Nonassociative Geometry and Non-Geometric Backgrounds

Richard Szabo

HERIOT WATT UNIVERSITY

MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES

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Quantum Structure of Spacetime

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D-branes and the Lie algebra $\mathfrak{su}(2)$

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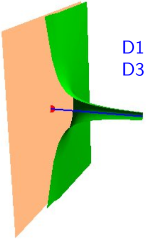

	0	1	2	3	4	5	6
D1	×						×
D3	×	×	×	×			

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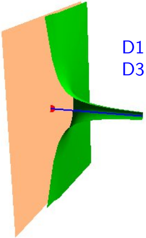

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(BPS) monopoles $\equiv$ D1-branes ending on D3-branes

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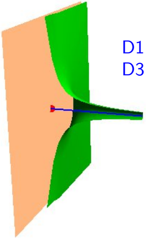

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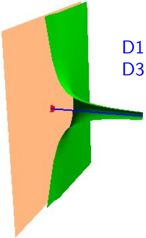

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Solution: $T^i(s) = \frac{1}{s}\,X^i$, $\qquad [X^i, X^j] = \varepsilon^{ijk}\,X^k$ fuzzy S^2 (Myers '99)

M-branes and the 3-Lie algebra A_4

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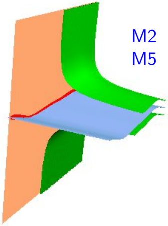

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M2	×					×	×
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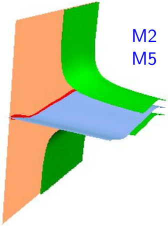

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$\equiv$ self-dual strings

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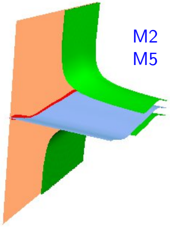

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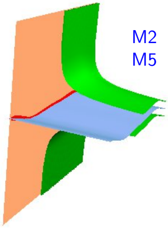

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- Low-energy dynamics described by noncommutative gauge theory on D3-brane

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- Quantization unknown

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- $M = T^3$ with H-flux gives geometric and non-geometric fluxes via T-duality (Hull '05; Shelton, Taylor & Wecht '05)

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- In Double Field Theory: $H_{ijk} = \partial_{[i} B_{jk]} \xrightarrow{T_{ijk}} R^{ijk} = \hat{\partial}^{[i} \beta^{jk]}$ (Andriot, Hohm, Larfors, Lüst & Patalong '12)

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- Closure, 3-cyclicity: $\int f\star g = \int f\,g \ , \ \int (f\star g)\star h = \int f\star(g\star h)$

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- Positivity, reality, ... using closure, 3-cyclicity,
  Hermiticity $(f \star g)^* = g^* \star f^*$, and unitality $f \star 1 = f = 1 \star f$

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- Minimal area and volume:

$$\langle A^{x^i, p_j} \rangle \ = \ \hbar\, \delta^i{}_j \ , \qquad \langle A^{ij} \rangle \ = \ \tfrac{\ell_s^3}{3\hbar}\, R^{ijk}\, \langle p_k \rangle \ , \qquad \langle V^{ijk} \rangle \ = \ \tfrac{\ell_s^3}{2}\, R^{ijk}$$

Quantized spacetime with cells of minimal volume $\frac{\ell_s^3}{2}\, R^{ijk}$

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Shifted coordinates $\widetilde{x}^I := x^I - \langle x^I \rangle$

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$$\langle A^{x^i, p_j} \rangle = \hbar\, \delta^i{}_j\ , \qquad \langle A^{ij} \rangle = \tfrac{\ell_s^3}{3\hbar}\, R^{ijk}\, \langle p_k \rangle\ , \qquad \langle V^{ijk} \rangle = \tfrac{\ell_s^3}{2}\, R^{ijk}$$

Quantized spacetime with cells of minimal volume $\frac{\ell_s^3}{2}\, R^{ijk}$

- Freed–Witten anomaly: No D3-branes on T^3 with H-flux $\xrightarrow{T_{ijk}}$ No D0-branes in R-flux background (Wecht '07)

Nonassociative geometry

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- On-shell associativity: $\int f \vartriangle g \vartriangle h \ = \ \int f\, g\, h$

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- Choice $R^{4,\mu\nu\alpha\beta} = R\,\varepsilon^{\mu\nu\alpha\beta}$ breaks $SL(5) \longrightarrow SO(4)$

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Restored for $R = 0$: $H_2(T^3, \mathbb{Z}) = \mathbb{Z}^3$

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$$
\begin{aligned}
[x^i, x^j] &= \frac{\mathrm{i}\,\ell_s^3}{3\hbar}\, R^{4,ijk4}\, p_k \ , \qquad [x^4, x^i] = \frac{\mathrm{i}\,\lambda\,\ell_s^3}{3\hbar}\, R^{4,1234}\, p^i \\
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- Reduces to string R-flux algebra at $\lambda = 0$ (with $x^4 = 1$ central)

Octonionic phase space

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- Originates from nonassociative, alternative, octonion algebra $\mathbb{O}$:

$$(x^A) \;=\; (x^i, x^4, p_i) \;=\; \Lambda\,(e_A) \;=\; \tfrac{1}{2\hbar}\,\big(\sqrt{\lambda\,\ell_s^4\,R/3}\;f_i\,,\;\sqrt{\lambda^3\,\ell_s^4\,R/3}\;e_7\,,\;-\lambda\,\hbar\,e_i\big)$$

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- Imaginary unit octonions:

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- Octonionic commutator $[e_A, e_B] \;=\; 2\,\eta_{ABC}\,e_C$ has components $(f_i \;=\; e_{i+3}\;,\; i=1,2,3)$:

$$[e_i, e_j] \;=\; 2\,\varepsilon_{ijk}\,e_k\;, \qquad [e_7, e_i] \;=\; 2\,f_i$$

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- Jacobiator:

$$[e_A, e_B, e_C] \;=\; -12\,\eta_{ABCD}\,e_D \;=\; 6\big((e_A\,e_B)\,e_C - e_A\,(e_B\,e_C)\big)$$

$$\eta_{ABCD} \;=\; +1 \;\; \text{for} \;\; ABCD \;=\; 1267,\ 1425,\ 1346,\ \dots$$

G_2-structures

(Kupriyanov & RS '17)

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$(\mathbb{R}^0, 0)$	$(\mathbb{R}^1, 0)$	$(\mathbb{R}^3, \times)$	$(\mathbb{R}^7, \times_\eta)$
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- Alternativity $(X\, X)\, X = X\, (X\, X)$ defines octonion exponential:

$$\mathrm{e}^{\,X_{\vec{k}}} = \cos |\vec{k}|\ \mathbb{1} + \frac{\sin |\vec{k}|}{|\vec{k}|}\ X_{\vec{k}}$$

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- Noncommutativity/nonassociativity:

$$\begin{aligned} \vec{p} \circledast_\eta \vec{p}\,' - \vec{p}\,' \circledast_\eta \vec{p} &= -2\,\vec{p}\times_\eta \vec{p}\,' \\ \vec{A}_\eta(\vec{p},\vec{p}\,',\vec{p}\,'') &= (\vec{p} \circledast_\eta \vec{p}\,') \circledast_\eta \vec{p}\,'' - \vec{p} \circledast_\eta (\vec{p}\,' \circledast_\eta \vec{p}\,'') = \tfrac{2}{3}\,\vec{J}_\eta(\vec{p},\vec{p}\,',\vec{p}\,'') \end{aligned}$$

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$$\begin{aligned} \vec{p} \circledast_\eta \vec{p}\,' - \vec{p}\,' \circledast_\eta \vec{p} \quad &= \quad -2\,\vec{p} \times_\eta \vec{p}\,' \\ \vec{A}_\eta(\vec{p},\vec{p}\,',\vec{p}\,'') \quad &= \quad (\vec{p} \circledast_\eta \vec{p}\,') \circledast_\eta \vec{p}\,'' - \vec{p} \circledast_\eta (\vec{p}\,' \circledast_\eta \vec{p}\,'') \;=\; \tfrac{2}{3}\,\vec{J}_\eta(\vec{p},\vec{p}\,',\vec{p}\,'') \end{aligned}$$

- Extend to all $\vec{k} \in \mathbb{R}^7$:

$$\vec{\mathcal{B}}_\eta(\vec{k},\vec{k}') \quad = \quad \frac{\sin^{-1}|\vec{p} \circledast_\eta \vec{p}\,'|}{\hbar\,|\vec{p} \circledast_\eta \vec{p}\,'|}\;\vec{p} \circledast_\eta \vec{p}\,' \,\Big|_{\vec{p}=\vec{k}\,\sin(\hbar\,|\vec{k}|)/|\vec{k}|}$$

Octonionic BCH formula

- BCH formula $\mathrm{e}^{X_{\vec k}}\,\mathrm{e}^{X_{\vec k'}} = \mathrm{e}^{X_{\vec{\mathcal{B}}_\eta(\vec k,\vec k')}}$ can be computed explicitly in terms of **vector star sums** of $\vec p,\vec p\,' \in B^7 \subset \mathbb{R}^7$:

$$\vec p \circledast_\eta \vec p\,' = \epsilon_{\vec p,\vec p\,'}\Big(\sqrt{1-|\vec p\,'|^2}\;\vec p + \sqrt{1-|\vec p\,|^2}\;\vec p\,' - \vec p \times_\eta \vec p\,'\Big)$$

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Quantization of M-theory phase space

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- Nonassociative phase space star product:

$$(f \star_\lambda g)(\vec{x}) \;=\; \int_{\vec{k},\vec{k}'} \tilde{f}(\vec{k})\,\tilde{g}(\vec{k}'\,)\;\mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\eta(\Lambda\,\vec{k},\Lambda\,\vec{k}'\,)\cdot\Lambda^{-1}\,\vec{x}}$$

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- Nonassociative BCH formula captured by "2-group addition":

$$\big(\mathrm{e}^{\,\mathrm{i}\,\vec{k}\cdot\vec{x}}\star_\lambda \mathrm{e}^{\,\mathrm{i}\,\vec{k}'\cdot\vec{x}}\big)\star_\lambda \mathrm{e}^{\,\mathrm{i}\,\vec{k}''\cdot\vec{x}} \ = \ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{A}}_\eta(\Lambda\,\vec{k},\Lambda\,\vec{k}',\Lambda\,\vec{k}'')}\ \mathrm{e}^{\,\mathrm{i}\,\vec{k}\cdot\vec{x}}\star_\lambda\big(\mathrm{e}^{\,\mathrm{i}\,\vec{k}'\cdot\vec{x}}\star_\lambda \mathrm{e}^{\,\mathrm{i}\,\vec{k}''\cdot\vec{x}}\big)$$

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- Gauge-equivalent closed, 3-cyclic star product:

$$f\bullet_\lambda g \ = \ \mathcal{D}^{-1}\big(\mathcal{D}f\star_\lambda\mathcal{D}g\big)\ , \qquad \mathcal{D} \ = \ 1+O(\lambda)$$

Spacetime quantization in M-theory

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- Minimal areas:

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\begin{aligned}
\langle A^{ij} \rangle &= \tfrac{\ell_s^3}{3\hbar} \left| R^{4,ijk4} \langle p_k \rangle \right| \, , \qquad \langle A^{4i} \rangle \;=\; \tfrac{\lambda \, \ell_s^3}{3\hbar} \left| R^{4,1234} \langle p^i \rangle \right| \\
\langle A^{x^i,p_j} \rangle &= \hbar \left| \delta^i_j \langle x^4 \rangle + \lambda \, \varepsilon^i{}_{jk} \langle x^k \rangle \right| \, , \qquad \langle A^{x^4,p_i} \rangle \;=\; \lambda^2 \, \hbar \, \langle x_i \rangle \\
\langle A^{p_i,p_j} \rangle &= \lambda \, \hbar \left| \varepsilon_{ijk} \langle p^k \rangle \right|
\end{aligned}
$$

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$$\langle A^{x^i,p_j}\rangle = \hbar\left|\delta^i_j\,\langle x^4\rangle + \lambda\,\varepsilon^i{}_{jk}\,\langle x^k\rangle\right| , \qquad \langle A^{x^4,p_i}\rangle = \lambda^2\,\hbar\,\langle x_i\rangle$$

$$\langle A^{p_i,p_j}\rangle = \lambda\,\hbar\left|\varepsilon_{ijk}\,\langle p^k\rangle\right|$$

- Minimal volumes:

$$\langle V^{ijk}\rangle = \tfrac{\ell_s^3}{2}\left|R^{4,ijk4}\,\langle x^4\rangle\right| , \qquad \langle V^{ij4}\rangle = \tfrac{\lambda^2\,\ell_s^3}{2}\left|R^{4,ijk4}\,\langle x_k\rangle\right|$$

$$\langle V^{p_i,x^j,x^k}\rangle = \tfrac{\lambda\,\ell_s^3}{2}\left|R^{4,1234}\left(\delta^j_i\,\langle p^k\rangle - \delta^k_i\,\langle p^j\rangle\right)\right| , \qquad \langle V^{p_i,x^j,x^4}\rangle = \tfrac{\lambda^2\,\ell_s^3}{2}\,R^{4,ijk4}\,\langle p_k\rangle$$

$$\langle V^{p_i,p_j,x^k}\rangle = \tfrac{\lambda\,\hbar^2}{2}\left|\lambda\,\varepsilon_{ij}{}^k\,\langle x^4\rangle + \delta^k_j\,\langle x_i\rangle - \delta^k_i\,\langle x_j\rangle\right| , \qquad \langle V^{p_i,p_j,x^4}\rangle = \tfrac{\lambda^3\,\hbar^2}{2}\left|\varepsilon_{ijk}\,\langle x^k\rangle\right|$$

Nonassociative geometry in M-theory

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- Configuration space triproducts:

$$
\begin{aligned}
\big(f \vartriangle_\lambda g \vartriangle_\lambda h\big)(\vec x\,) &= \big((f \star_\lambda g) \star_\lambda h\big)(x^\mu, p_i)\big|_{p=0} \\
&= \int_{\vec k, \vec k', \vec k''} \tilde f(\vec k\,)\, \tilde g(\vec k'\,)\, \tilde h(\vec k''\,)\ \mathrm{e}^{\,\mathrm{i}\, \vec{\mathcal{T}}_\Lambda(\vec k, \vec k', \vec k''\,) \cdot \Lambda^{-1}\, \vec x}
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$$

$$
\begin{aligned}
&\vec{\mathcal{T}}_\Lambda(\vec{k},\vec{k}',\vec{k}''\,) \\
&= \frac{\sin^{-1}\big|(\vec{p} \circledast_\eta \vec{p}\,') \circledast_\eta \vec{p}\,''\big|}{\hbar\,\big|(\vec{p} \circledast_\eta \vec{p}\,') \circledast_\eta \vec{p}\,''\big|}\, \Big(\vec{A}_\eta(\vec{p},\vec{p}\,',\vec{p}\,''\,) + \epsilon_{\vec{p}\,',\vec{p}\,''}\, \sqrt{1-\big|\vec{p}\,' \circledast_\eta \vec{p}\,''\big|^2}\ \vec{p} \\
&\quad + \epsilon_{\vec{p},\vec{p}\,''}\, \sqrt{1-\big|\vec{p} \circledast_\eta \vec{p}\,''\big|^2}\ \vec{p}\,' + \epsilon_{\vec{p},\vec{p}\,'}\, \sqrt{1-\big|\vec{p} \circledast_\eta \vec{p}\,'\big|^2}\ \vec{p}\,''\Big)\Big|_{\vec{p}=\Lambda\vec{k}\ \sin(\hbar\,|\Lambda\vec{k}|)/|\Lambda\vec{k}|}
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- Quantizes 3-bracket $[f,g,h]_{\triangle_\lambda} = \mathrm{Asym}(f \,\triangle_\lambda\, g \,\triangle_\lambda\, h)$ for A_4:

$$[x^\mu, x^\nu, x^\alpha]_{\triangle_1} = \ell_s^3\, R\, \varepsilon^{\mu\nu\alpha\beta}\, x^\beta$$

Noncommutative M-theory momentum space

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- Setting $x^\mu = 0$ reveals noncommutative associative deformation of momentum space with $\times_\eta \longrightarrow \times$, independent of R-flux:

(Guedes, Oriti & Raasakka '13; Kupriyanov & Vitale '15)

$$(f \star_\lambda g)(\boldsymbol{p}) = \int_{\boldsymbol{l},\boldsymbol{l}'} \tilde{f}(\boldsymbol{l})\, \tilde{g}(\boldsymbol{l}')\, \mathrm{e}^{-\frac{2\,\mathrm{i}}{\lambda}\, \boldsymbol{\mathcal{B}}(-\frac{\lambda}{2}\,\boldsymbol{l}, -\frac{\lambda}{2}\,\boldsymbol{l}')\cdot \boldsymbol{p}}$$

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- Familiar from 3D quantum gravity (Freidel & Livine '06)

Spin(7)-**structures**

$Spin(7)$-structures

- **Triple cross product** of $K = (K_{\hat{A}}) = (k_0, \vec{k}) \in \mathbb{R}^8 = \mathbb{R} \oplus \mathbb{R}^7$:

$$(K \times_\phi K' \times_\phi K'')_{\hat{A}} := \phi_{\hat{A}\hat{B}\hat{C}\hat{D}} K_{\hat{B}} K'_{\hat{C}} K''_{\hat{D}}$$

$$\phi_{0ABC} = \eta_{ABC} \ , \qquad \phi_{ABCD} = \eta_{ABCD}$$

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$$= \left(\vec{k} \cdot (\vec{k}' \times_\eta \vec{k}'') \ , \ \tfrac{1}{12} \vec{J}_\eta(\vec{k}, \vec{k}', \vec{k}'') - k_0 (\vec{k}' \times_\eta \vec{k}'') - k_0' (\vec{k}'' \times_\eta \vec{k}) - k_0'' (\vec{k} \times_\eta \vec{k}')\right)$$

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- Preserved by $Spin(7) \subset SO(8)$, extends representation $X_{\vec{k} \times_\eta \vec{k}'} = \frac{1}{2}\left[X_K, X_{K'}\right]$ on octonions $X_K = k_0\, \mathbb{1} + k^A\, e_A \in \mathbb{O}$:

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- $\mathit{Spin}(7) \longrightarrow G_2$: $\vec{k} \times_\eta \vec{k}' = \vec{k} \times_\phi (1, \vec{0}\,) \times_\phi \vec{k}'$

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$$(K \times_\phi K' \times_\phi K''\,)_{\hat{A}} := \phi_{\hat{A}\hat{B}\hat{C}\hat{D}}\, K_{\hat{B}}\, K'_{\hat{C}}\, K''_{\hat{D}}$$

$$\phi_{0ABC} = \eta_{ABC} \ , \qquad \phi_{ABCD} = \eta_{ABCD}$$

$$K \times_\phi K' \times_\phi K'' \\ = \ \big(\vec{k} \cdot (\vec{k}' \times_\eta \vec{k}''\,)\,,\ \tfrac{1}{12}\, \vec{J}_\eta(\vec{k}, \vec{k}', \vec{k}''\,) - k_0\, (\vec{k}' \times_\eta \vec{k}''\,) - k_0'\, (\vec{k}'' \times_\eta \vec{k}\,) - k_0''\, (\vec{k} \times_\eta \vec{k}'\,)\big)$$

- Preserved by $Spin(7) \subset SO(8)$, extends representation $X_{\vec{k}\times_\eta \vec{k}'} = \frac{1}{2}\big[X_K, X_{K'}\big]$ on octonions $X_K = k_0\, \mathbb{1} + k^A\, e_A \in \mathbb{O}$:

$$X_{K\times_\phi K'\times_\phi K''} = \tfrac{1}{2}\big((X_K\, \bar{X}_{K'})\, X_{K''} - (X_{K''}\, \bar{X}_{K'})\, X_K\big)$$

- $Spin(7) \longrightarrow G_2$: $\vec{k} \times_\eta \vec{k}' = \vec{k} \times_\phi (1, \vec{0}\,) \times_\phi \vec{k}'$
- Trivector: $[\xi_{\hat{A}}, \xi_{\hat{B}}, \xi_{\hat{C}}]_\phi = \phi_{\hat{A}\hat{B}\hat{C}\hat{D}}\, \xi_{\hat{D}}$ for $\xi = (\xi_0, \vec{\xi}\,) = (\mathbb{1}, e_A)$

“Covariant” M-theory phase space 3-algebra

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- 8D phase space coordinates $X = (x^\mu, p_\mu) = (\Lambda\vec{\xi}, -\frac{\lambda}{2}\xi_0)$ have $SO(4)\times SO(4)$-symmetric 3-brackets:

$$
\begin{aligned}
[x^i, x^j, x^k]_\phi &= -\frac{\ell_s^3}{2}\, R^{4,ijk4}\, x^4 \ , \quad [x^i, x^j, x^4]_\phi = \frac{\lambda^2\,\ell_s^3}{2}\, R^{4,ijk4}\, x_k \\
[p^i, x^j, x^k]_\phi &= -\frac{\lambda^2\,\ell_s^3}{2}\, R^{4,ijk4}\, p_4 - \frac{\lambda\,\ell_s^3}{2}\, R^{4,ijk4}\, p_k \\
[p_i, x^j, x^4]_\phi &= -\frac{\lambda^2\,\ell_s^3}{2}\, R^{4,1234}\, \delta_i^j\, p_4 - \frac{\lambda^2\,\ell_s^3}{2}\, R^{4,ijk4}\, p_k \\
[p_i, p_j, x^k]_\phi &= \frac{\lambda^2}{2}\, \varepsilon_{ij}{}^k\, x^4 + \frac{\hbar^2\,\lambda}{2}\, \big(\delta_j^k\, x_i - \delta_i^k\, x_j\big) \\
[p_i, p_j, x^4]_\phi &= -\frac{\hbar^2\,\lambda^3}{2}\, \varepsilon_{ijk}\, x^k \ , \quad [p_i, p_j, p_k]_\phi = -2\hbar^2\,\lambda\,\varepsilon_{ijk}\, p_4 \\
[p_4, x^i, x^j]_\phi &= \frac{\lambda\,\ell_s^3}{2}\, R^{4,ijk4}\, p_k \ , \quad [p_4, x^i, x^4]_\phi = -\frac{\lambda^2\,\ell_s^3}{2}\, R^{4,1234}\, p^i \ , \\
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- $SO(4)$-invariance: Trivector modelled on negative chirality spinors $S_-(\mathbb{R}^4)$ (Günaydin, Lüst & Malek '16)

Vector trisums

Vector trisums

- Restrict $X_P\, X_{P'} \;=\; \big(p_0\, p_0' - \vec{p}\cdot\vec{p}\,'\,\big)\, \mathbb{1} + p_0\, X_{\vec{p}\,'} + p_0'\, X_{\vec{p}} + X_{\vec{p}\times_\eta \vec{p}\,'}$ to $P, P' \in S^7 \;\cong\; Spin(7)/G_2 \subset \mathbb{R}^8$:

$$X_{\vec{p}\circledast_\eta \vec{p}\,'} \;=\; \mathrm{Im}\big(X_{P\,'}\, X_P\big) \ , \qquad \epsilon_{\vec{p},\vec{p}\,'} \;=\; \mathrm{sgn}\ \mathrm{Re}\big(X_{P\,'}\, X_P\big)$$

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- Extend to **vector trisum** $X_{\vec{p}\circledast_\phi \vec{p}\,'\circledast_\phi \vec{p}\,''} \;=\; \mathrm{Im}\big((X_P\, X_{P'})\, X_{P''}\big)$:

$$\begin{aligned}
&\vec{p}\circledast_\phi \vec{p}\,'\circledast_\phi \vec{p}\,'' \\
&= \epsilon_{\vec{p},\vec{p}\,',\vec{p}\,''}\Big(\epsilon_{\vec{p}\,',\vec{p}\,''}\sqrt{1-\big|\vec{p}\,'\circledast_\eta \vec{p}\,''\big|^2}\ \vec{p} + \epsilon_{\vec{p},-\vec{p}\,''}\sqrt{1-\big|\vec{p}\circledast_\eta (-\vec{p}\,'')\big|^2}\ \vec{p}\,' \\
&+ \epsilon_{\vec{p},\vec{p}\,'}\sqrt{1-\big|\vec{p}\circledast_\eta \vec{p}\,'\big|^2}\ \vec{p}\,'' + \vec{A}_\eta(\vec{p},\vec{p}\,',\vec{p}\,'') \\
&+ \sqrt{1-|\vec{p}\,|^2}\,(\vec{p}\,'\times_\eta \vec{p}\,'') + \sqrt{1-|\vec{p}\,'|^2}\,(\vec{p}\,''\times_\eta \vec{p}\,) + \sqrt{1-|\vec{p}\,''|^2}\,(\vec{p}\times_\eta \vec{p}\,')\Big)
\end{aligned}$$

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&\;=\; \epsilon_{\vec{p},\vec{p}\,',\vec{p}\,''}\,\Big(\epsilon_{\vec{p}\,',\vec{p}\,''}\,\sqrt{1-\big|\vec{p}\,'\circledast_\eta \vec{p}\,''\,\big|^2}\ \vec{p} + \epsilon_{\vec{p},-\vec{p}\,''}\,\sqrt{1-\big|\vec{p}\circledast_\eta(-\vec{p}\,'')\big|^2}\ \vec{p}\,' \\
&+\epsilon_{\vec{p},\vec{p}\,'}\,\sqrt{1-\big|\vec{p}\circledast_\eta \vec{p}\,'\,\big|^2}\ \vec{p}\,'' + \vec{A}_\eta(\vec{p},\vec{p}\,',\vec{p}\,'') \\
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\end{aligned}$$

- *Spin*(7) $\longrightarrow$ G_2: $\vec{p}\circledast_\eta \vec{p}\,' \;=\; \vec{p}\circledast_\phi \vec{0}\circledast_\phi \vec{p}\,'$

Vector trisums

- Restrict $X_P\, X_{P'} \;=\; \left(p_0\, p_0' - \vec{p}\cdot\vec{p}\,'\right) \mathbb{1} + p_0\, X_{\vec{p}\,'} + p_0'\, X_{\vec{p}} + X_{\vec{p}\times_\eta \vec{p}\,'}$ to $P, P' \in S^7 \;\cong\; Spin(7)/G_2 \subset \mathbb{R}^8$:

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- *Spin*(7) $\longrightarrow$ G_2: $\vec{p}\circledast_\eta \vec{p}\,' \;=\; \vec{p}\circledast_\phi \vec{0}\circledast_\phi \vec{p}\,'$

- Nonassociativity: $\mathrm{Asym}(\vec{p}\circledast_\phi \vec{p}\,'\circledast_\phi \vec{p}\,'') \;=\; \mathrm{Im}\left(X_{P\times_\phi P'\times_\phi P''}\right)$

Phase space nonassociative geometry

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- Extend vector trisum to all $\vec{k} \in \mathbb{R}^7 \subset \mathbb{R}^8$:

$$\vec{\mathcal{B}}_\phi(\vec{k}, \vec{k}', \vec{k}'') = \frac{\sin^{-1} |\vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,''|}{\hbar\, |\vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,''|}\; \vec{p} \circledast_\phi \vec{p}\,' \circledast_\phi \vec{p}\,'' \Big|_{\vec{p}=\vec{k}\,\sin(\hbar\,|\vec{k}|)/|\vec{k}|}$$

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&= \vec{k} + \vec{k}' + \vec{k}'' + \hbar\left(\vec{k} \times_\eta \vec{k}' + \vec{k}'' \times_\eta \vec{k} + \vec{k}' \times_\eta \vec{k}''\right) \\
&\quad + \tfrac{\hbar^2}{2}\left(2\,\vec{A}_\eta(\vec{k},\vec{k}\,',\vec{k}\,'') - |\vec{k}\,' + \vec{k}\,''|^2\,\vec{k} - |\vec{k} + \vec{k}\,''|^2\,\vec{k}\,' - |\vec{k}\,' + \vec{k}|^2\,\vec{k}\,''\right) \\
&\quad + O(\hbar^3)
\end{aligned}$$

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$$= \vec{k} + \vec{k}' + \vec{k}'' + \hbar\left(\vec{k}\times_\eta \vec{k}' + \vec{k}''\times_\eta \vec{k} + \vec{k}'\times_\eta \vec{k}''\right) + \tfrac{\hbar^2}{2}\left(2\,\vec{A}_\eta(\vec{k},\vec{k}\,',\vec{k}\,'') - |\vec{k}\,'+\vec{k}\,''|^2\,\vec{k} - |\vec{k}+\vec{k}\,''|^2\,\vec{k}\,' - |\vec{k}\,'+\vec{k}|^2\,\vec{k}\,''\right) + O(\hbar^3)$$

- Phase space triproducts:

$$(f \diamond_\lambda g \diamond_\lambda h)(\vec{x}) = \int_{\vec{k},\vec{k}',\vec{k}''} \tilde{f}(\vec{k})\,\tilde{g}(\vec{k}')\,\tilde{h}(\vec{k}'')\ \mathrm{e}^{\,\mathrm{i}\,\vec{\mathcal{B}}_\phi(\Lambda\,\vec{k},\Lambda\,\vec{k}',\Lambda\,\vec{k}'')\cdot\Lambda^{-1}\,\vec{x}}$$

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$$+ \tfrac{\hbar^2}{2}\left(2\,\vec{A}_\eta(\vec{k}, \vec{k}\,', \vec{k}\,'') - |\vec{k}\,' + \vec{k}\,''|^2\,\vec{k} - |\vec{k} + \vec{k}\,''|^2\,\vec{k}\,' - |\vec{k}\,' + \vec{k}|^2\,\vec{k}\,''\right)$$

$$+ O(\hbar^3)$$

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- $f \star_\lambda g = f \diamond_\lambda 1 \diamond_\lambda g\ , \qquad f \vartriangle_\lambda g \vartriangle_\lambda h = (f \diamond_\lambda g \diamond_\lambda h)\big|_{p=0}$

Phase space nonassociative geometry

- Extend vector trisum to all $\vec{k} \in \mathbb{R}^7 \subset \mathbb{R}^8$:

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- Phase space triproducts:

$$(f \diamond_\lambda g \diamond_\lambda h)(\vec{x}) = \int_{\vec{k}, \vec{k}', \vec{k}''} \tilde{f}(\vec{k})\, \tilde{g}(\vec{k}')\, \tilde{h}(\vec{k}'')\, \mathrm{e}^{\,\mathrm{i}\, \vec{\mathcal{B}}_\phi(\Lambda\, \vec{k}, \Lambda\, \vec{k}', \Lambda\, \vec{k}'') \cdot \Lambda^{-1}\, \vec{x}}$$

- $f \star_\lambda g = f \diamond_\lambda 1 \diamond_\lambda g \ , \qquad f \vartriangle_\lambda g \vartriangle_\lambda h = (f \diamond_\lambda g \diamond_\lambda h)\big|_{p=0}$
- 3-bracket $[f, g, h]_{\diamond_\lambda} = \mathrm{Asym}(f \diamond_\lambda g \diamond_\lambda h)$ obeys:

$$[f, g, 1]_{\diamond_\lambda} = -3\,[f, g]_{\star_\lambda} \ , \qquad \lim_{\lambda \to 0} [x^i, x^j, x^k]_{\diamond_\lambda} \Big|_{p=0} = \ell_s^3\, R^{ijk}$$