge conjugate of its left-handed component, $\psi_R = (\psi_L)^c$, where $\psi^c = C\overline{\psi}^T$, and C is the charge conjugation matrix. Then one has $\psi = \psi_L + (\psi_L)^c$. In such a case, the mass Lagrangian is $L_m = -\frac{m}{2}\left[\overline{(\psi_L)^c}\psi_L + \overline{\psi_L}(\psi_L)^c\right] = \frac{m}{2}\left[\psi_L^T C^{-1}\psi_L + h.c.\right]$, (being $\overline{\psi^c} = -\psi^T C^{-1}$), i.e. L_m has the structure $\psi_L\psi_L + h.c.$, which therefore, under the $U(1)$ transformations, breaks all the $U(1)$-charges of two units.

The coefficients a_{ij} of the Kossakowski matrix, could be derived by the properties of the environment [10] and F_i, with $i = N^2 - 1$, are a set of operators such that $Tr(F_k) = 0$ for any k and $Tr(F_i^\dagger F_j) = \delta_{ij}$. In the three flavor neutrino mixing case, F_i are represented by the Gell-Mann matrices λ_i. In the two flavor neutrino mixing F_i, are the Pauli matrices σ_i.

For the sake of simplicity, in the following we consider the mixing between two flavors (our results can be extended to the three flavor neutrino mixing case). Expanding Eqs. (4) and (5) in the bases of the $SU(2)$, Eq. (4) can be written as

$$\frac{d\rho_\lambda}{dt}\sigma_\lambda = 2\,\epsilon_{ijk}\,H_i\,\rho_j(t)\,\sigma_\lambda\,\delta_{\lambda k} + D_{\lambda\mu}\,\rho_\mu(t)\,\sigma_\lambda\,, \tag{6}$$

where $\rho_\mu = \mathrm{Tr}(\rho\,\sigma_\mu)$, with $\mu \in [0,3]$ and $D_{\lambda\mu}$ is a 4×4 matrix. The conservation of the probability implies $D_{\lambda 0} = D_{0\mu} = 0$, then we can consider a 3×3 matrix for the dissipator of the form

$$D_{ij} = -\begin{pmatrix} \gamma_1 & \alpha & \beta \\ \alpha & \gamma_2 & \delta \\ \beta & \delta & \gamma_3 \end{pmatrix}. \tag{7}$$

All the parameters of Eq. (7) are reals and the diagonal elements are positive in order to satisfy the condition, $Tr(\rho(t)) = 1$ for any time t.

In order to study the effects of a non-diagonal form of the decoherence matrix on neutrino physics, we consider, for simplicity the matrix D_{ij} given by

$$D_{ij} = -\begin{pmatrix} \gamma & \alpha & 0 \\ \alpha & \gamma & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}. \tag{8}$$

This dissipator is obtained by Eq. (7), by setting, $\gamma_1 = \gamma_2 = \gamma$, and $\beta = \delta = 0$. The condition of complete positivity of the density matrix $\rho(t)$, implies $\forall t$, the following condition $|\alpha| \le \gamma_3/2 \le \gamma$. By setting, $\Delta = \frac{\Delta m^2}{2E}$, and by taking into account Eq. (8), we have $\dot\rho_0(t) = 0$, which for two neutrino families implies $\rho_0(t) = 1$. Then the master equation (6) can be written as

$$\begin{pmatrix} \dot\rho_1(t) \\ \dot\rho_2(t) \\ \dot\rho_3(t) \end{pmatrix} = \begin{pmatrix} -\gamma & -\Delta-\alpha & 0 \\ \Delta-\alpha & -\gamma & 0 \\ 0 & 0 & -\gamma_3 \end{pmatrix} \begin{pmatrix} \rho_1(t) \\ \rho_2(t) \\ \rho_3(t) \end{pmatrix}. \tag{9}$$

By solving Eq. (9), we get

$$\rho_1(t) = e^{-\gamma t}\Big[\rho_1(0)\cosh(\Omega_\alpha t) - \rho_2(0)\,\frac{\sinh(\Omega_\alpha t)}{\Omega_\alpha}\,\Xi_-\Big]$$
$$\rho_2(t) = e^{-\gamma t}\Big[\rho_1(0)\,\frac{\sinh(\Omega_\alpha t)}{\Omega_\alpha}\,\Xi_+ + \rho_2(0)\cosh(\Omega_\alpha t)\Big]$$
$$\rho_3(t) = \rho_3(0)e^{-\gamma_3 t}\,, \tag{10}$$

where $\Xi_\pm = \alpha \pm \Delta$ and $\Omega_\alpha = \sqrt{\alpha^2 - \Delta^2}$. Hence, the matrix density, at any time, t reads

$$\rho(t) = \frac{1}{2}\begin{pmatrix} \rho_0(t)+\rho_3(t) & \rho_1(t)-i\rho_2(t) \\ \rho_1(t)+i\rho_2(t) & \rho_0(t)-\rho_3(t) \end{pmatrix}. \tag{11}$$

By using the mixing relations for Majorana neutrinos, given in Eq. (3), the matrix density of the neutrino with flavor σ, (ν_σ is for example the electron neutrino in the case of $\nu_e - \nu_\mu$ mixing), at time $t = 0$, is

$$\rho_\sigma(0) = \begin{pmatrix} \cos^2\theta & \frac{1}{2}\sin 2\theta\, e^{i\phi} \\ \frac{1}{2}\sin 2\theta\, e^{-i\phi} & \sin^2\theta \end{pmatrix}, \tag{12}$$

and similar for neutrino with flavor ϱ. Then at time t, we have

$$\rho_\sigma(t) = \begin{pmatrix} \Lambda_+ & \Theta^* \\ \Theta & \Lambda_- \end{pmatrix}, \tag{13}$$

where, $\Lambda_\pm = \frac{1}{2}\left[1 \pm \cos 2\theta e^{-\gamma_3 t}\right]$, and

$$\Theta = \frac{\sin 2\theta\, e^{-\gamma t - i\phi}}{2\Omega_\alpha t}\Big\{\Omega_\alpha\cosh(\Omega_\alpha t) - i\Upsilon_{\alpha,\phi}\sinh(\Omega_\alpha t)\Big\},$$

with $\Upsilon_{\alpha,\phi} = e^{2i\phi}\alpha + \Delta$.

The probabilities of transition $P_{\nu_\sigma\to\nu_\varrho}(t)$, with σ and ϱ neutrino flavors, are given by $P_{\nu_\sigma\to\nu_\varrho}(t) = Tr\left[\rho_\varrho(t)\rho_\sigma(0)\right]$. Explicitly, we have

$$P_{\nu_\sigma\to\nu_\varrho}(t) = \frac{1}{2}\Big\{1 - e^{-\gamma_3 t}\cos^2 2\theta - e^{-\gamma t}\sin^2 2\theta$$
$$\times\Big[\cosh(\Omega_\alpha t) + \frac{\alpha\sin(2\phi)\sinh(\Omega_\alpha t)}{\Omega_\alpha}\Big]\Big\}. \tag{14}$$

In a similar way, for anti-neutrino, we have $P_{\overline\nu_\sigma\to\overline\nu_\varrho}(t) = Tr\left[\rho_{\overline\varrho}(t)\rho_{\overline\sigma}(0)\right]$, i.e.

$$P_{\overline\nu_\sigma\to\overline\nu_\varrho}(t) = \frac{1}{2}\Big\{1 - e^{-\gamma_3 t}\cos^2 2\theta - e^{-\gamma t}\sin^2 2\theta$$
$$\times\Big[\cosh(\Omega_\alpha t) - \frac{\alpha\sin(2\phi)\sinh(\Omega_\alpha t)}{\Omega_\alpha}\Big]\Big\}. \tag{15}$$

Eqs. (14) and (15) show an asymmetry between the transitions $\nu_\sigma \to \nu_\varrho$ and $\overline\nu_\sigma \to \overline\nu_\varrho$, i.e. $P_{\nu_\sigma\to\nu_\varrho}(t) \ne P_{\overline\nu_\sigma\to\overline\nu_\varrho}(t)$. This asymmetry is due to Majorana phase ϕ and appears also in the probability of an electron, muon or tau neutrino preserving its flavor σ, ($\sigma = e, \mu, \tau$), i.e. $P_{\nu_\sigma\to\nu_\sigma}(t) \ne P_{\overline\nu_\sigma\to\overline\nu_\sigma}(t)$. The CP violation, induced by the oscillation formulas Eqs. (14) and (15) is explicitly given by,

$$\Delta^M_{CP}(t) = P_{\nu_\sigma\to\nu_\varrho}(t) - P_{\overline\nu_\sigma\to\overline\nu_\varrho}(t)$$
$$= -\sin^2 2\theta\,\frac{\alpha\sin(2\phi)\sinh(\Omega_\alpha t)}{\Omega_\alpha}\,e^{-\gamma t}. \tag{16}$$

Notice that in the three neutrino system, it appears a further CP violation in $\nu_\sigma \to \nu_\varrho$ oscillations (with $\sigma \ne \varrho$), due to the presence of the δ phase in the PMNS mixing matrix. Such a violation affects both Majorana and Dirac neutrinos and it is compensated by the T violation, $\Delta_{CP} = \Delta_T$ in order to preserve CPT symmetry. The CP violation due to the δ phase in three flavor mixing could cover in part the effect presented in Eq. (16).

However, for Majorana neutrinos one has a CP violation also in the transitions preserving their flavors,

$$\Delta^{M(\sigma\leftrightarrow\sigma)}_{CP}(t) = P_{\nu_\sigma\to\nu_\sigma}(t) - P_{\overline\nu_\sigma\to\overline\nu_\sigma}(t) \tag{17}$$
$$= \sin^2 2\theta\,\frac{\alpha\sin(2\phi)\sinh(\Omega_\alpha t)}{\Omega_\alpha}\,e^{-\gamma t},$$

with $\sigma = e, \mu, \tau$. In three flavor mixing case the δ phase of the PMNS mixing matrix does not affect the transitions preserving their flavor. Therefore the CP violation in Eq. (17) is due only to the decoherence and characterizes only the Majorana neutrinos. No CP violation is indeed exhibited by Dirac neutrinos in the oscillation probabilities $P_{\nu_\sigma\leftrightarrow\nu_\sigma}$, with $\sigma = e, \mu, \tau$. The violation $\Delta^{M(\sigma\leftrightarrow\sigma)}_{CP}(t)$ presented in Eq. (17), which is purely due to the decoherence of Majorana neutrinos, can be analyzed also in experiments like DUNE which should be sensitive to the δ phase of the PMNS matrix. In fact, δ does not affect $\Delta^{M(\sigma\leftrightarrow\sigma)}_{CP}(t)$ and the result $\Delta^{M(\sigma\leftrightarrow\sigma)}_{CP}(t) \ne 0$ only implies that the neutrinos are Majorana particles. A detailed study of the three flavor neutrino mixing case in the presence of decoherence will be done in a forthcoming paper.

The definition of the T-violating quantity in the case of dissipative matter is more delicate. Indeed, the decoherence and the dissipation induce an explicit violation of the T symmetry, which is independent on the nature of the particle. Here we are interested to the study of T symmetry in neutrino oscillation.

In QM mixing treatment, the T violating asymmetry can be obtained by means of two equivalent definitions, as follows

$$\Delta_T^M(t) = P_{\nu_\sigma\to\nu_\varrho}(t) - P_{\nu_\varrho\to\nu_\sigma}(t) = P_{\nu_\sigma\to\nu_\varrho}(t) - P_{\nu_\sigma\to\nu_\varrho}(-t)\,. \tag{18}$$

However, in the presence of decoherence, the definition $\Delta_T^M(t) = P_{\nu_\sigma\to\nu_\varrho}(t) - P_{\nu_\sigma\to\nu_\varrho}(-t)$, cannot be used, since the complete positivity of the matrix density is not satisfied for any time. Indeed, one has

$$\Delta_T^M(t) = P_{\nu_\sigma\to\nu_\varrho}(t) - P_{\nu_\sigma\to\nu_\varrho}(-t) = -\sin^2 2\theta\Big[\frac{\alpha\sin(2\phi)\,\sinh(\Omega_\alpha t)\cosh(\gamma t)}{\Omega_\alpha} - \sinh(\gamma t)\cosh(\Omega_\alpha t)\Big] + \sinh(\gamma_3 t)\cos^2 2\theta\,. \tag{19}$$

The presence of hyperbolic functions in Eq. (19) induces, for sufficiently long time, a violation of the positivity of ρ, that generates values of Δ_T^M not included in the interval $[-1, 1]$. This result is not physically acceptable.

On the contrary, the relation, $\Delta_T^M(t) = P_{\nu_\sigma\to\nu_\varrho}(t) - P_{\nu_\varrho\to\nu_\sigma}(t)$, is defined properly at any time t. By using such a relation, we have,

$$\Delta_T^M(t) = 0\,, \tag{20}$$

i.e., the two flavor Majorana neutrino oscillation, in the presence of decoherence, does not violate the T symmetry. This result holds also for the transitions preserving the flavors, $\Delta_T^{M(\sigma\leftrightarrow\sigma)}(t) = 0$.

The CPT invariance imposes the relationship $\Delta_{CP} = \Delta_T$. However, by comparing Eq. (16) with Eq. (20), we have $\Delta_{CP}^M \neq \Delta_T^M$, which implies the violation of the CPT symmetry for Majorana neutrinos, $\Delta_{CPT}^M \neq 0$. Since $\Delta_T^M = 0$, the analysis of the Δ_{CP}^M induced by the decoherence allows also the study of Δ_{CPT}^M.

Let us consider now Dirac neutrinos. The phase ϕ can be set equal to zero, then the oscillation formulas $P_{\nu_\sigma\to\nu_\varrho}(t)$ and $P_{\overline{\nu}_\sigma\to\overline{\nu}_\varrho}(t)$ are equivalent, $P_{\nu_\sigma\to\nu_\varrho}(t) = P_{\overline{\nu}_\sigma\to\overline{\nu}_\varrho}(t)$ and reduce to

$$P_{\nu_\sigma\to\nu_\varrho}(t) = P_{\overline{\nu}_\sigma\to\overline{\nu}_\varrho}(t) = \frac{1}{2}\Big[1 - e^{-\gamma_3 t}\cos^2 2\theta - e^{-\gamma t}\sin^2 2\theta\cosh(\Omega_\alpha t)\Big]\,. \tag{21}$$

In this case, the neutrino oscillation preserves the CP and T symmetries, $\Delta_{CP}^D = \Delta_T^D = 0$. The above results show that the decoherence could produce another difference between Majorana and Dirac neutrinos, i.e. the CPT symmetry is violated by Majorana neutrinos, while it is preserved by Dirac neutrinos.

The kind of CPT violation here presented is due to the mixing in the presence of the decoherence. It could represent an effect induced by the quantum gravity [23]. We emphasize that such a violation is different from an explicit CPT symmetry breaking in the Hamiltonian dynamics such that $[CPT, H] \neq 0$. In this case, a possible cause of the CPT breaking can be represented by the Lorentz violation due to a propagation in a curved space violating the Lorentz invariance. In the present framework, the decoherence may lead to an effectively ill-defined CPT quantum mechanical operator [24].

Another result here shown is the dependence on the choice of the mixing matrix U of the oscillation formulas for Majorana neutrinos. Different choices of U lead to different oscillation formulas. Indeed, Eqs. (14) and (15) are obtained by using the mixing relations given in Eq. (3). By contrast, by using the mixing matrix of Eq. (2), the oscillation formula for neutrinos Eq. (14) is replaced by that for antineutrinos Eq. (15) and vice versa. The dependence of the oscillation formulas on the choice of the mixing matrix representation characterizes the Majorana neutrinos. Therefore, if the neutrinos are Majorana particles, the study of the oscillation formulas in long baseline experiments could also allow the determination of the right mixing matrix.

Notice that similar effects are produced by the following dissipator

$$D_{\lambda\mu} = -\begin{pmatrix}\gamma_1 & 0 & 0\\ 0 & \gamma_2 & \delta\\ 0 & \delta & \gamma_3\end{pmatrix}. \tag{22}$$

On the contrary, the dissipator

$$D_{\lambda\mu} = -\begin{pmatrix}\gamma_1 & 0 & \beta\\ 0 & \gamma_2 & 0\\ \beta & 0 & \gamma_3\end{pmatrix}, \tag{23}$$

generates oscillation formulas depending on the phase ϕ, but there is no CP and CPT violations, being, in this case, $P_{\nu_\sigma\to\nu_\varrho}(t) = P_{\overline{\nu}_\sigma\to\overline{\nu}_\varrho}(t)$.

In order to emphasize the role of off-diagonal elements in the dissipator, we compare the above results with that obtained in the case of diagonal dissipator, i.e. $\alpha = 0$, and $D_{ij} = -\mathrm{diag}(\gamma, \gamma, \gamma_3)$. In this case, we have $\frac{\gamma_3}{2} \leq \gamma$. The oscillation formulas for Majorana neutrinos are independent on ϕ and coincide with those of Dirac neutrinos,

$$P_{\nu_\sigma\to\nu_\varrho}(t) = P_{\overline{\nu}_\sigma\to\overline{\nu}_\varrho}(t) = \frac{1}{2}\Big[1 - e^{-\gamma_3 t}\cos^2 2\theta - \sin^2 2\theta\cos(\Delta t)\,e^{-\gamma t}\Big]. \tag{24}$$

Then, for two flavor neutrino oscillations we have $\Delta_{CP} = \Delta_T = \Delta_{CPT} = 0$. The Pontecorvo formulas [21] are recovered by setting $\gamma = \gamma_3 = 0$ in Eq. (24).

All the results here presented hold for neutrino propagation in vacuum.

In the case of neutrino oscillation through media, the Earth is not charge-symmetric, indeed it contains electrons, protons and neutrons, but it does not contain their antiparticles. Then, (even in absence of decoherence) the behavior of neutrinos is different with respect to that of antineutrinos (we have to consider R_- in Δm_m^2 and $\sin 2\theta_m$ for neutrinos and R_+ for antineutrinos). This fact implies that the $\nu_e \leftrightarrow \nu_\mu$ and $\nu_e \leftrightarrow \nu_\tau$ oscillations in matter break the CP and CPT symmetry also in absence of decoherence. Therefore, the study of the CPT violation induced by the decoherence, together with that on the neutrino nature can be better done by analyzing neutrino oscillations in vacuum.

However, for completeness, we also consider the neutrino propagation through the matter and we follow the procedure introduced in Ref. [18] to describe the quantum decoherence for neutrino oscillations in matter. Then, the decoherence matrix in the matter mass eigenstates basis corresponding to the dissipator in Eq. (8) is

$$\begin{pmatrix}\Gamma_+ + \Gamma_-\cos 4\psi & \alpha\cos 2\psi & \Gamma_-\sin 4\psi\\ \alpha\cos 2\psi & \gamma & \alpha\sin 2\psi\\ \Gamma_-\sin 4\psi & 2\sin 2\psi & \Gamma_+ - \Gamma_-\cos 4\psi\end{pmatrix}, \tag{25}$$

where $\Gamma_\pm = \frac{\gamma\pm\gamma_3}{2}$, $\cos 2\psi = -\frac{\mu}{\sqrt{\mu^2+\nu^2}}$, and $\sin 2\psi = -\frac{\nu}{\sqrt{\mu^2+\nu^2}}$, with $\mu = \left(\sqrt{2}G_F n_e\cos 2\theta - \Delta\right)$ and $\nu = \sqrt{2}G_F n_e\sin 2\theta$. Eq. (25) will be used to derive numerically the CP violation in the matter.

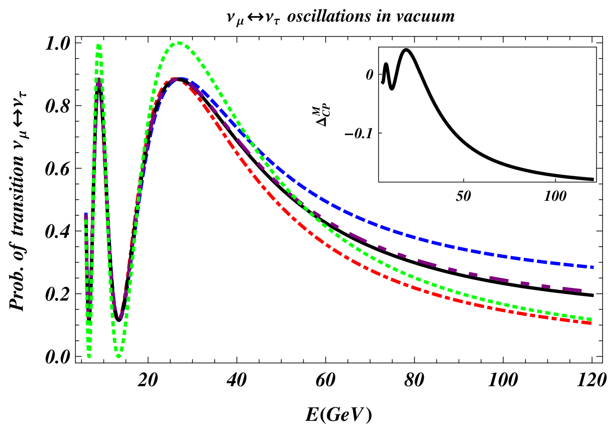

Fig. 1. Plots of the oscillation formulas $P_{\nu_\mu \to \nu_\tau}$ (the red dot dashed line) and $P_{\overline{\nu}_\mu \to \overline{\nu}_\tau}$ (the blue dashed line) for Majorana neutrinos and for Dirac neutrinos ($\phi = 0$, the black line), as a function of the energy E, in vacuum. The purple, dashed line is obtained by setting $\alpha = 0$. In this case $P_{\nu_\mu \to \nu_\tau} = P_{\overline{\nu}_\mu \to \overline{\nu}_\tau}$ and one has the same formula for Majorana and for Dirac neutrinos. The Pontecorvo formula is represented by the green dotted line. We assume $\phi = \frac{\pi}{4}$, $x = 1.3 \times 10^4$ km and we use the following experimental values of the parameters: $\sin^2\theta_{23} = 0.51$, $\Delta m_{23}^2 = 2.5 \times 10^{-3}$ eV^2. Moreover, we set $\gamma = 4 \times 10^{-24}$ GeV, $\gamma_3 = 7.9 \times 10^{-24}$ GeV, $\alpha = 3.8 \times 10^{-24}$ GeV. Picture in the inset: plot of $\Delta_{CP}^M(x)$ for the same values of the parameters used in the main plots. Such plots describe the propagation in vacuum and through Earth.

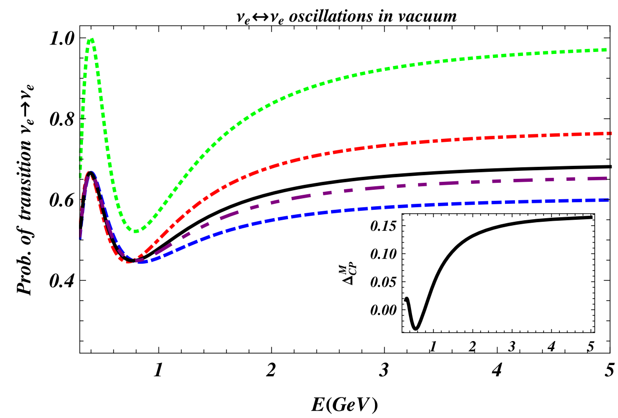

Fig. 2. Plots of $P_{\nu_e \to \nu_e}$ (red dot dashed line) and $P_{\overline{\nu}_e \to \overline{\nu}_e}$ (blue dashed line) for Majorana neutrinos and for Dirac neutrinos ($\phi = 0$, black line), as a function of E, in vacuum. The purple, dashed line is obtained by setting $\alpha = 0$. In this case $P_{\nu_e \to \nu_\mu} = P_{\overline{\nu}_e \to \overline{\nu}_\mu}$. The Pontecorvo formula is represented by the green dotted line. We use the same values of ϕ and x of Fig. 1, moreover, we consider: $\sin^2\theta_{12} = 0.861$, $\Delta m_{12}^2 = 7.56 \times 10^{-5}$ eV^2, $\gamma = 1.2 \times 10^{-23}$ GeV, $\gamma_3 = 2.3 \times 10^{-23}$ GeV, $\alpha = 1.1 \times 10^{-23}$ GeV. Picture in the inset: plot of $\Delta_{CP}^M(x)$.

4. Numerical analysis

We now present a numerical analysis of Eqs. (14), (15), (21) in order to study the nature of neutrinos. Moreover, we consider Eqs. (16) and (17) to analyze the CP and CPT violations in Majorana neutrinos. Finally, we include the matter effect and we compute the CP violation due to the charge asymmetry of the media in the cases of absence and of presence of decoherence.

More precisely, we consider the neutrino oscillation in vacuum. We analyze the mixing between ν_μ and ν_τ and we compute the probability of transition $P_{\nu_\mu \to \nu_\tau}(x)$ and the corresponding oscillation formula for the anti-neutrinos $P_{\overline{\nu}_\mu \to \overline{\nu}_\tau}(x)$ in the range of energies $(6 - 100)$ GeV. We also study the mixing between ν_e and ν_μ and the oscillation formulas $P_{\nu_e \to \nu_e}(x)$ and $P_{\overline{\nu}_e \to \overline{\nu}_e}(x)$ in vacuum. The range of neutrino energy analyzed for $\nu_e \leftrightarrow \nu_\mu$ oscillation is $(0.3 - 5)$ GeV, which is characteristic of DUNE experiment. Moreover, we analyze the neutrino propagation through the matter and we compare the CP asymmetry in absence of decoherence (due to the charge asymmetry of the Earth) with the CP violation induced by the decoherence. We consider the case of constant matter density, which represents a realistic approximation in experiments such as DUNE experiment, whose beam passes through roughly constant matter density. By contrast, for a complete analysis of matter effects in experiments such as IceCube [25], which detects neutrino oscillations from atmospheric cosmic rays over a baseline across the Earth, a strongly changing matter profile must be considered and the PREM model would be used. This analysis is out of the purposes of the present paper and it will be an object of a forthcoming work. In the following, we use in natural units the approximation, $x \approx t$, where x is the distance traveled by neutrinos.

In Fig. 1, we plot the oscillation formulas $P_{\nu_\mu \to \nu_\tau}$ and $P_{\overline{\nu}_\mu \to \overline{\nu}_\tau}$ in vacuum, as a function of the neutrino energy E. The plots refer to Majorana neutrinos and to Dirac neutrinos, (cf. Eqs. (14), (15), (21), respectively). The comparison with the oscillation formula for diagonal dissipator, $\alpha = 0$ (cf. Eq. (24)) and the Pontecorvo-Bilenky oscillation formula is also analyzed. In the inset, we plot the value of the CP asymmetry $\Delta_{CP}^M(x) = P_{\nu_\mu \to \nu_\tau}(x) - P_{\overline{\nu}_\mu \to \overline{\nu}_\tau}(x)$ as a function of E for the same values of the parameters used in the main plot. The plots are derived by assuming $\phi = \frac{\pi}{4}$. We used a distance $x = 1.3 \times 10^4$ km, considered the energy interval $[6 - 120]$ GeV and the following values of the parameters: $\sin^2\theta_{23} = 0.51$, $\Delta m_{23}^2 = 2.55 \times 10^{-3}$ eV^2, $\gamma = 4 \times 10^{-24}$ GeV, $\gamma_3 = 7.9 \times 10^{-24}$ GeV, $\alpha = 3.8 \times 10^{-24}$ GeV [14].

In Fig. 2, we plot the oscillation formulas in vacuum, $P_{\nu_e \to \nu_e}$ and $P_{\overline{\nu}_e \to \overline{\nu}_e}$ and in the inset the CP asymmetry $\Delta_{CP}^M = P_{\nu_e \to \nu_e}(t) - P_{\overline{\nu}_e \to \overline{\nu}_e}(t)$. We use the same values of ϕ and x considered in Fig. 1, moreover we use $\sin^2\theta_{12} = 0.861$, $\Delta m_{12}^2 = 7.56 \times 10^{-5}$ eV^2, $\gamma = 1.2 \times 10^{-23}$ GeV, $\gamma_3 = 2.23 \times 10^{-23}$ GeV, $\alpha = 1.1 \times 10^{-23}$ GeV [15].

By analyzing the plots in Figs. 1 and 2, one can see that the differences between Majorana and Dirac neutrinos, the CP and CPT violations are, in principle, detectable. Indeed, considering the CP violation, which, in the case of two flavor neutrino mixing, is different from zero only for Majorana neutrinos, one finds: a) for $\nu_\mu \leftrightarrow \nu_\tau$ neutrino oscillation in vacuum, in particular ranges of the energy, $\Delta_{CP}^{M(\mu-\tau)} \sim 0.18$, b) for $\nu_e \leftrightarrow \nu_\mu$ neutrino oscillation in vacuum, $\Delta_{CP}^{M(e-e)} \sim 0.16$ (see Figs. 1 and 2). Notice that $\Delta_{CP}^{M(e-e)}$ in the three flavor neutrino mixing case, is not affected by the phase δ of the PMNS matrix. Therefore such a violation, can be analyzed in experiments sensitive to δ phase, like the DUNE experiment.

In Fig. 3, we include the matter effect for the oscillation $\nu_e \leftrightarrow \nu_\mu$ and we plot the CP asymmetry $\Delta_{CP} = P_{\nu_e \to \nu_e}(t) - P_{\overline{\nu}_e \to \overline{\nu}_e}(t)$, for Majorana and for Dirac neutrinos in the presence of decoherence. Moreover, we plot Δ_{CP} in absence of decoherence. In our computations, we consider the range of energy $[0.3 - 1]$ GeV and the electron number density $n_e = 2.2$ cm$^{-3}N_A$, which is the electron densities in the Earth mantle and which fits with the DUNE baseline parameters. The plots show different behaviors of Δ_{CP} for Majorana and for Dirac neutrinos in the presence of decoherence with off-diagonal term. These behaviors are different with respect to that of Δ_{CP} obtained by considering the two flavor neutrino mixing without decoherence. We point out that in the range of energy which we consider to study the CP symmetry in the matter, i.e. $E \in [0, 3 - 1]$ GeV, the differences between the results obtained by using the correct procedure presented in Ref. [18] and those obtained by replacing Δm^2 with Δm_m^2 and $\sin 2\theta$ with $\sin 2\theta_m$, are negligible.

5. Conclusions

We have studied different features of the phenomenon of the decoherence in neutrino oscillations. In particular, we have shown

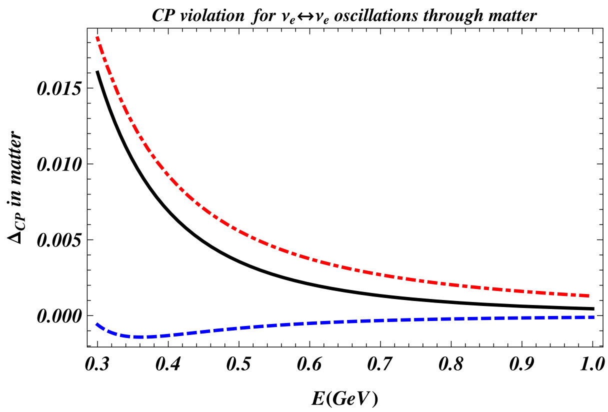

Fig. 3. Plots of $\Delta_{CP} = P_{\nu_e\to\nu_e}(t) - P_{\overline{\nu}_e\to\overline{\nu}_e}(t)$, through matter, for Majorana neutrinos (the red dot dashed line), for Dirac neutrinos (the blue dashed line) in the presence of decoherence with off-diagonal term and for neutrinos in absence of decoherence (the black dotted line). In the plots, we consider the same parameters of Fig. 2 and the energy interval $E \in (0.3 - 1)$ GeV.

the possible CPT symmetry breaking in the Majorana neutrino oscillation and we have shown that the probability of transitions for Majorana neutrinos depend on the representation of the mixing matrix. Moreover, we have studied the phenomenological differences between Majorana and Dirac neutrinos in their oscillations.

By considering the constraints on decoherence parameters [14,15], we have analyzed the oscillation formulas $P_{\nu_\mu\to\nu_\tau}$ and $P_{\overline{\nu}_\mu\to\overline{\tau}_\mu}$, and for neutrinos produced in accelerator, $P_{\nu_e\to\nu_e}$ and $P_{\overline{\nu}_e\to\overline{\nu}_e}$. We have studied the behaviors of CP and CPT violations in neutrino oscillation and we have shown that, the differences between Majorana and Dirac neutrinos, together with the CPT violation could be detected if the phenomenon of decoherence is taken into account during the neutrino propagation in long baseline experiments. Moreover, the oscillation formulas could provide a tool to determine the choice of the mixing matrix for Majorana neutrinos, if the neutrino is a Majorana fermion.

We have shown that the CPT violation, the difference between the oscillation formulas of Majorana and Dirac neutrinos, and the dependence of such formulas on the representation of the mixing matrix appear only in the cases of a not diagonal form of the dissipator, similar to that presented in Eq. (8). In the case of diagonal dissipator, such effects disappear. We consider the neutrino propagation in vacuum and through the matter. Long baseline experiments could allow the determination of the correct form of the matrix describing the decoherence, if such phenomenon is relevant in neutrino oscillation.

Neutrino decoherence and CPT violation could be signals of quantum gravity. Therefore, our analysis could open new interesting scenarios not only in the study of neutrinos, but also in other fields of fundamental physics.

Notice also that, non-perturbative field theoretical effects of particle mixing [26], [27], can be neglected in our treatment.

Acknowledgements

A.C. and G.L. acknowledge partial financial support from MIUR and INFN and the COST Action CA1511 Cosmology and Astrophysics Network for Theoretical Advances and Training Actions (CANTATA). S.M.G. acknowledge support by the H2020 CSA Twinning project No. 692194, “RBI-T-WINNING”.

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