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Lie-deformed quantum Minkowski spaces from twists: Hopf-algebraic versus Hopf-algebroid approach

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ABSTRACT

We consider new Abelian twists of Poincare algebra describing nonsymmetric generalization of the ones given in [1], which lead to the class of Lie-deformed quantum Minkowski spaces. We apply corresponding twist quantization in two ways: as generating quantum Poincare–Hopf algebra providing quantum Poincare symmetries, and by considering the quantization which provides Hopf algebroid describing class of quantum relativistic phase spaces with built-in quantum Poincare covariance. If we assume that Lorentz generators are orbital i.e. do not describe spin degrees of freedom, one can embed the considered generalized phase spaces into the ones describing the quantum-deformed Heisenberg algebras.

1. Introduction

Due to quantum gravity (see [2–6]) as well as quantized strings effects (see e.g. [7,8]) at Planck distances the notion of classical space-time can not be maintained. The quantum-mechanical space-time localization in the presence of gravitational interactions are constrained by new type of bounds, extending Heisenberg uncertainty relations to the measurements of pairs of space-time coordinates (see e.g. [4]).[1] Algebraically DSR uncertainty relations can be derived from the noncommutative structure of quantum Minkowski space $\widehat{\mathcal{M}}(\widehat{x}_\mu \in \widehat{\mathcal{M}})$ with nonvanishing commutator $[\widehat{x}_\mu, \widehat{x}_\mu]_{|\mu\neq\nu}$ proportional to λ_{pl}^2 (Planck length $\lambda_{pl} \simeq 10^{-33}$ cm).

The noncommutative structures linked with quantum-deformed dynamical theories (e.g. quantum gravity) appeared recently in two-fold way:

i.) as quantum generalization of Lie-algebraic symmetries, described by noncommutative Hopf algebras [9],[10],
ii.) as deformed quantum phase spaces, with modified deformed canonical Heisenberg relations, described in the formalism of noncommutative geometry by a generalization of Hopf algebras, called Hopf algebroids [11–13].

Both noncommutative structures can be generated by twist quantization procedure. If twist $\mathcal{F}$ of classical Poincare–Hopf algebra is generated by classical r-matrix with the terms $P \wedge M$, where P describes the fourmomenta and M the Lorentz generators, the noncommutativity of quantum space-time takes the Lie-algebraic form (see e.g. [1]). In this paper we shall consider new class of such twists and present explicitly twisted quantum Poincare symmetries as well as Hopf algebroid structure of corresponding quantum phase spaces. We add that Hopf algebroid structures of quantum phase spaces with Lie-algebraic space-time sector were already discussed (see e.g. [14–17]), however mostly either without considering the twist quantizations [14] or with twisted Hopf algebroids considered not explicitly but as a part of general mathematical framework [15],[16]; see however [17].

The classical Minkowski space $\mathcal{M}(x_\mu \in \mathcal{M})$[2] is fully determined if it is given as the irreducible four-vector representation of Lorentz algebra $\mathcal{O}(3,1)$, with the action of Lorentz generators

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[1] This new type of uncertainty relations will be further called Doplicher–Fredenhagen–Roberts (DFR) uncertainty relations.

[2] Further we shall consider physical $D=4$ case, i.e. $\mu = 0, 1, 2, 3$.

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$M_{\mu\nu}$ given by the adjoint action describing the semidirect product $\mathcal{O}(3,1) \rtimes \mathcal{M}$

$$M_{\mu\nu} \rhd x_\rho \equiv [M_{\mu\nu}, x_\rho] = \eta_{\nu\rho} x_\mu - \eta_{\mu\rho} x_\nu . \tag{1}$$

One can supplement as well the relation

$$M_{\mu\nu} \rhd p_\rho \equiv [M_{\mu\nu}, p_\rho] = \eta_{\nu\rho} p_\mu - \eta_{\mu\rho} p_\nu , \tag{2}$$

by observing that Poincare algebra with classical generators $g = (M_{\mu\nu}, p_\mu)$ is also endowed with a semidirect product structure $\mathcal{O}(3,1) \rtimes \mathcal{T}$ ($p_\mu \in \mathcal{T}$). The relations (1)–(2) can be extended further by the formula[3]

$$p_\mu \rhd x_\rho = -i\eta_{\mu\rho} . \tag{3}$$

In such a way we obtain consistent classical action of Poincare algebra $\mathcal{P}$ on the Minkowski space $\mathcal{M}$, describing the cross product $\mathcal{P}\#\mathcal{M}$.[4]

Important class of quantum Poincare algebras are described by twist quantizations of classical Poincare algebra, with all deformation located only in coalgebraic sector of Poincare–Hopf algebra $\mathbb{H}$. In such a case the twist $\mathcal{F} \equiv \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \in \mathcal{U}(g) \otimes \mathcal{U}(g)$ depends on classical Poincare generators, and the formula for coproducts

$$\Delta_{\mathcal{F}}(g) = \mathcal{F} \Delta_0(g) \mathcal{F}^{-1} , \tag{4}$$

can be calculated using only the classical Poincare algebra commutators.

The quantum Minkowski space $\widehat{\mathcal{M}}(\widehat{x}_\mu \in \widehat{\mathcal{M}})$ in the case of twist quantization is again fully specified if it is given as the irreducible four-dimensional module of the respective twisted Poincare–Hopf algebra $\mathbb{H}_{\mathcal{F}}$

$$\mathbb{H}_{\mathcal{F}} = (\mathcal{U}(\mathcal{P}), m, \Delta_{\mathcal{F}}, S_{\mathcal{F}}, \epsilon), \tag{5}$$

where $S_{\mathcal{F}}$ denotes twisted antipode (coinverse) and ϵ is a counit.

The standard Hopf-algebraic way of introducing the quantum Minkowski coordinates $\widehat{x}_\mu$ is to consider quantum Poincare group and use the Hopf-algebraic duality between the coproducts of p_μ and the coordinates $\widehat{x}_\mu$ and introduce the notion of Heisenberg double [10,18]. However, the twist $\mathcal{F}$ provides directly the formula linking $x_\mu \in \mathcal{M}$ with $\widehat{x}_\mu \in \widehat{\mathcal{M}}$, by means of the relation which is a special case of derived in Sect. 2 star-product relation (34) (for $f(x) = x$)

$$\widehat{x}_\mu = \left[\left(\mathcal{F}^{-1}\right)^{(1)} \rhd x_\mu \right] \left(\mathcal{F}^{-1}\right)^{(2)} , \tag{6}$$

with the action $\rhd$ in (6) provided by the formulae (1), (3). In such a way one can express the noncommutative $\widehat{x}_\mu$ in terms of classical phase space coordinates (x_μ, p_μ) and generators $M_{\mu\nu}$.

In this paper we plan to consider the generalization of twist considered in [1], with arbitrary symmetry of $\log \mathcal{F}$ as tensor product. In such a way we introduce additional real parameter u which for $u = \frac{1}{2}$ leads to antisymmetric $\log \mathcal{F}$ tensor and antisymmetric classical r-matrix (this was the case considered in [1] and [19]); other cases $u = 0$ and $u = 1$ correspond to maximally nonsymmetric twists, which for $u = 0$ was discussed as well in the literature [20].

[3] The relation (3) can be linked to Hopf-algebraic scheme (see e.g. [10]) if we consider $\mathcal{P}$ and $\mathcal{T}$ as dual bialgebras, in classical case with primitive coproducts.

[4] The semidirect product of two Lie algebras is again a Lie algebra, what is generalized by the notion of smash product, which describes the algebraic structure on the vector space $H \oplus V$, where $H = (M_H, \Delta_H, \epsilon_H, 1_H)$ is (unital and counital) bialgebra (in particular Lie bialgebra) and V is a unital H-module algebra, which may be noncommutative. Cross-product algebra can be endowed with Hopf algebroid structure [11],[12].

The aim of our paper is to describe the twist quantizations in two frameworks: first is based entirely on Hopf-algebraic techniques, which provides quantum Poincare–Hopf algebra $\mathbb{H}_{\mathcal{F}}$ and second, which leads for any value of parameter u to the embedding of $\mathbb{H}_{\mathcal{F}}$ into the Hopf algebroids describing suitably deformed smash product $\mathcal{T}\#\mathcal{M}$. New results in the present paper are provided by the second method by providing the Hopf algebroid structure: construction of source, target and antipode maps (for their definition see [15],[16]) and by determining the coproduct freedom for twisted bialgebroids $\mathcal{H}_{\mathcal{F}} = (\mathcal{P}\#\mathcal{M})_{\mathcal{F}}$ (so-called coproduct gauges, see [21]). We add that the Hopf algebroid techniques were extensively studied in mathematics (see e.g. [22],[11–16],[23]) and applied to the description of quantum-deformed relativistic phase spaces by some of the present authors [24–28],[21]. The novelty of our discussion of Hopf algebroid structure in comparison with our earlier efforts [24–28] is to consider Lorentz generators $M_{\mu\nu}$ as independent – we shall not assume the standard orbital phase space realization of $M_{\mu\nu}$

$$M_{\mu\nu} = i(x_\mu p_\nu - x_\nu p_\mu). \tag{7}$$

If relation (7) is valid, the Hopf-algebraic Poincare algebra twist $\mathcal{F}$ as well as noncommutative Minkowski space coordinates $\widehat{x}_\mu$ can be expressed in terms of phase space variables i.e. the Hopf-algebraic formulae are realized in terms of canonical Heisenberg algebra, which provides a classical example of Hopf algebroid. The noncommutative Minkowski coordinates $\widehat{x}_\mu$ can be expressed as the following functions of classical phase space variables (x_μ, p_μ)

$$\widehat{x}_\mu = x_\nu \varphi^\nu{}_\mu(p). \tag{8}$$

We mention that the formula (8) was considered [20], [29–31] as well for other Lie-algebraic quantum deformations of Poincare algebra, not necessarily described by twist quantization[5] (see [28,14]).

2. Twist-deformed Poincaré Hopf algebra and quantum Minkowski spaces

Poincaré algebra $\mathcal{P}$, generated by Lorentz generators $M_{\mu\nu}$ and momentum generators p_μ is defined by

$$[p_\mu, p_\nu] = 0, \tag{9}$$

$$[M_{\mu\nu}, p_\rho] = \eta_{\nu\rho} p_\mu - \eta_{\mu\rho} p_\nu , \tag{10}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = \eta_{\nu\rho} M_{\mu\tau} - \eta_{\mu\rho} M_{\nu\tau} - \eta_{\nu\tau} M_{\mu\rho} + \eta_{\mu\tau} M_{\nu\rho}, \tag{11}$$

where $\eta_{\mu\nu} = diag(-1, 1, ..., 1)$.

Classical Poincaré Hopf algebra is defined by the universal enveloping algebra $\mathcal{U}(\mathcal{P})$ of the Poincaré algebra $\mathcal{P}$, together with the coproduct Δ_0, antipode S_0 and counit ϵ_0, given by

$$\Delta_0(p_\mu) = p_\mu \otimes 1 + 1 \otimes p_\mu , \tag{12}$$

$$\Delta_0(M_{\mu\nu}) = M_{\mu\nu} \otimes 1 + 1 \otimes M_{\mu\nu} ,$$

$$S_0(p_\mu) = -p_\mu , \qquad S_0(M_{\mu\nu}) = -M_{\mu\nu} , \tag{13}$$

$$\epsilon_0(p_\mu) = 0, \qquad \epsilon_0(M_{\mu\nu}) = 0. \tag{14}$$

Twist $\mathcal{F}$ is an invertible element of $\mathcal{U}(\mathcal{P}) \otimes \mathcal{U}(\mathcal{P})$, satisfying the cocycle condition

[5] Such example of quantization which can not be obtained by using twist is the κ-deformation of Poincare algebra [32,33].

$$(\mathcal{F}\otimes 1)(\Delta_0\otimes 1)\mathcal{F}=(1\otimes\mathcal{F})(1\otimes\Delta_0)\mathcal{F}, \tag{15}$$

and the normalization condition

$$(\epsilon\otimes 1)\mathcal{F}=(1\otimes\epsilon)\mathcal{F}=1\otimes 1. \tag{16}$$

Twists deform coproducts (12) and antipodes (13) of $h\in\mathcal{U}(\mathcal{P})$ as follows:

$$\Delta_{\mathcal{F}}h=\mathcal{F}\Delta_0 h\mathcal{F}^{-1}, \tag{17}$$

$$S_{\mathcal{F}}(h)=\chi_{\mathcal{F}}S_0(h)\chi_{\mathcal{F}}^{-1}, \tag{18}$$

where $\chi_{\mathcal{F}}=m[(S_0\otimes 1)\mathcal{F}]$ and the deformed coproduct $\Delta_{\mathcal{F}}$ is coassociative due to the cocycle condition (15).

Here we consider the following families of Abelian twists, for all dimensions $n\geq 3$:

$$\begin{aligned}\mathcal{F}_u&=\exp\left((1-u)\frac{a\cdot p}{2\kappa}\otimes\theta^{\alpha\beta}M_{\alpha\beta}-u\theta^{\alpha\beta}M_{\alpha\beta}\otimes\frac{a\cdot p}{2\kappa}\right)\\&=\exp\left(\frac{1-u}{2}\mathcal{K}^{\alpha\beta}\otimes M_{\alpha\beta}-\frac{u}{2}M_{\alpha\beta}\otimes\mathcal{K}^{\alpha\beta}\right),\end{aligned} \tag{19}$$

where $a\cdot p=a^\mu\eta_{\mu\nu}p^\nu$.

The parameter $u\in[0,1]$, κ is the deformation parameter with dimension of mass, $a^2\in\{-1,0,1\}$,

$$\mathcal{K}^{\mu\nu}=\frac{a\cdot p}{\kappa}\theta^{\mu\nu}, \tag{20}$$

and the following conditions hold:

$$\theta^{\mu\nu}=-\theta^{\nu\mu},\qquad a_\mu\theta^{\mu\nu}=0. \tag{21}$$

These twists are a generalization of the twist proposed in [1] – being Abelian, they automatically satisfy the cocycle condition. Because under flip transformation $(a\otimes b)^\tau=b\otimes a$, we obtain the following *u-independent* formula for universal $\mathcal{R}$-matrix ($a\wedge b=a\otimes b-b\otimes a$)

$$\mathcal{R}=\mathcal{F}_u^\tau\mathcal{F}_u^{-1}=\exp[\frac{1}{2}(M_{\alpha\beta}\wedge\mathcal{K}^{\alpha\beta})]. \tag{22}$$

Using equation (17) for the family of twists (19) we obtain the following deformed coproducts ($\mathcal{F}\equiv\mathcal{F}_u$)

$$\Delta_{\mathcal{F}}(p_\mu)=p_\alpha\otimes(e^{-u\mathcal{K}})^\alpha{}_\mu+(e^{(1-u)\mathcal{K}})^\alpha{}_\mu\otimes p_\alpha, \tag{23}$$

$$\begin{aligned}\Delta_{\mathcal{F}}(M_{\mu\nu})=&M_{\alpha\beta}\otimes(e^{-u\mathcal{K}})^\alpha{}_\mu(e^{-u\mathcal{K}})^\beta{}_\nu\\&+(e^{(1-u)\mathcal{K}})^\alpha{}_\mu(e^{(1-u)\mathcal{K}})^\beta{}_\nu\otimes M_{\alpha\beta}\\&-\frac{\theta^{\alpha\beta}}{2\kappa}(a_\mu\delta^\gamma_\nu-a_\nu\delta^\gamma_\mu)\Big[(1-u)p_\delta\otimes M_{\alpha\beta}(e^{-u\mathcal{K}})^\delta{}_\gamma\\&-uM_{\alpha\beta}(e^{(1-u)\mathcal{K}})^\delta{}_\gamma\otimes p_\delta\Big],\end{aligned} \tag{24}$$

where $\mathcal{K}_{\mu\nu}$ is given in equation (20).

Corresponding antipodes (18) are:

$$S_{\mathcal{F}}(p_\mu)=-(e^{-(1-2u)\mathcal{K}})^\alpha{}_\mu p_\alpha, \tag{25}$$

$$\begin{aligned}S_{\mathcal{F}}(M_{\mu\nu})=&-M_{\alpha\beta}(e^{-(1-2u)\mathcal{K}})^\alpha{}_\mu(e^{-(1-2u)\mathcal{K}})^\beta{}_\nu\\&+\frac{1}{\kappa}(a_\mu\delta^\alpha_\nu-a_\nu\delta^\alpha_\mu)\left[S(p_\alpha)+(1-2u)\theta_{\alpha\beta}S(p^\beta)\right],\end{aligned} \tag{26}$$

The counit is trivial:

$$\epsilon(p_\mu)=0,\qquad\epsilon(M_{\mu\nu})=0. \tag{27}$$

It is interesting to note that coproduct and antipode of $\mathcal{K}_{\mu\nu}$ remain classical, i.e.

$$\Delta_{\mathcal{F}}(\mathcal{K}_{\mu\nu})=\mathcal{K}_\nu\otimes 1+1\otimes\mathcal{K}_{\mu\nu}=\Delta_0(\mathcal{K}_{\mu\nu}), \tag{28}$$

$$S_{\mathcal{F}}(\mathcal{K}_{\mu\nu})=-\mathcal{K}_{\mu\nu}=S_0(\mathcal{K}_{\mu\nu}). \tag{29}$$

Coproduct and antipode of $(e^{\mathcal{K}})_\mu{}^\nu$ are

$$\Delta_{\mathcal{F}}(e^{\mathcal{K}})_\mu{}^\nu=(e^{\mathcal{K}})_\mu{}^\alpha\otimes(e^{\mathcal{K}})_\alpha{}^\nu, \tag{30}$$

$$S_{\mathcal{F}}((e^{\mathcal{K}})_\mu{}^\nu)=(e^{-\mathcal{K}})_\mu{}^\nu. \tag{31}$$

If noncommutativity is introduced through Hopf-algebraic twist quantization one can introduce the star product realization of the algebra $\hat{A}$ of functions on quantum Minkowski space in terms of $\star$-algebra of classical functions $f(x), g(x)$

$$f(x)\star_{\mathcal{F}}g(x)=m\left[\mathcal{F}^{-1}(\triangleright\otimes\triangleright)(f(x)\otimes g(x))\right], \tag{32}$$

where the action $\triangleright$ is defined by eq. (1), (3) and $(\hat{A},\cdot)$ algebra is represented as $(A,\star)$ algebra.

Alternatively, one can write equation (32) in the following form

$$f(x)\star_{\mathcal{F}}g(x)=\hat{f}_{\mathcal{F}}\triangleright g(x), \tag{33}$$

where

$$\hat{f}_{\mathcal{F}}=m\left[\mathcal{F}^{-1}(\triangleright\otimes 1)(f(x)\otimes 1)\right], \tag{34}$$

is a noncommutative counterpart of $f(x)$, described as the functions of classical generators $(x_\mu,p_\nu,M_{\rho\sigma})\in\mathcal{P}\#\mathcal{M}$. Because $p_\mu\triangleright 1=M_{\mu\nu}\triangleright 1=0$ one gets that $\hat{x}_\mu\triangleright 1=x_\mu$ and subsequently $\hat{f}_{\mathcal{F}}\triangleright 1=f(x)$.

Following (6), if we put in (34) $f=x_\mu$ and $\mathcal{F}=\mathcal{F}_u$ the noncommutative coordinates are given by

$$\begin{aligned}\hat{x}_\mu&=m\left[\mathcal{F}_u^{-1}(\triangleright\otimes 1)(x_\mu\otimes 1)\right]\\&=x_\alpha(e^{-u\mathcal{K}})_\mu{}^\alpha+(1-u)\frac{ia_\mu}{2\kappa}\theta^{\alpha\beta}M_{\alpha\beta}.\end{aligned} \tag{35}$$

The non-commutative coordinates (35) close to a Lie algebra

$$[\hat{x}_\mu,\hat{x}_\nu]=\frac{i}{\kappa}(a_\mu\theta_{\nu\alpha}-a_\nu\theta_{\mu\alpha})\hat{x}^\alpha. \tag{36}$$

Note that the structure constants $C_{\mu\nu}{}^\alpha=\frac{1}{\kappa}(a_\mu\theta_\nu{}^\alpha-a_\nu\theta_\mu{}^\alpha)$ do not depend on the parameter u and satisfy Jacobi identities.

If $u=1$ the twist $\mathcal{F}_1$ is special, because only such value of u gives the particular choice described by formula (8)

$$\hat{x}_\mu=x_\alpha(e^{-\mathcal{K}})_\mu{}^\alpha, \tag{37}$$

for any choice of the Lorentz generators $M_{\alpha\beta}$. If $u=0$, eq. (35) reduces to

$$\hat{x}_\mu=x_\mu+\frac{ia_\mu}{2\kappa}\theta^{\alpha\beta}M_{\alpha\beta.} \tag{38}$$

This case was considered in [20] and it was related to twisted statistics. For $u=1/2$, twist $\mathcal{F}^\tau=\mathcal{F}^{-1}$ and because $\mathcal{F}^\dagger=\mathcal{F}^\tau$ (where † denotes Hermitian conjugation), for such a choice of u the twist $\mathcal{F}_{\frac{1}{2}}$ is unitary. This case was considered in [1],[19]. In [19] it was related to non-Pauli effects in noncommutative space-times.

In order to obtain the coproduct sector and consistent bialgebroid relations for $\hat{x}_\mu$ defined by (35) we should look for the Hopf algebroid structure of twist-deformed cross product $(\mathcal{P}\#\mathcal{M})_{\mathcal{F}}$.

3. Deformed Heisenberg algebras and twisted cross products in algebroid approach

Quantum-mechanical phase-space coordinates x^μ and momenta p_μ describe canonical undeformed Heisenberg algebra, given by:

$$\begin{aligned} &[x^\mu, x^\nu] = 0, \\ &[p_\mu, x^\nu] = -i\delta_\mu^\nu, \\ &[p_\mu, p_\nu] = 0. \end{aligned} \tag{39}$$

If we deal with Hopf-algebraic scheme of Poincare symmetries the relations (39) can be derived by the identification of $x_\mu \in \mathcal{M}$ with Abelian space-time translations of classical Poincare group and $p_\mu \in \mathcal{T}$ with the generators of dual Abelian fourmomenta subalgebra acting on $\mathcal{M}$. The standard quantum-mechanical phase-space with basic algebra (39) is provided by smash product $\mathcal{H}_0 = \mathcal{T}\#\mathcal{M}$, defining Heisenberg double with undeformed (canonical) Heisenberg Hopf algebroid structure [6] of two Abelian dual Hopf algebras which describe respectively the functions of coordinates x_μ and momenta p_μ. The cross multiplication rules in $\mathcal{H}_0$ are given by the Heisenberg double formula [10],[18]

$$p_\mu x_\nu = x_\nu^{(1)} \langle p_\mu^{(1)}, x_\nu^{(2)} \rangle p_\mu^{(2)}, \qquad p_\mu \in \mathcal{T}, \quad x_\nu \in \mathcal{M}, \tag{40}$$

where $\langle \cdot,\cdot \rangle$ describes the canonical duality pairing, with $\Delta_0(p_\mu) = p_\mu^{(1)} \otimes p_\mu^{(2)}$ given by (12) and

$$\Delta_0(x_\mu) = x_\mu^{(1)} \otimes x_\mu^{(2)} = x_\mu \otimes 1 + 1 \otimes x_\mu. \tag{41}$$

The relation (41) follows as well from the coproduct of classical Poincare group describing space-time translations, after contraction of the Lorentz group parameters $\Lambda_\nu^\mu \longrightarrow \delta_\nu^\mu$. The action $p_\mu \triangleright x_\nu$, given by formula (3), in Hopf-algebraic scheme can be identified with the binary duality map $\mathcal{T} \otimes \mathcal{M} \longrightarrow C : p \otimes x \to \langle p, x \rangle$, which provides the differential realization of fourmomenta p_μ

$$p_\mu \triangleright f(x) = \langle p_\mu, f(x) \rangle = \frac{1}{i}\partial_\mu f(x). \tag{42}$$

Finally, one can easily deduce from (40)–(41) and (3) the set of canonical commutation relations given by eq. (39).

In $\mathcal{H}_0$ one can choose different bases, in particular it is possible to incorporate the change $x_\mu \longrightarrow \widehat{x}_\mu = x_\rho \varphi^\rho{}_\mu(p)$ (see (8)) and employ the fourmomenta p_μ, satisfying the relations (9) with twisted coproduct (23) of p_μ denoted as follows

$$\Delta_{\mathcal{F}}(p_\nu) \equiv \Delta_{\mathcal{F}}^{(1)}(p_\nu) \otimes \Delta_{\mathcal{F}}^{(2)}(p_\nu) = \Delta_0(p_\nu) + \delta\Delta_{\mathcal{F}}(p_\nu). \tag{43}$$

As follows from (35) the deformed Heisenberg algebra basis $(\widehat{x}^\mu, p_\mu)$ satisfies the standard duality relations $\langle p_\mu, \widehat{x}_\nu \rangle = -i\eta_{\mu\nu}$. Further one can show that the algebraic relation (36) are dual to the coproducts (23) in accordance with Hopf-algebraic duality, e.g.

$$\langle \Delta_{\mathcal{F}}(p_\rho), \widehat{x}_\mu \otimes \widehat{x}_\nu \rangle = \langle p_\rho, \widehat{x}_\mu \widehat{x}_\nu \rangle. \tag{44}$$

Subsequently, introducing the coproduct

$$\Delta(\widehat{x}_\mu) = \widehat{x}_\mu^{(1)} \otimes \widehat{x}_\mu^{(2)} = \widehat{x}_\mu \otimes 1 + 1 \otimes \widehat{x}_\mu, \tag{45}$$

which is dual to commuting fourmomenta p_μ, one can show that we deal with Heisenberg double H with the basis $(\widehat{x}_\mu, p_\mu)$ describing deformed Heisenberg algebra. Therefore, the basic relation (40) remains valid, i.e.

$$p_\mu \widehat{x}_\nu = \widehat{x}_\nu^{(1)} \langle \Delta_{\mathcal{F}}^{(1)}(p_\mu), \widehat{x}_\nu^{(2)} \rangle \Delta_{\mathcal{F}}^{(2)}(p_\mu). \tag{46}$$

We get from decomposition (43) that the term $\Delta_0(p_\nu)$ gives the contribution $\eta_{\mu\nu} + \widehat{x}_\nu p_\mu$, and relation (46) takes the form of deformed canonical commutation relations

$$\begin{aligned} [p_\mu, \widehat{x}_\nu] = -i\eta_{\mu\nu} + \left\{ (\Delta^{(1)}(p) - \Delta_{(0)}^{(1)}(p)) \triangleright \widehat{x}_\nu \right\} (\Delta^{(2)}(p) \\ - \Delta_{(0)}^{(2)}(p)), \end{aligned} \tag{47}$$

where due to the duality of coordinates $\widehat{x}_\nu$ and momenta p_μ, we use the formula

$$p_\mu \triangleright \widehat{x}_\nu = \langle p_\mu, \widehat{x}_\nu \rangle = -i\eta_{\mu\nu}. \tag{48}$$

The formula (47) can be also written in the form

$$[p_\mu, \widehat{x}_\nu] = -i\eta_{\mu\nu} + m[(\Delta - \Delta_0)(p_\mu)(\triangleright \otimes 1)(\widehat{x}_\nu \otimes 1)], \tag{49}$$

which was derived in alternative way also in [14].

If we use the formulae (35), (2) and (39) we can directly calculate the commutator (49). Such a method leads to the same deformed Heisenberg algebra $\hat{\mathcal{H}}$, given by relations (36), i.e. the cross commutator $[p_\mu, \widehat{x}_\nu]$ which for any u does not depend on Lorentz generators $M_{\mu\nu}$. We obtain

$$\begin{aligned} &[p_\mu, \hat{x}_\nu] = -i(e^{-u\mathcal{K}})_{\nu\mu} + (1-u)\frac{ia_\nu}{\kappa}\theta_\mu{}^\alpha p_\alpha, \\ &[p_\mu, p_\nu] = 0. \end{aligned} \tag{50}$$

Additionally, commutation relations between $(e^{\mathcal{K}})_\mu{}^\nu$ and $\hat{x}^\lambda$ and $M_{\alpha\beta}$ after using (1)–(3) are given by

$$[(e^{\mathcal{K}})_\mu{}^\nu, \hat{x}^\lambda] = -\frac{ia^\lambda}{\kappa}\theta_\mu{}^\alpha (e^{\mathcal{K}})_\alpha{}^\nu, \tag{51}$$

$$[(e^{\mathcal{K}})_\mu{}^\nu, M_{\alpha\beta}] = \frac{a_\alpha p_\beta - a_\beta p_\alpha}{\kappa}(\theta e^{\mathcal{K}})_\mu{}^\nu, \tag{52}$$

and the commutation relations between Lorentz generators $M_{\mu\nu}$ and non-commutative coordinates $\hat{x}_\rho$ are the following form

$$\begin{aligned} [M_{\mu\nu}, \hat{x}_\rho] = (\delta_\mu^\alpha \eta_{\beta\nu} - \delta_\nu^\alpha \eta_{\beta\mu}) \Big[\hat{x}_\gamma (e^{u\mathcal{K}})_\alpha{}^\gamma \\ -(1-u)\frac{ia_\alpha}{2\kappa}\theta^{\gamma\delta} M_{\gamma\delta} \Big] (e^{-u\mathcal{K}})_\rho{}^\beta \\ + \frac{1}{\kappa} u \theta_\rho{}^\alpha \hat{x}_\alpha (a_\mu p_\nu - a_\nu p_\mu) \\ + (1-u) i a_\rho (M_\mu{}^\alpha \theta_{\alpha\nu} - \theta_\mu{}^\alpha M_{\alpha\nu}). \end{aligned} \tag{53}$$

The relation (53) in generalized quantum-deformed phase space $(\hat{x}_\mu, p_\mu, S_{\mu\nu})$ where

$$M_{\mu\nu} = i(x_\mu p_\nu - x_\nu p_\mu) + S_{\mu\nu}, \tag{54}$$

describe the noncommutativity of translational and spin degrees of freedom.

Let us consider now the Hopf algebroid $\mathcal{H}_{\mathcal{F}}$ with algebraic structure described by classical Poincare algebra $\mathcal{P}$ supplemented by the noncommutative space-time coordinates $\hat{x}_\mu \in \hat{\mathcal{M}}$ satisfying the relations (36), (50) and (53)

$$\mathcal{H}_{\mathcal{F}} = (\mathcal{A}, m; \mathcal{B}_{\mathcal{F}}, s_{\mathcal{F}}, t_{\mathcal{F}}, \tilde{\Delta}_{\mathcal{F}}, \tilde{\epsilon}_{\mathcal{F}}, S_{\mathcal{F}}). \tag{55}$$

[6] It has been shown (see [11], Sect. 6) that Heisenberg doubles are endowed with Hopf algebroid structure.

The total algebra $\mathcal{A}$ with the basis $(\hat{x}_\mu, p_\mu, M_{\mu\nu})$ is given by the smash product $\mathcal{U}(\mathcal{P})\#\mathcal{U}(\hat{\mathcal{M}})$ and base algebra $\mathcal{B}_\mathcal{F}$ $(\hat{x}_\mu \in \mathcal{B}_\mathcal{F})$ is provided by the algebra of functions on $\hat{\mathcal{M}}$ with the multiplication in $\mathcal{B}_\mathcal{F}$ represented by star product formula (32). The source map $s_\mathcal{F} : \mathcal{B}_\mathcal{F} \to \mathcal{A}$ (algebra homomorphism) and target map $t_\mathcal{F} : \mathcal{B}_\mathcal{F} \to \mathcal{A}$ (algebra antihomomorphism) introduce in $\mathcal{A}$ the $(\mathcal{B}_\mathcal{F}, \mathcal{B}_\mathcal{F})$ bimodule structure, namely for any $a \in \mathcal{A}$ and $b, b' \in \mathcal{B}_\mathcal{F}$ one gets the formula $bab' = s_\mathcal{F}(b)t_\mathcal{F}(b')a$, i.e. we consider $\mathcal{H}_\mathcal{F}$ as left bialgebroid [13],[11]. The comultiplication map $\tilde{\Delta}_\mathcal{F} : \mathcal{A} \to \mathcal{A}\otimes_{\mathcal{B}_\mathcal{F}}\mathcal{A}$ with nonstandard tensor product introduced firstly in [22] is a coassociative bimodule map with the elements $a\otimes_{\mathcal{B}_\mathcal{F}}a' \in \mathcal{A}\otimes_{\mathcal{B}_\mathcal{F}}\mathcal{A}$ defined in the description using standard tensor product $\mathcal{A}\otimes\mathcal{A}$ by the equivalence class generated by the following condition [11]

$$m(\mathcal{I}_\mathcal{F}(b\otimes b')) = 0, \qquad \mathcal{I}_\mathcal{F} = (t_\mathcal{F} \otimes 1 - 1 \otimes s_\mathcal{F}), \tag{56}$$

where $\mathcal{A}\otimes_{\mathcal{B}_\mathcal{F}}\mathcal{A} = (\mathcal{A}\otimes\mathcal{A})/\mathcal{I}_\mathcal{F}$ and $s_\mathcal{F}$ and $t_\mathcal{F}$ are respectively the twisted source and target maps defined below (see (61)–(62)). If we describe coproducts[7] $\tilde{\Delta}_\mathcal{F}$ using standard tensor products one can treat the elements $(a\otimes a')$ satisfying (56) as defining coproduct gauges, with gauge-invariant elements described by $a\otimes_{\mathcal{B}_\mathcal{F}}a'$. In particular for $\hat{x}_\mu \in \mathcal{B}_\mathcal{F}$ we shall choose the special coproduct gauge defined by the formula (see e.g. [11],[21])

$$\tilde{\Delta}_\mathcal{F}(\widehat{x}_\mu) = \widehat{x}_\mu \otimes 1. \tag{57}$$

The canonical choice of the coproduct given by the formula (57) can be obtained if we insert the twisted coproducts (23)–(24) and

$$\tilde{\Delta}_\mathcal{F}(x_\mu) = \mathcal{F}\tilde{\Delta}_0(x_\mu)\mathcal{F}^{-1}, \qquad \tilde{\Delta}_0(x_\mu) = x_\mu \otimes 1, \tag{58}$$

into the relation (35), in accordance with the equality

$$\begin{aligned} \widehat{x}_\mu &\equiv \hat{x}_\mu(x_\mu, p_\mu, M_{\mu\nu}) \\ &\longrightarrow \tilde{\Delta}_\mathcal{F}(\widehat{x}_\mu) \equiv \hat{x}_\mu(\tilde{\Delta}_\mathcal{F}(x_\mu), \Delta_\mathcal{F}(p_\mu), \Delta_\mathcal{F}(M_{\mu\nu})). \end{aligned} \tag{59}$$

One can check further that the coproducts (57) and (23)–(24) describe the homomorphic map $\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ of the algebraic relations (36), (50) and (53) i.e. if $a = (\hat{x}_\mu, p_\mu, M_{\mu\nu}) \in \mathcal{A}$ we get the following canonical full set of coproducts for bialgebroid (55) (see (57) and (23)–(24))

$$\begin{aligned} \tilde{\Delta}_\mathcal{F}(a) = (\tilde{\Delta}_\mathcal{F}(\widehat{x}_\mu), \tilde{\Delta}_\mathcal{F}(p_\mu) = \Delta_\mathcal{F}(p_\mu), \tilde{\Delta}_\mathcal{F}(M_{\mu\nu}) \\ = \Delta_\mathcal{F}(M_{\mu\nu})). \end{aligned} \tag{60}$$

The twisted source and target maps are introduced as follows [11],[15],[16]

$$s_0(x_\mu) = x_\mu \quad \overset{\mathcal{F}}{\rightarrow} \quad s_\mathcal{F}(\widehat{x}_\mu) = m[\mathcal{F}^{-1}(\triangleright\otimes 1)(s_0(x_\mu)\otimes 1)] = \widehat{x}_\mu, \tag{61}$$

$$\begin{aligned} t_0(x_\mu) = x_\mu \quad \overset{\mathcal{F}}{\rightarrow} \quad t_\mathcal{F}(\widehat{x}_\mu) &= m[(\mathcal{F}^{-1})^\tau(\triangleright\otimes 1)(t_0(x_\mu)\otimes 1)] \\ &= \widehat{x}_\alpha(e^\mathcal{K})_\mu{}^\alpha - i\frac{a_\mu}{2\kappa}\theta^{\alpha\beta}M_{\alpha\beta}. \end{aligned} \tag{62}$$

Due to the model-independent relation (see e.g. [11], proof of preposition (2.4))

$$\tilde{\Delta}_\mathcal{F}(s_\mathcal{F}(\widehat{x}_\mu)) = s_\mathcal{F}(\widehat{x}_\mu) \otimes 1, \tag{63}$$

it follows that the formulae (61) and (57) are consistent as expected. Further, it can be shown that the source and target maps satisfy the relations $(C_{\mu\nu}{}^\alpha = \frac{i}{\kappa}a_{[\mu}\theta_{\nu]}{}^\alpha)$

$$[s(\widehat{x}_\mu), s(\widehat{x}_\nu)] = C_{\mu\nu}{}^\alpha s(\widehat{x}_\alpha), \tag{64}$$

$$[t(\widehat{x}_\mu), t(\widehat{x}_\nu)] = -C_{\mu\nu}{}^\alpha t(\widehat{x}_\alpha), \tag{65}$$

$$[s(\widehat{x}_\mu), t(\widehat{x}_\nu)] = 0. \tag{66}$$

The coproduct $\tilde{\Delta}_\mathcal{F} : \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ is only coassociative when $\mathcal{A}\otimes\mathcal{A}$ is projected into equivalence classes $\mathcal{A}\otimes_{\mathcal{B}_\mathcal{F}}\mathcal{A}$ generated by the ideal $\mathcal{I}_\mathcal{F}$. The choice of representatives in the equivalence class defines the coproduct gauge.

The simplest choice of the coproduct gauge transformation is obtained by adding to (57) the ideal $\mathcal{I}_\mathcal{F}$ multiplied by a constant α

$$\tilde{\Delta}_{(\alpha)}(\widehat{x}_\mu) = \widehat{x}_\mu \otimes 1 + \alpha(t_\mathcal{F}(\widehat{x}_\mu) \otimes 1 - 1 \otimes s_\mathcal{F}(\widehat{x}_\mu)). \tag{67}$$

One can check that the coproducts $\tilde{\Delta}_{(\alpha)}(\widehat{x}_\mu)$ together with the Hopf-algebraic coproducts (23)–(24) satisfy the algebraic relations which are homomorphic to the relations (36), (50) and (53). The coproduct gauge can be generalized by introducing powers of ideal $\mathcal{I}_\mathcal{F}$ as well as powers of coproducts (60). In such a case the homomorphism between the algebraic and coalgebraic relations of Hopf algebroid $\mathcal{H}_\mathcal{F}$ will be only valid in standard tensor notation modulo the choices of coproduct gauge transformation [28],[21].

Finally we complete the description of Hopf bialgebroid $\mathcal{H}_\mathcal{F}$ structure if we define $(\epsilon_0(x_\mu) = x_\mu$ for undeformed case) (see e.g. [28])

$$\epsilon_\mathcal{F}(\widehat{x}_\mu) = m[\mathcal{F}^{-1}(\triangleright\otimes 1)(\epsilon_0(x_\mu)\otimes 1)] = \widehat{x}_\mu, \tag{68}$$

$$\epsilon_\mathcal{F}(p_\mu) = \epsilon_\mathcal{F}(M_{\mu\nu}) = 0, \qquad \epsilon_\mathcal{F}(1) = 1. \tag{69}$$

Hopf algebroid is a bialgebroid with antipode (coinverse). In order to obtain the antipode $S_\mathcal{F}$ one can use the formula $(a \in \mathcal{A})$ (see (18))

$$S_\mathcal{F}(a) = \chi_\mathcal{F} S_0(a)\chi_\mathcal{F}^{-1}, \tag{70}$$

with $\chi_\mathcal{F} = \exp[-(1-2u)\frac{a\cdot p}{2\kappa}\theta^{\alpha\beta}M_{\alpha\beta}]$, one gets

$$S_\mathcal{F}(\widehat{x}_\mu) = (e^\mathcal{K})_\mu{}^\alpha\widehat{x}_\alpha - i\frac{a_\mu}{2\kappa}\theta^{\alpha\beta}M_{\alpha\beta} = t_\mathcal{F}(\widehat{x}_\mu). \tag{71}$$

Note that $S_\mathcal{F}^2 = 1$. The antipodes $S_\mathcal{F}(p_\mu)$, $S_\mathcal{F}(M_{\mu\nu})$ in Hopf algebroid (55) remain the same as for the twisted Poincare -Hopf algebra (see (25)–(26)) and are also involutive (see (18)). Further it can be shown that

$$S_\mathcal{F}(t_\mathcal{F}(\widehat{x}_\mu)) = s_\mathcal{F}(\widehat{x}_\mu) = \widehat{x}_\mu, \tag{72}$$

$$m[(1\otimes S_\mathcal{F}) \circ \tilde{\Delta}_\mathcal{F}] = s_\mathcal{F}\epsilon_\mathcal{F} = \epsilon_\mathcal{F}, \tag{73}$$

$$m[(S_\mathcal{F}\otimes 1) \circ \tilde{\Delta}_\mathcal{F}] = t_\mathcal{F}\epsilon_\mathcal{F}S_\mathcal{F}. \tag{74}$$

Note that in second formula the introduction of anchor projection γ $(\tilde{\Delta}_\mathcal{F} \to \gamma\tilde{\Delta}_\mathcal{F}$, where γ is a section of the projection $\mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes_{\mathcal{B}_\mathcal{F}}\mathcal{A}$, see [11,12]) is not needed.[8]

Finally we recall that for spinless systems one can introduce the orbital realization of Lorentz generators $M_{\mu\nu}$, described by formula (7). Inserting (7) in formula (19) one gets the u-dependent twist of canonical Heisenberg–Hopf algebroid $\mathcal{H}_\mathcal{F}$

$$\mathcal{F}_u(p_\mu, M_{\mu\nu}) \to \widetilde{\mathcal{F}}_u(p_\mu, x_\nu) = \mathcal{F}_u(p_\mu, i(x_\mu p_\nu - x_\nu p_\mu)), \tag{75}$$

[7] We denote the bialgebroid coproducts with tilde.

[8] The anchor projection restricts the coproduct gauge for which the formula (73) is valid.

with the coproducts $\tilde{\Delta}_{\tilde{\mathcal{F}}}$ of x_μ and p_ν obtained from (23) and (57) after inserting the substitution (75). Further, it follows that the two-cocycle condition of $\mathcal{F}$ (see [10]) is becoming a two-cocycle condition for bialgebroid twist $\tilde{\mathcal{F}}$ (see [9]), given by (75).

It is easy to see that after inserting (7) in relation (35) the formula (8) becomes valid for all values of u. Concluding, from the Hopf algebroid (55) with independent Lorentz generators by using (7) one obtains twisted Heisenberg–Hopf algebroid with the formulae for source and target maps, antipodes and the ideal describing coproduct gauges expressed only in terms of phase space variables $(\hat{x}_\mu, p_\mu)$ or (x_μ, p_μ).[10]

4. Final remarks

The cross product algebra $\mathcal{P}\#\mathcal{M}$, with the algebra basis described by generators $(p_\mu, M_{\mu\nu}, \hat{x}_\mu)$, can be named Poincare–Heisenberg algebra [34] or DSR algebra [35],[36].[11] In this paper we provide a particular example of quantum twist-deformed DSR algebra $\mathcal{P}\#\hat{\mathcal{M}}$ and present explicitly its algebraic and coalgebraic Hopf algebroid structure. It should be observed that DSR algebra can be obtained by the contraction of full generalized relativistic quantum phase space described as the Heisenberg double (see e.g. [21]), i.e. the cross product $\mathbb{H}\#\tilde{\mathbb{H}}$ of deformed Poincare–Hopf algebra $\mathbb{H}$ (with basis $(p_\mu, M_{\mu\nu})$) and quantum Poincare–Hopf quantum group $\tilde{\mathbb{H}}$ (with basis $(x_\mu, \Lambda_{\mu\nu})$, where $\Lambda_{\mu\nu}$ $(\Lambda_{\mu\alpha}\Lambda^{\alpha}_{\ \nu} = \eta_{\mu\nu})$ describe the Lorentz 4×4 matrix group elements).

Because $D = 4$ Heisenberg algebra is described as well by the cross-product $\mathcal{T}_4\#\mathcal{M}_4$, one can represent $D = 4$ DSR algebraic structure as the following composition of cross products

$$\textit{DSR algebra} = SO(3,1)\#(\mathcal{T}_4\#\mathcal{M}_4). \tag{76}$$

It follows from (76) that the Lorentz generators $SO(3,1)$ act covariantly on the standard (without spin degrees of freedom) quantum phase space $\mathcal{T}_4\#\mathcal{M}_4$. We add that the cross-product structures presented in (76) are preserved for twist quantum-deformed phase space $(\mathcal{T}_4\#\hat{\mathcal{M}}_4)_{\mathcal{F}}$ endowed with quantum-relativistic covariance under the action of twisted Poincare -Hopf algebra

There remain some questions which should be further studied. In particular one should elaborate more on the role of Heisenberg algebra twists (see e.g. (75)), in the construction of Hopf algebroids which provide the quantum deformed relativistic phase space frameworks. In this paper the advantage of our approach to quantum phase space formulation is the appearance of spin degrees of freedom $S_{\mu\nu}$ as independent phase space coordinates. The extension of quantum phase spaces with spin degrees of freedom still remains quite open subject, and we plan to study the relation of such extended phase spaces (see e.g. [37],[38] in undeformed case) with the Hopf algebroid constructions.

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[9] The two-cocycle condition for bialgebroid twists is given e.g. in [15],[16].

[10] The canonical coordinates x_μ can be expressed in terms of $\hat{x}_\mu$ if the formula (8) is invertible.

[11] DSR ≡ Doubly Special Relativity or Deformed Special Relativity.

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