A direct numerical approach to one-loop amplitudes

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Introduction

High statistics data from the current experiments push the particle physics into the precision realm.

It is necessary to have a complete control over the radiative corrections, starting with the one-loop contributions.

For one-loop contributions, the problem has been completely solved, both analytically and numerically and applied in calculating many experimentally interesting cross sections.

We raise the question: is the standard approach optimal?

The standard approach to one-loop amplitudes

The calculation of the amplitude is performed in a reductionistic, diagram-by-diagram, approach.

$$\mathrm{i}\mathcal{M} = \sum \text{diagram}$$

$$\text{diagram} = \sum \text{coefficient} \times \text{Master integral}$$

The master integrals are universal scalar integrals and have been tabulated.

The coefficients can be calculated in a number of different ways: Passarino-Veltman, Integration-by-parts, Tensor reduction, Unitarity cuts, . . .

The standard approach: shortcomings

For large number of external particles, the number of contributing diagrams grows rapidly $\rightsquigarrow$ the calculation of the coefficients is a very demanding and repetitive procedure $\rightsquigarrow$ needs to be automatized.

The final expression for the amplitude is usually much simpler than what one would expect from the diagrams themselves $\rightsquigarrow$ large cancellations between different diagrams $\rightsquigarrow$ the question: are these laborious calculations really necessary?

In general, the reduction methods are analytically oriented, avoiding numerical calculation as long as possible. However, to get the cross section, one must numerically integrate the square of the amplitude over the kinematical phase space.

The numerical approach to one-loop amplitudes

The main problem of any numerical attempt at calculating diagrams/amplitudes are singularities.

UV/IR divergent contributions have to be subtracted before the numerical integration.

The remaining singularities, coming from the vanishing of the propagators, are integrable but cause numerical instabilities if not treated properly.

The numerical approach: existing methods

- The contour deformation method (**Soper** et al., **Weinzierl** et al.) displaces the integration path into the complex plane $\ell \to \tilde{\ell} = \ell + \mathrm{i}\kappa(\ell)$, avoiding the singularities.
  - Pros: numerical stability; direct 4D Monte Carlo integration, regardless of the number of external particles.
  - Cons: one diagram at a time; the exact contour shape is highly process dependent; works best with massless propagators.
- The extrapolation method (**de Doncker** et al.) regulates the integrable singularities by keeping the $\mathrm{i}\epsilon$ prescription finite, then numerically calculates $\mathcal{M}(\epsilon)$ and extrapolates to $\mathcal{M}(0)$.
  - Pros: numerical stability for finite ϵ; integration over Feynman parameters - finite integration domain.
  - Cons: one diagram at a time; # of integrations = # of external particles; high integration precision needed for subsequent extrapolation.

The numerical approach: a proposed method (I.)

We propose a completely numerical method that

- calculates the sum of diagrams, i.e. the complete amplitude, at once
- integrates directly over the 4D loop-momenta
- uses a process-independent way of avoiding the integrable singularities by combining the contour deformation approach with the finite $i\epsilon$ approach
- can handle massive propagators with ease

The numerical approach: a proposed method (II.)

The starting point of our approach is the expression for the complete amplitude in the form

$$\mathrm{i}\mathcal{M} = \int \frac{\mathrm{d}^4\ell}{(2\pi)^4} \sum_{\text{diagrams}} \frac{\mathcal{N}(\ell)}{\prod_{k=1}^{n} \left((\ell - r_k)^2 - m_k^2 + \mathrm{i}\epsilon \right)},$$

where

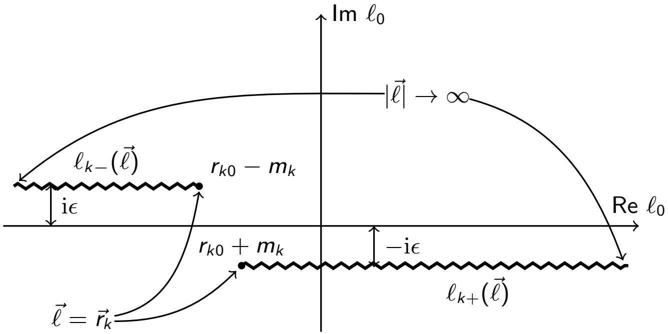
- $\mathcal{N}(\ell)$ is some well behaved function which *we do not have to specify in details*
- r_k is the relative momentum between the two external lines
- m_k is the mass of the propagator

If the *complete amplitude* is UV/IR divergent, we assume the appropriate subtraction has been made.

The numerical approach: a proposed method (III.)

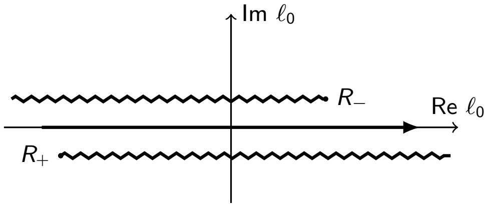
The position of the integrable singularities of a particular propagator is determined by the condition

$$\ell_0 = \ell_{k\pm} \equiv r_{k0} \pm \sqrt{(\vec{\ell} - \vec{r}_k)^2 + m_k^2 - \mathrm{i}\epsilon}$$



The numerical approach: a proposed method (IV.)

The singularities of the complete amplitude integrand lie on the same lines



In general, the integration path is pinched between the singularity lines $\rightsquigarrow$ no contour deformation is possible at the level of the amplitude integrand.

The numerical approach: a proposed method (V.)

Idea: separate the two contributions to the amplitude as follows

$$\mathrm{i}\mathcal{M} = \mathrm{i}\mathcal{M}_{\mathsf{UV}} + \mathrm{i}\mathcal{M}_{\mathsf{IR}},$$

with

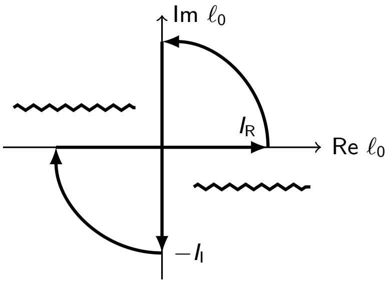
$$\mathrm{i}\mathcal{M}_{\mathsf{UV}} = \int_{|\vec{\ell}|\geq\Lambda} \frac{\mathrm{d}^4\ell}{(2\pi)^4} \sum_{\text{diagrams}} \frac{\mathcal{N}(\ell)}{\prod_{k=1}^{n}\left((\ell - r_k)^2 - m_k^2 + \mathrm{i}\epsilon\right)}$$

$$\mathrm{i}\mathcal{M}_{\mathsf{IR}} = \int_{|\vec{\ell}|\leq\Lambda} \frac{\mathrm{d}^4\ell}{(2\pi)^4} \sum_{\text{diagrams}} \frac{\mathcal{N}(\ell)}{\prod_{k=1}^{n}\left((\ell - r_k)^2 - m_k^2 + \mathrm{i}\epsilon\right)}$$

where Λ is some conveniently chosen scale.

The numerical approach: a proposed method (VI.)

If we choose $\Lambda \gtrapprox 2\sqrt{s}$, we can perform the Wick rotation on the UV part of the amplitude.



The numerical approach: a proposed method (VII.)

The transformed integral for the UV part of the amplitude is

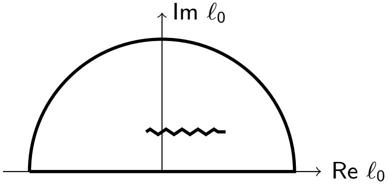
$$\mathcal{M}_{\mathrm{UV}} = \iiint\limits_{|\vec{\ell}|>2\sqrt{s}} \frac{\mathrm{d}^3\ell}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{\mathrm{d}\ell_0}{2\pi} \sum_{\text{diagrams}} \frac{\mathcal{N}(\ell)}{\mathcal{D}(\ell)}\bigg|_{\ell=(\mathrm{i}\ell_0,\vec{\ell})}$$

and it

- avoids the singularity lines
- can be calculated with $\epsilon = 0$
- has Euclidean metric
- monotonically vanishes as $\ell \to \infty$
- can be numerically integrated by quadratures

The numerical approach: a proposed method (VIII.)

In the remaining IR part of the amplitude, the singularities are localized and we can close the contour with an infinite semicircle and employ the Cauchy's residue theorem to analytically integrate over ℓ_0.



The numerical approach: a proposed method (IX.)

After the ℓ_0 integration, the remaining integral for the IR part of the amplitude is

$$\mathcal{M}_{\mathrm{IR}}(\epsilon) = \iiint\limits_{|\vec{\ell}|<2\sqrt{s}} \frac{\mathrm{d}^3\ell}{(2\pi)^3} \sum_{\text{diagrams}} \sum_{k=1}^{n} \frac{\mathcal{N}(\ell)}{\mathcal{D}_k(\ell)}\bigg|_{\ell=(\ell_{k-},\vec{\ell})}$$

and it

- is a *3D integral with finite integration domain*
- has to be calculated with $\epsilon > 0$ in order to have numerical stability
- is suitable for an adaptive Monte Carlo integration

The numerical approach: a proposed method (X.)

To summarize, in order to calculate the one-loop amplitude completely numerically, we

- integrate the sum of all diagrams over the 4D loop-momenta keeping the $i\epsilon$ finite
- introduce the scale Λ separating the UV and the IR part of the amplitude
- perform Wick rotation on the UV part of the amplitude, put $\epsilon \to 0$ and integrate by quadratures
- use residue theorem in the IR part of the amplitude and perform a 3D Monte Carlo integration with finite ϵ
- calculate $\mathcal{M}_{\mathrm{IR}}(\epsilon)$ for different ϵ and extrapolate the result for $\epsilon \to 0$

Implementation of the method

We apply our method to calculate the N-scalar and N-photon amplitudes for various masses of the loop propagators.

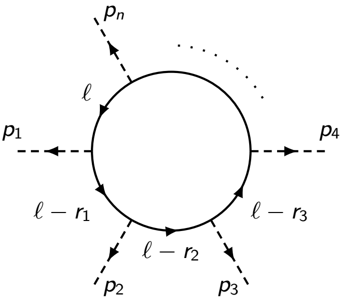
For the IR part of the amplitude, we use 10^6 Monte Carlo points for the *complete amplitude* and aim for a precision of 10^{-2}.

We vary the ϵ in the range $\epsilon/s \in [10^{-4}, 10^{-2}]$.

We fit the $\mathcal{M}_{\mathrm{IR}}(\epsilon)$ to a 2/1 Pade approximant $\frac{a_0+a_1\epsilon+a_2\epsilon^2}{1+b_1\epsilon}$

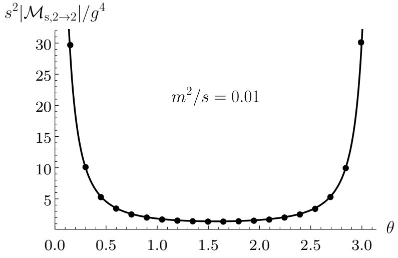
Results: N-scalar amplitudes

A general $2\varphi \to (N-2)\varphi$ amplitude consists of $(N-1)!$ diagrams shown below

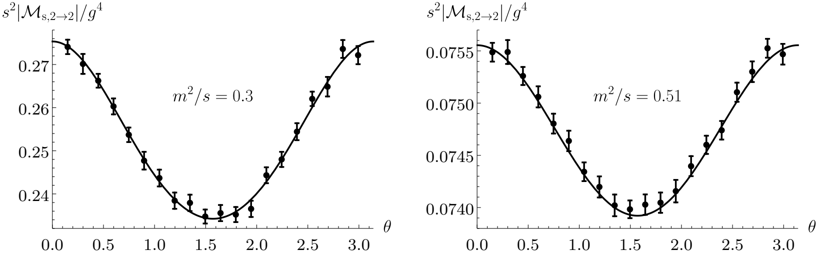


We take the external fields φ to be massless, while we give the internal fields ϕ mass m.

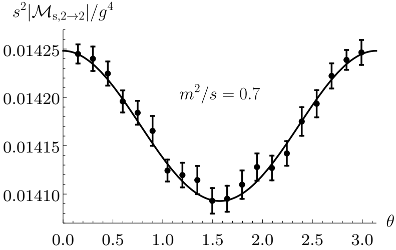
Results: 4-scalar amplitudes (I.)

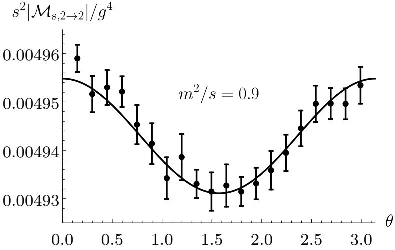


Results: 4-scalar amplitudes (II.)

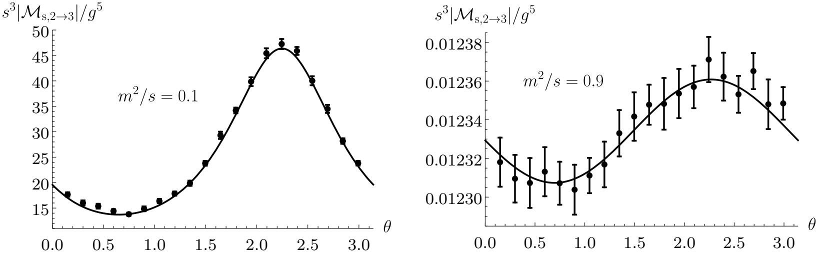


Results: 4-scalar amplitudes (III.)

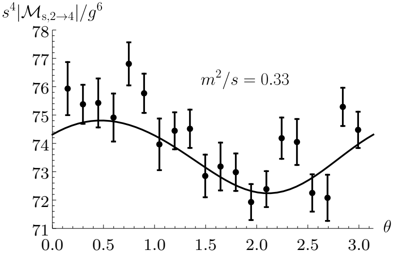


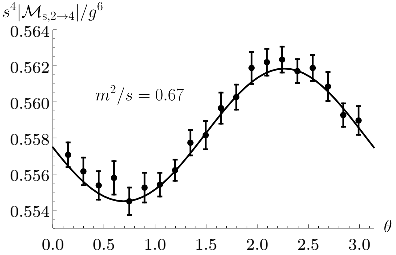


Results: 5-scalar amplitudes



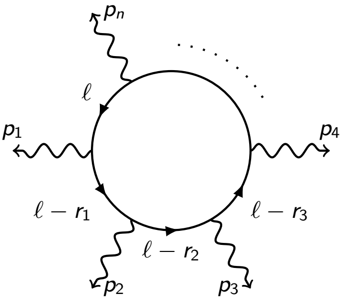
Results: 6-scalar amplitudes





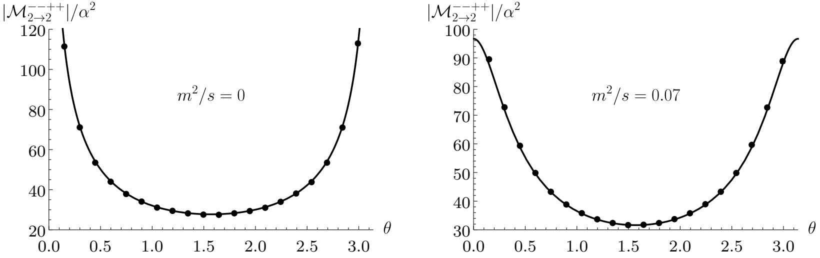
Results: N-photon amplitudes

A general $2\gamma \to (N-2)\gamma$ amplitude consists of $(N-1)!$ diagrams shown below

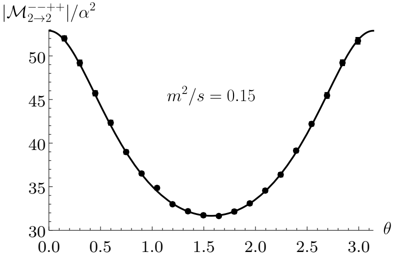


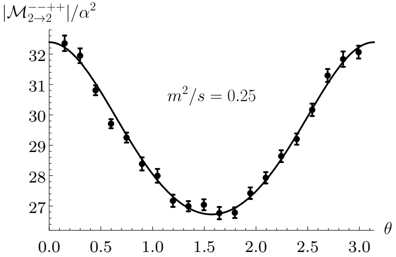
The fermion mass is taken to be m.

Results: 4-photon amplitudes $--++$ (I.)

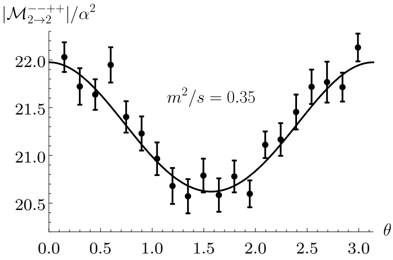


Results: 4-photon amplitudes $- - ++$ (II.)

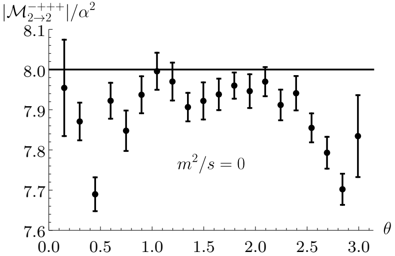




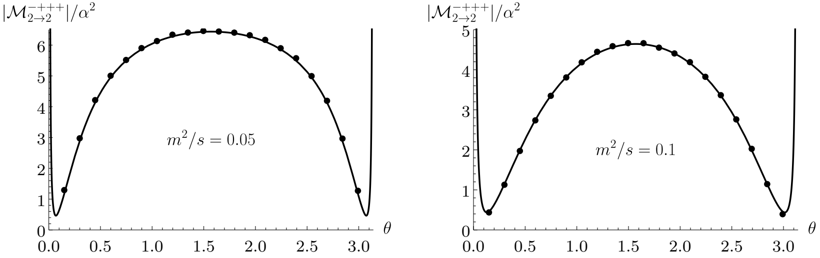
Results: 4-photon amplitudes $--++$ (III.)



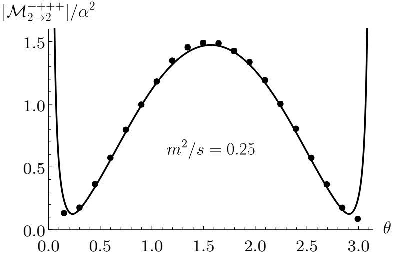
Results: 4-photon amplitudes $- + + +$ (I.)

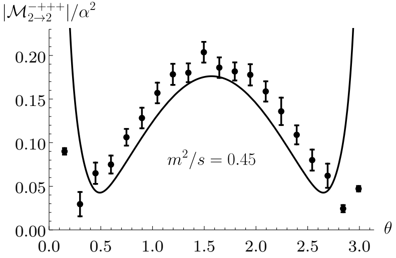


Results: 4-photon amplitudes $-+++$ (II.)

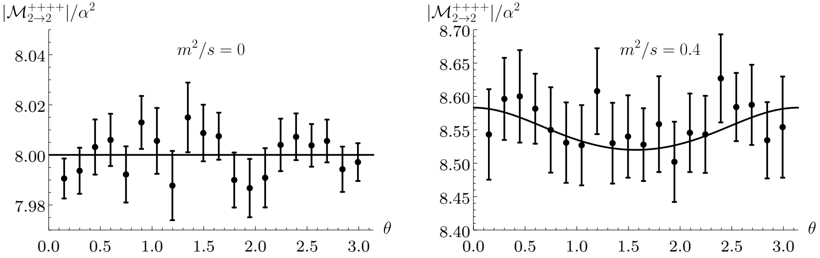


Results: 4-photon amplitudes $-+++$ (III.)

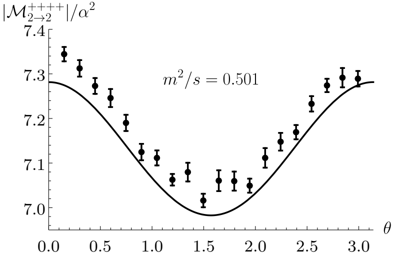


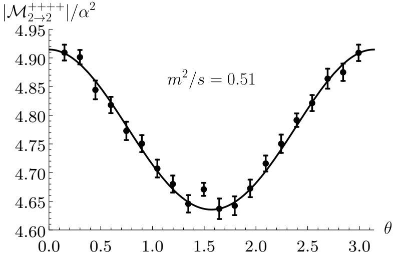


Results: 4-photon amplitudes $++++$ (I.)

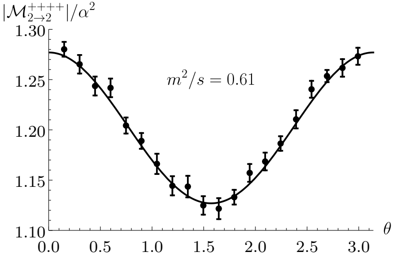


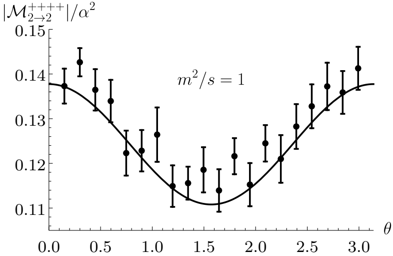
Results: 4-photon amplitudes $++++$ (II.)





Results: 4-photon amplitudes $++++$ (III.)





Results: 5-photon amplitudes

Due to Furry's theorem, the odd-N-photon amplitudes have to vanish.

In our approach we can verify this by checking that

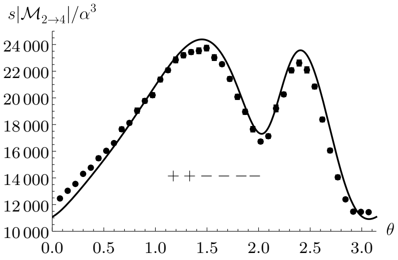
$$\mathcal{M}_{\mathrm{UV}} = -\lim_{\epsilon\to 0} \mathcal{M}_{\mathrm{IR}}(\epsilon)$$

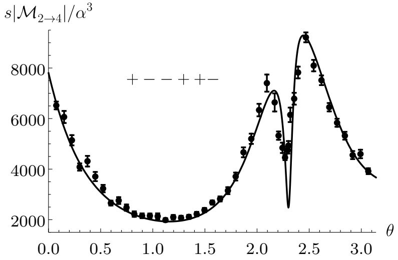
This is much more efficient than having to numerically integrate the whole amplitude to zero.

Our calculations are in full agreement with Furry's theorem.

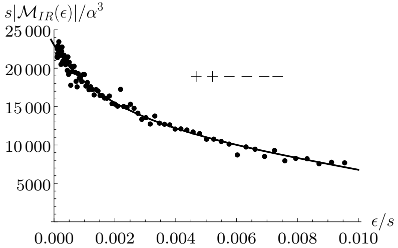
Results: 6-photon amplitudes

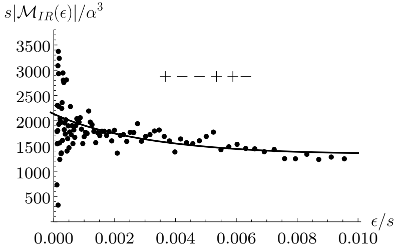
For the 6-photon case, we calculate the massless amplitudes which were analytically calculated by **Mastrolia** et al.





Discussion: scattering of integration points and extrapolation





Conclusions

- We have devised a completely numerical approach to one-loop amplitudes.
- Our method combines previously developed contour deformation and extrapolation methods.
- The method is very general and process-independent.
- All the results shown were generated by a mere 10^6 Monte Carlo points per amplitude.
- The method was implemented in Mathematica using NIntegrate.
- The question remains: How far could a more serious implementation take us?

Conclusions

Thanks for the attention!