# Large N behaviour of Macdonald-deformed 2D Yang-Mills theory

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based on

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JHEP 1310 (2013) 067, arXiv:1306.1707 [hep-th]

with A. Sinkovics and R. J. Szabo

RBI, Zagreb, 20. March 2019.

# Why 2D Yang-Mills at all?

- Partition function of 4D BPS black holes (BH) compactified in IIA string theory is given by the partition function of topological strings (TS):

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$$\mathcal{Z}_{q\mathrm{YM}}^{SU(N)} \xrightarrow{N\to\infty} \mathcal{Z}_{\mathrm{TS}}$$

$\rightsquigarrow$ Large $N$ expansion of qYM should be related to quantities in TS

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- Phase transition in planar limit
- Expansion of the partition function on closed surface
- New deformation of Hurwitz theory of branched covers
- Expansion of the partition function on surface with boundary terms (not in this talk)
- Expansion of the expectation value of non-intersecting Wilson loop (not in this talk)

# Basics of 2D $U(N)$ YM

- Partition function on 2 dim manifold $\Sigma$:

$$\mathcal{Z}_{\mathrm{YM}}(N;\Sigma,g_{\mathrm{YM}}) = \int [\mathcal{D}A] \mathrm{e}^{-1/4g_{\mathrm{YM}}^2 \int_\Sigma \mathrm{d}^2x \, \mathrm{Tr}(*F \wedge F)}$$

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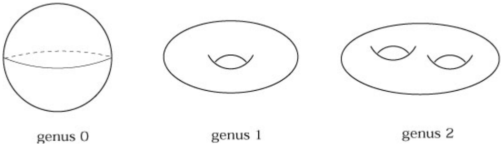
- $\mathcal{Z}_{\mathrm{YM}}$ only depends on the topology and the area: $\mathcal{Z}_{\mathrm{YM}}(N; g_{\mathrm{YM}}, g, A)$

$g$ : genus of $\Sigma$ $\qquad$ $A$ : area of $\Sigma$



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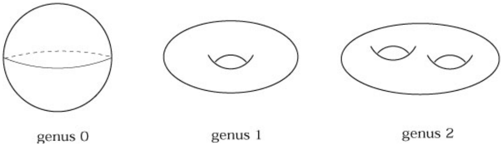


$\Rightarrow$ The 2D YM is exactly solvable

# Basics of 2D $U(N)$ YM

Representation theory of $U(N)$:

- Irreps are labeled by Young diagrams:

$$\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_N] \qquad \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0 \qquad \lambda_i \in \mathbb{Z}_{\geq 0}$$

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$\rightsquigarrow$ Heat kernel expansion for closed surface $\Sigma$:

$$\mathcal{Z}_{\mathrm{YM}}(G; g, g_{\mathrm{YM}}^2 A) = \sum_{\lambda \in \Lambda_+} (\dim(R_\lambda))^{2-2g}\, \mathrm{e}^{-g_{\mathrm{YM}}^2 A/2\, C_2(R_\lambda)}$$

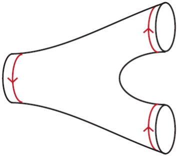
[Migdal '90]

# Basics of 2D $U(N)$ YM

Fundamental building blocks of $\Sigma$:

Cup:

Pants:



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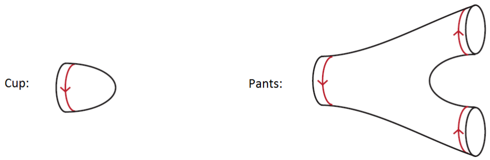


$$\mathcal{Z}_{\mathrm{YM}}(g^2_{\mathrm{YM}}A_{cap}; U) = \sum_{\lambda\in\Lambda_+} \dim(R_\lambda)\,\chi_\lambda(U)\,\mathrm{e}^{\frac{-g^2_{\mathrm{YM}}A_{cap}}{2}C_2(R_\lambda)}$$

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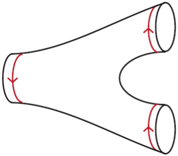

$$\mathcal{Z}_{\mathrm{YM}}(g_{\mathrm{YM}}^2 A_{cap}; U) = \sum_{\lambda \in \Lambda_+} \dim(R_\lambda)\, \chi_\lambda(U)\, \mathrm{e}^{\frac{-g_{\mathrm{YM}}^2 A_{cap}}{2} C_2(R_\lambda)}$$

$$\mathcal{Z}_{\mathrm{YM}}(g_{\mathrm{YM}}^2 A_{pants}; U_1, U_2, U_3) = \sum_{\lambda \in \Lambda_+} \frac{\chi_\lambda(U_1)\, \chi_\lambda(U_2)\, \chi_\lambda(U_3)}{\dim(R_\lambda)}\, \mathrm{e}^{\frac{-g_{\mathrm{YM}}^2 A_{pants}}{2} C_2(R_\lambda)}$$

where $U = \mathcal{P}\mathrm{e}^{\mathrm{i}\oint \mathrm{d}x\, A}$ is the holonomy along the boundary.

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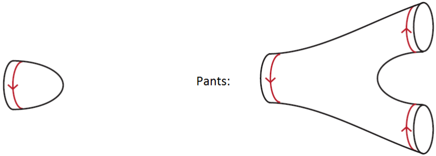


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Gluing two boundaries gives:

$$\int \mathrm{d}U\,\mathcal{Z}_{\rm YM}(g_{\rm YM}^2 A_1; U, \dots)\,\mathcal{Z}_{\rm YM}(g_{\rm YM}^2 A_2; U^\dagger, \dots) = \mathcal{Z}_{\rm YM}(g_{\rm YM}^2 (A_1 + A_2); \dots)$$

# Planar limit of 2D YM on $S^2$

2D YM on $S^2$ $(g = 0)$

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$$S_{\mathrm{eff}}(h) = -\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \log |h(x) - h(y)| + \frac{A}{2} \int_0^1 \mathrm{d}x \, h(x)^2 - \frac{A}{24} - \frac{3}{2}$$

Solution of the saddle-point equation $\Rightarrow$ matrix model techniques

[Douglas, Kazakov '93]

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Phase transition:

- Non-analytical behavior of the solution ⇒ Two phases

small area phase : $A < \pi^2$

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$$\text{small area phase} \;:\; A < \pi^2$$
$$\text{large area phase} \;:\; A > \pi^2$$

- Reason: instantons are suppressed in the small area phase

$$\text{instanton suppression factor}:\; \mathrm{e}^{-N\gamma(A)} \quad \text{with} \quad \gamma(A = \pi^2) = 0$$

# Large $N$ expansion of 2D $U(N)$ YM

- For large $N$, the $SU(N)$ representations factorise to chiral and antichiral parts:

$$R_{\mathrm{SU(N)}} \xrightarrow{N \gg 1} R_{\mathrm{chiral}} \otimes R_{\mathrm{antichiral}}$$

$$R_{\mathrm{chiral}} = \bigoplus_{n \geq 0} \bigoplus_{\lambda \in \Lambda_+^n} R_\lambda$$

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- $\mathcal{Z}_{\mathrm{YM}}$ factorize for large N

$$\mathcal{Z}_{\mathrm{YM}} = \sum_{n,\overline{n}=0}^{\infty} \sum_{\lambda \in \Lambda_+^n} \sum_{\mu \in \Lambda_+^{\overline{n}}} \left(\dim(\overline{R}_\mu R_\lambda)\right)^{2-2g} \mathrm{e}^{-g_{\mathrm{YM}}^2 A/2\left(C_2(R_\lambda)+C_2(R_\mu)+\frac{2n\overline{n}}{N}\right)}$$

$$\dim(\overline{R}_\mu R_\lambda) = \dim(R_\mu)\dim(R_\lambda)\left[1 + \mathcal{O}\left(\frac{1}{N^2}\right)\right] ,$$

where $\overline{R}_\mu R_\lambda$ is a composite partition.

[Gross, Taylor '93]

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$$\mathcal{Z}^{+}_{\mathrm{YM}}(A \to 0) = \sum_{n,i=0}^{\infty} \frac{1}{N^{(2g-2)n-i}} \sum_{s_1, t_1, \ldots, s_g, t_g \in \mathcal{S}_n} \frac{1}{n!}\, \delta\left(\Omega_n^{2-2g} \prod_{i=1}^{g} s_i\, t_i\, s_i^{-1} t_i^{-1}\right)$$

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$$\delta(s) = \sum_{\lambda \in \Lambda^n_+} \dim(r_\lambda) \, \chi_{r_\lambda}(s) \qquad \Omega_n = \sum_{v \in \mathcal{S}_n} N^{K_v - n} v \qquad K_v : \#\{\text{cycles in } v\}$$

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$$\delta(s) = \sum_{\lambda \in \Lambda^n_+} \dim(r_\lambda)\, \chi_{r_\lambda}(s) \qquad \Omega_n = \sum_{v \in \mathcal{S}_n} N^{K_v - n} v \qquad K_v : \#\{\text{cycles in } v\}$$

$\rightsquigarrow$ For $A \to 0$ we get a topological theory

## Branched covers of $\Sigma_g$

- Hurwitz-space :

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- Large $N$ expansion of the chiral part

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$\chi(\Sigma_{g,L})$ is the Euler character of the configuration space of $L$ points on $\Sigma_g$

# Topological string theory

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## $q$-deformed 2D $SU(N)$ YM from TS

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$$X \;=\; \mathcal{O}(2g-2+p)\,\oplus\,\mathcal{O}(-p)\;\longrightarrow\;\Sigma_g \qquad \text{(2 line bundles)}$$

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$$[x]_q = \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \qquad \text{is the } q\text{-number with } [x]_{q \to 1} = x$$

[Aganagic, Ooguri, Saulina, Vafa '05]

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- For large $N$ it factorises to a chiral and antichiral part:

$$\mathcal{Z}_{\rm BH} \sim \left|\mathcal{Z}_{\rm top,\, str}\right|^2$$

# Large $N$ chiral expansion of 2D $U(N)$ qYM

- Chiral partition function for zero area

$$\mathcal{Z}^{+}_{q\mathrm{YM}}(N,g) = \sum_{n=0}^{\infty} \sum_{\lambda \in \Lambda^n_+} \left( \frac{1}{\dim_q(R_\lambda)} \right)^{2g-2}$$

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$$R_{\omega_1}^{\otimes n} \cong \bigoplus_{\lambda \in \Lambda^n_+} R_\lambda \otimes r_\lambda \qquad\qquad r_\lambda \ : \ \text{irrep. of Hecke algebra } \mathcal{H}_q(\mathcal{S}_n)$$

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- Hecke-algebra $\mathcal{H}_q(\mathcal{S}_n)$

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$$\mathrm{g}_i\, \mathrm{g}_{i+1}\, \mathrm{g}_i = \mathrm{g}_{i+1}\, \mathrm{g}_i\, \mathrm{g}_{i+1} \ , \qquad \mathrm{g}_i\, \mathrm{g}_j = \mathrm{g}_j\, \mathrm{g}_i \quad \text{for} \quad |i-j|>1$$

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Linear map $\mathrm{h} : \mathbb{C}\mathcal{S}_n \to \mathcal{H}_q(\mathcal{S}_n)$:

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- This formula has not got a nice geometrical interpretation like the original one ☹

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Macdonald- or $(q,t)$-deformed 2D $SU(N)$ YM

## Refinement of TS and the BPS BH

- Non-refined string theory: one string coupling: $g_s$ ($q = \mathrm{e}^{-g_s}$)
- Refinement: one additional parameter, $\epsilon_1, \epsilon_2$ ($q = \mathrm{e}^{-\epsilon_1}$, $t = \mathrm{e}^{-\epsilon_2}$)

[Iqbal, Kozcaz, Vafa '09]

- Motivation: M theory on $TN \times \mathrm{S}^1 \times X$ ($TN$ is the 4D Taub–NUT space):

$$\mathcal{Z}_M\big(TN \times_q \mathrm{S}^1 \times X\big) = \mathcal{Z}^X_{\mathrm{top}}(g_s)$$

$$\mathcal{Z}_M\big(TN \times_{(q,t)} \mathrm{S}^1 \times X\big) = \mathcal{Z}^X_{\mathrm{ref\ top}}(\epsilon_1, \epsilon_2)$$

- The gauge theory dual ($X$ as before) is

Macdonald- or $(q,t)$-deformed 2D $SU(N)$ YM

- Refined BPS BH partition function:

$\rightsquigarrow$ Counts BPS states with spin

with

$$\mathcal{Z}_{\mathrm{ref\ BH}} = \mathcal{Z}^{U(N)}_{qt\mathrm{YM}} \quad \text{and} \quad \mathcal{Z}_{\mathrm{ref\ BH}} \sim \left|\mathcal{Z}^{\mathrm{ref\ top}}\right|^2$$

[Aganagic, Schaeffer '12]

# The 2D $U(N)$ qtYM

Partition function

$$\mathcal{Z}_{qt\mathrm{YM}}(N,g,p) = \sum_{\lambda\in\Lambda_+} \frac{\dim_{q,t}(R_\lambda)^{2-2g}}{(g_\lambda)^{1-g}}\; q^{\frac{p}{2}(\lambda,\lambda)}\, t^{p\,(\rho,\lambda)} \qquad (\lambda,\mu) := \sum_i (\lambda_i,\mu_i)$$

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$$\dim_{q,t}(R_\lambda) = \prod_{m=0}^{\beta-1} \prod_{1\le i<j\le N} \frac{\left[\lambda_i - \lambda_j + \beta(j-i) + m\right]_q}{\left[\beta(j-i)+m\right]_q}$$

Refined quantum dimension

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$$g_\lambda = \prod_{m=0}^{\beta-1} \prod_{1\le i<j\le N} \frac{\left[\lambda_i-\lambda_j+\beta(j-i)+m\right]_q}{\left[\lambda_i-\lambda_j+\beta(j-i)-m\right]_q}$$

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$$\beta = \frac{\log t}{\log q} = \frac{\epsilon_2}{\epsilon_1}$$ and $$\rho_i = \frac{N+1}{2} - i$$ Weyl vector

$\beta$ can be analytically continued away later

# Planar limit of qtYM on $S^2$

- Planar limit with $p \neq 0$:

$$N \to \infty, \qquad \epsilon_1, \epsilon_2 \to 0, \qquad \tau_{1,2} = \epsilon_{1,2} N \quad \text{fixed 't Hooft parameters}$$

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$$S_{\mathrm{eff}}[h] = -\,\beta \int_0^1 \int_0^1 \mathrm{d}x\mathrm{d}y \,\log\sinh\left|\frac{\tau_2(h(x)-h(y))}{2}\right| + \\ + \frac{\beta p \tau_2}{2} \int_0^1 \mathrm{d}x\, h^2(x) + + \frac{2\beta}{\tau_2^2} F_0^{\mathrm{CS}}(\tau_2) - \frac{\beta \tau_2 p}{24},$$

$F_0^{\mathrm{CS}}(\tau)$: planar Chern-Simons free energy on $S^3$

## Saddle point equation of qtYM on $S^2$

- Density function

$$\rho(h) = \frac{\mathrm{d}x(h)}{\mathrm{d}h}$$

- From the definition of the Young diagram:

$$0 < \rho(h) \leq 1 \qquad \text{and} \qquad \int \rho(h)\mathrm{d}h = 1.$$

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- Saddle-point equation:

$$\left. \frac{\delta S_{\text{eff}}[h]}{\delta h} \right|_{h=h_0} = 0$$

$$\Rightarrow \quad ph = \mathrm{P} \int \mathrm{d}h' \, \rho(h') \coth \left( \frac{\tau_2}{2}(h - h') \right)$$

# Phase transition of qtYM on $S^2$

- Solution of the saddle-point equation (Using matrix model techinques)

$$\rho(h) = \frac{p}{\pi} \arctan\left( \frac{\sqrt{e^{\tau_2/p} - \cosh^2\left(\frac{\tau_2 h}{2}\right)}}{\cosh\left(\frac{\tau_2 h}{2}\right)} \right) \qquad \rho(h)\big|_{\max} = \frac{p}{2}$$

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Phase transition

- $p \leq 2$: no phase transition
- $p > 2$: we have a critical line:

$$A^*(p^*) = {p^*}^2 \log\left(1 + \tan^2 \frac{\pi}{p^*}\right),$$

$$A := \tau_2 p$$

- $p \rightarrow \infty$: the original Douglas-Kazakov phase transition

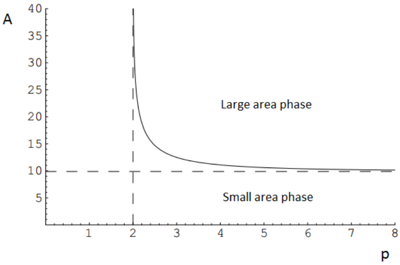


Figure: Phase diagram

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- Macdonald polynomials can be expressed as

$$\chi_{\Phi_\lambda}(\mathrm{e}^{(z,H)}) = \mathrm{Tr}_{R_{\lambda_\beta}}\big(\Phi_\lambda\, \mathrm{e}^{(z,H)}\big) = \frac{M_\lambda(\mathrm{e}^z; q, t)}{\sqrt{g_\lambda}} \qquad \text{and} \qquad \lambda_\beta = \lambda + (\beta - 1)\rho$$

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- The refined quantum dimension:

$$\dim_{q,t}(R_\lambda) = \chi_{\Phi_\lambda}(t^{(\rho,H)})$$

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$$P_\lambda = \frac{d_\lambda(q)}{q^{\frac{n\,(n-1)}{4}}\,[n]_q!} \sum_{\sigma \in \mathcal{S}_n} q^{-\ell(\sigma)} \, \chi_{r_\lambda}\left(\mathrm{h}(\sigma^{-1})\right) \mathrm{h}(\sigma)$$

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$\Rightarrow$ We can use this $\Phi_n$ to express the generalized character:

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\left(\Phi_n \, U \, P_\lambda\right) \; = \; \chi_{\lambda_{\beta-2}}(U) \, d_\lambda(1)$$

# Refined quantum dimension

- Using $P_\lambda$ and take $U = t^{(\rho,H)}$

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} = \frac{q^{-\frac{n(n-1)}{4}}}{[n]_q!} \frac{d_{\lambda_\beta + a\mathbb{I}}(q)}{d_{\lambda_\beta + a\mathbb{I}}(1)} \sum_{\sigma \in \mathcal{S}_n} q^{-\ell(\sigma)} \chi_{r_{\lambda_\beta + a\mathbb{I}}}(\mathsf{h}(\sigma^{-1})) \operatorname{Tr}_{R_{\omega_1}^{\otimes n}}(\Phi_n t^{(\rho,H)} \mathsf{h}(\sigma))$$

where $a = (\beta - 1)\frac{N+1}{2}$ and $\mathbb{I} = (1, \ldots, 1)$

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- Reduce the trace $\operatorname{Tr}_{R_{\omega_1}^{\otimes n}}(\Phi_n \, U \, \mathrm{h}(\sigma))$ using cyclicity

$$\operatorname{Tr}_{R_{\omega_1}^{\otimes n}}(\Phi_n U \, x \, y) = \operatorname{Tr}_{R_{\omega_1}^{\otimes n}}(\Phi_n U \, y \, x) \qquad \text{for} \quad x, y \in \mathcal{H}_q(\mathcal{S}_n)$$

to trace of minimal words $m_\mu \in \mathcal{S}_n$:

$$\operatorname{Tr}(\mathrm{h}(\sigma)) = \sum_{\mu \in \Lambda_+^n} \alpha_\mu(\sigma) \operatorname{Tr}(\mathrm{h}(m_\mu))$$

A minimal word contains each generator $\mathrm{g}_i$ of $\mathcal{H}_q(\mathcal{S}_n)$ at most once

$$\operatorname{Tr}(\mathrm{g}_2\,\mathrm{g}_1\,\mathrm{g}_3\,\mathrm{g}_2\,\mathrm{g}_1) = (q^2 - q + 1)\operatorname{Tr}(\mathrm{g}_1\,\mathrm{g}_2\,\mathrm{g}_3) + q(q-1)\operatorname{Tr}(\mathrm{g}_2\,\mathrm{g}_3)$$

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The expansion with minimal words :

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$$C_\mu = q^{n-\ell(\mu)} \sum_{\mu \in \Lambda^n_+} q^{-\ell(\sigma)} \alpha_\mu(\sigma^{-1}) \, \mathrm{h}(\sigma) \qquad \text{central elements}$$

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- While $g_i$ acts on $R_{\omega_1}^{(i)} \otimes R_{\omega_1}^{(i+1)}$ as $PR$, where

$$P(v \otimes w) = w \otimes v \qquad \text{flipping operator}$$

$$\mathrm{R} = q^{1/2} \sum_{i=1}^{N} H_i \otimes H_i + \sum_{i \neq j} H_i \otimes H_j + \left(q^{1/2} - q^{-1/2}\right) \sum_{i>j} E_{ij} \otimes E_{ji}$$

is the $R$-matrix with $E_{ji} e_k = \delta_{ik} e_j$.

# Refined quantum dimension

We calculate a minimal world recursively

$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\left(\Phi_n\, U\, \mathrm{g}_1\, \mathrm{g}_2 \cdots \mathrm{g}_{n-1}\right)$$
$$= (g_{\omega_1})^{-n/2}\ \mathrm{Tr}_{R_{\omega_1}}\Big(U\left(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}\right)\left(\left(U \otimes \mathbb{1}_{R_{\omega_1}}\right)\mathrm{g}_1\right)\ldots$$
$$\times\ \left(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-2)}\right)\left(\left(U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-2)}\right)\mathrm{g}_1\right)\left(\mathrm{Tr}_{R_{\omega_1}} \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)}\right)\left(\left(U \otimes \mathbb{1}_{R_{\omega_1}}^{\otimes(n-1)}\right)\mathrm{g}_1\right)\Big)$$

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Solution of the recurrence equations

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$$\mathrm{Tr}_{R_{\omega_1}^{\otimes n}}\left(\Phi_n\, U\, g_1\, g_2 \cdots g_{n-1}\right) = (g_{\omega_1})^{-n/2} \sum_{k=0}^{n-1} f_k^{(n-1)}\ \mathrm{Tr}_{R_{\omega_1}}\left(U^{k+1}\right)$$

$$f_0^{(m+1)} = \sum_{k=0}^{m} f_k^{(m)}\ t^{(k+1)\frac{N+1}{2}}\ \frac{q-1}{t^{k+1}-1}$$
$$f_{k+1}^{(m+1)} = f_k^{(m)}\ \frac{t^{k+1}-q}{t^{k+1}-1}$$

# Refined quantum dimension

$$\frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} = \frac{q^{-\frac{n(n-1)}{4}}}{[n]_q!} \frac{d_{\lambda_\beta + a\mathbb{I}}(q)}{d_{\lambda_\beta + a\mathbb{I}}(1)} \left( \frac{[\beta N]_q}{\sqrt{g_{\omega_1}}\,[\beta]_q} \right)^n \chi_{r_{\lambda_\beta + a\mathbb{I}}}(\Omega_n(q,t))$$

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$$\Omega_n(q,t) = \left( \frac{[\beta]_q}{[\beta N]_q} \right)^n t^{n\frac{N+1}{2}} \sum_{\mu \in \Lambda_+^n} q^{\ell(\mu)} \times \prod_{i=1}^{\ell(\mu)} \left\{ \sum_{k=1}^{\mu_i} t^{-k\,\frac{N+1}{2}} \frac{(q^{-1};t)_k}{(q-1)\,(t;t)_{k-1}} \zeta_{\mu_i - k}(q,t) \frac{[k\beta N]_q}{[k\beta]_q} \right\} C_\mu$$

$$\zeta_m(q,t) := \sum_{\lambda \in \Lambda_+^m} L_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{(q^{-1};t)_{\lambda_i}}{(t^{\lambda_i}-1)\,(t;t)_{\lambda_i - 1}} \qquad \text{for} \quad m > 0\,,$$

$$\zeta_0(q,t) := 1$$

$$(a;q)_k := \prod_{j=0}^{k-1} (1 - a\,q^j) \qquad \text{for} \quad 0 < k \le \infty \qquad q\text{-Pochhammer symbols}$$

$L_\lambda$: number of distinguishable orderings of $\lambda_i$: $L_{(2,1)} = 2$ and $L_{(1,1)} = 1$

# Partition function for zero area

- Chiral part of the partition funciton

$$\mathcal{Z}^{+}_{qt\mathrm{YM}}(N,g,0) = \sum_{n=0}^{\infty} \sum_{\lambda \in \Lambda^n_+} \left( \frac{\dim_{q,t}(R_\lambda)}{\sqrt{g_\lambda}} \right)^{2-2g}$$

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$$\chi_{r_\lambda}(C)\,\chi_{r_\lambda}(\mathsf{h}(\sigma)) = d_{r_\lambda}(1)\,\chi_{r_\lambda}\big(C\,\mathsf{h}(\sigma)\big) \qquad \forall C\in Z(\mathcal{H}_q(\mathcal{S}_n))$$

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- Using the expansion of the refined quantum dimension gives

$$\mathcal{Z}^{+}_{qt\mathrm{YM}}(N,g,0) = \sum_{n=aN}^{\infty} \left(\frac{q^{-\frac{n\,(n-1)}{4}}}{[n]_q!}\right)^{2} \left(\frac{[\beta N]_q}{\sqrt{g_{\omega_1}}\,[\beta]_q}\right)^{(2-2g)n} \sum_{\{s_i,t_i\},\in\mathcal{S}_n} q^{-\sum_i\big(\ell(s_i)+\ell(t_i)\big)}$$

$$\times \sum_{\lambda\in\Lambda_+^{n-aN}} d_{\lambda_\beta+a\mathbb{I}}(q)\,\chi_{r_{\lambda_\beta+a\mathbb{I}}}\left(D_n\,\Omega_n^{2-2g}\prod_{i=1}^{g}\mathrm{h}(s_i)\,\mathrm{h}(t_i)\,\mathrm{h}(s_i^{-1})\,\mathrm{h}(t_i^{-1})\right)$$

# Partition function for zero area

- Step-function $\Theta_n$ on Young diagrams

$$\chi_\mu(\Theta_n(a,\beta)) = \begin{cases} d_\mu(1) & \mu_i \geq (\beta-1)\rho_i + a \quad \forall i \\ 0 & \text{otherwise} \end{cases}$$

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- Final expression for $\mathcal{Z}^+_{q\mathrm{tYM}}(N,g,0)$

$$\mathcal{Z}^+_{q\mathrm{tYM}}(N,g,0) = \sum_{n=\lceil aN\rceil}^{\infty} \frac{q^{-\frac{n(n-1)}{4}}}{[n]_q!} \left(\frac{[\beta N]_q}{\sqrt{g_{\omega_1}}\,[\beta]_q}\right)^{(2-2g)n} \sum_{\{s_i,t_i\},\in\mathcal{S}_n} q^{-\sum_i \left(\ell(s_i)+\ell(t_i)\right)} \times \delta\left(\Theta_n\, D_n\, \Omega_n^{2-2g} \prod_{i=1}^{g} \mathrm{h}(s_i)\,\mathrm{h}(t_i)\,\mathrm{h}(s_i^{-1})\,\mathrm{h}(t_i^{-1})\right)$$

$\rightsquigarrow$ $\beta$ can be analytically continued away from $\mathbb{Z}_{\geq 1}$

## New deformation of Hurwitz theory

Jack polynomial limit : $t = q^\beta$ then $q \to 1$

$$\lim_{q\to 1} \mathcal{Z}^{+}_{qt\mathrm{YM}}(N, g, 0)\Big|_{t=q^\beta} = \sum_{n=\lceil a\,N\rceil}^{\infty} \sum_{d=0}^{\infty} \left(\frac{1}{N}\right)^{2g-2} \sum_{L=0}^{d} \overbrace{\chi_{n,d,g,L}(\beta)}^{\text{parametrized Euler characters}}$$

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$\Rightarrow$ Generating function of $\chi_{n,d,g,L}(\beta)$

$$\chi_{n,d,g,L}(\beta) := \sum_{\lambda\in\Lambda^n_+} \frac{\omega_\lambda(\beta)}{\beta^{n\,(h-1)}} \sum_{\substack{\mu^1,\dots,\mu^L\in\Lambda^n_+ \\ \mu^l\neq(1^n)\,,\,\sum_{l=1}^L \ell^*(\mu^l)=d}} \varDelta_{\mu^1}(\beta)\cdots\varDelta_{\mu^L}(\beta) \times \chi(\Sigma_{h,L})\, H_{h,n}(\lambda,\mu^1,\dots,\mu^L)$$

$$\varDelta_\mu(\beta) = \prod_{i=1}^{\ell(\mu)} \sum_{k=1}^{\mu_i} \frac{\Gamma(k-\frac{1}{\beta})}{\Gamma(1-\frac{1}{\beta})\,\Gamma(k)} \sum_{\lambda\in\Lambda^{\mu_i-k}_+} \beta^{-\ell(\lambda)}\,\frac{\ell(\lambda)!}{z_\lambda} \prod_{j=1}^{\ell(\lambda)} \frac{\Gamma(\lambda_j-\frac{1}{\beta})}{\Gamma(1-\frac{1}{\beta})\,\Gamma(\lambda_j)}\,,$$

$$\omega_\lambda(\beta) = \frac{1}{n!} \sum_{\mu\in\Lambda^n_+} d_\mu(1)\,\chi_{r_\mu}(m_{T_\lambda})$$

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1605.03748; 1306.1707

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- Expand other observable, the defect holonomy punctures on the surface.
- Interpret geometrically the deformed Hurwitz theory.

Thank you for your attention!