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Through the Big Bang

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The Shape-Dynamical interpretation of General Relativity

Lichnerowicz–York–Choquet-Bruhat... (70's): there is **choice of foliation** in which the physical degrees of freedom of GR are **3D conformal invariants.**

Maybe those degrees of freedom are **fundamental**, and GR with its refoliation-invariant spacetime is **emergent.**

This assumption allows to solve some **conceptual problems in quantum gravity** (problem of many-fingered time, problem of observables) and leads to what should be an **inequivalent quantum theory.**

At the classical level there are spacetimes in which York's foliation cannot be continued everywhere. But if the conformally invariant degrees of freedom can be continued everywhere, **these are solutions of SD but not of GR.**

Arnowitt–Deser–Misner's Hamiltonian formulation of GR

3+1 split of the metric:

$$ {}^{(4)}g_{\mu\nu} = \begin{pmatrix} -N^2 + g_{ij}\,\xi^i\,\xi^j & g_{ik}\,\xi^k \\ g_{jk}\,\xi^k & g_{ij} \end{pmatrix}, $$

Einstein action:

$$ \int d^4x \sqrt{{}^{(4)}g}\;{}^{(4)}R = \int dt d^3x \left(\dot{g}_{ij} p^{ij} + N\,\mathcal{H}[g,p] + \xi^i \mathcal{D}_i[g,p] \right)\;, $$

3D-Diffeomorphism constraint:

$$ \mathcal{D}_i = -2\,\nabla_j p^j{}_i \approx 0\,, $$

Hamiltonian constraint:

$$ \mathcal{H} = \frac{1}{\sqrt{g}} \left(p^{ij} p_{ij} - \tfrac{1}{2} (\mathrm{tr}\, p)^2 \right) - \sqrt{g}\,R \approx 0\,, $$

York's conformal method for initial value problem

In a closed spatial hypersurface, in *CMC slicing:*

$$\mathrm{tr}\, p = g_{ij}p^{ij} = \tfrac{3}{2}\tau\sqrt{g} = \textit{const.}\,\sqrt{g}\ ,$$

$\mathcal{H} \approx 0$ and $\mathcal{D}_i \approx 0$ **decouple** and turn into **elliptic equations.**

Start with a reference metric g_{ij}, a tensor density p^{ij} and a real number τ, by solving the two equations we get a conformally transformed metric γ_{ij}, and a transverse-constant-trace momentum π^{ij} satisfying all of the constraints.

Conformal invariance: $\begin{cases} g_{ij} \to \phi^4\, g_{ij} \\ p^{ij} \to \phi^{-4}\, p^{ij} \end{cases} \quad \Rightarrow \quad \begin{cases} \gamma_{ij}[\phi^4 g, \phi^{-4} p] = \gamma_{ij}[g,p] \\ \pi^{ij}[\phi^4 g, \phi^{-4} p] = \pi^{ij}[g,p] \end{cases}$

$$\boxed{\text{CMC solutions of GR completely specified by a conformal class of metrics (conformal geometry), a transverse-traceless tensor and } \tau \in \mathbb{R}}$$

Shape Dynamics

Reformulate GR as an intrinsically 3-dimensional conformal field theory.

(CMC) time evolution is generated by the conformally-invariant Hamiltonian:

$$\boxed{H_{\mathsf{SD}}[g_{ij}, p^{ij}, \tau] := \int d^3x \sqrt{g}\, \Omega^6}$$

where Ω is the solution of the LY equation:

$$\frac{\Omega^{-6}}{\sqrt{g}} \left(p^{ij} - \tfrac{1}{3} g^{ij} \mathrm{tr}\, p\right)^2 - \tfrac{3}{8} \sqrt{g}\, \Omega^6 \tau^2 - \sqrt{g} \left(R\, \Omega^2 - 8\, \Omega \Delta \Omega\right) = 0\,.$$

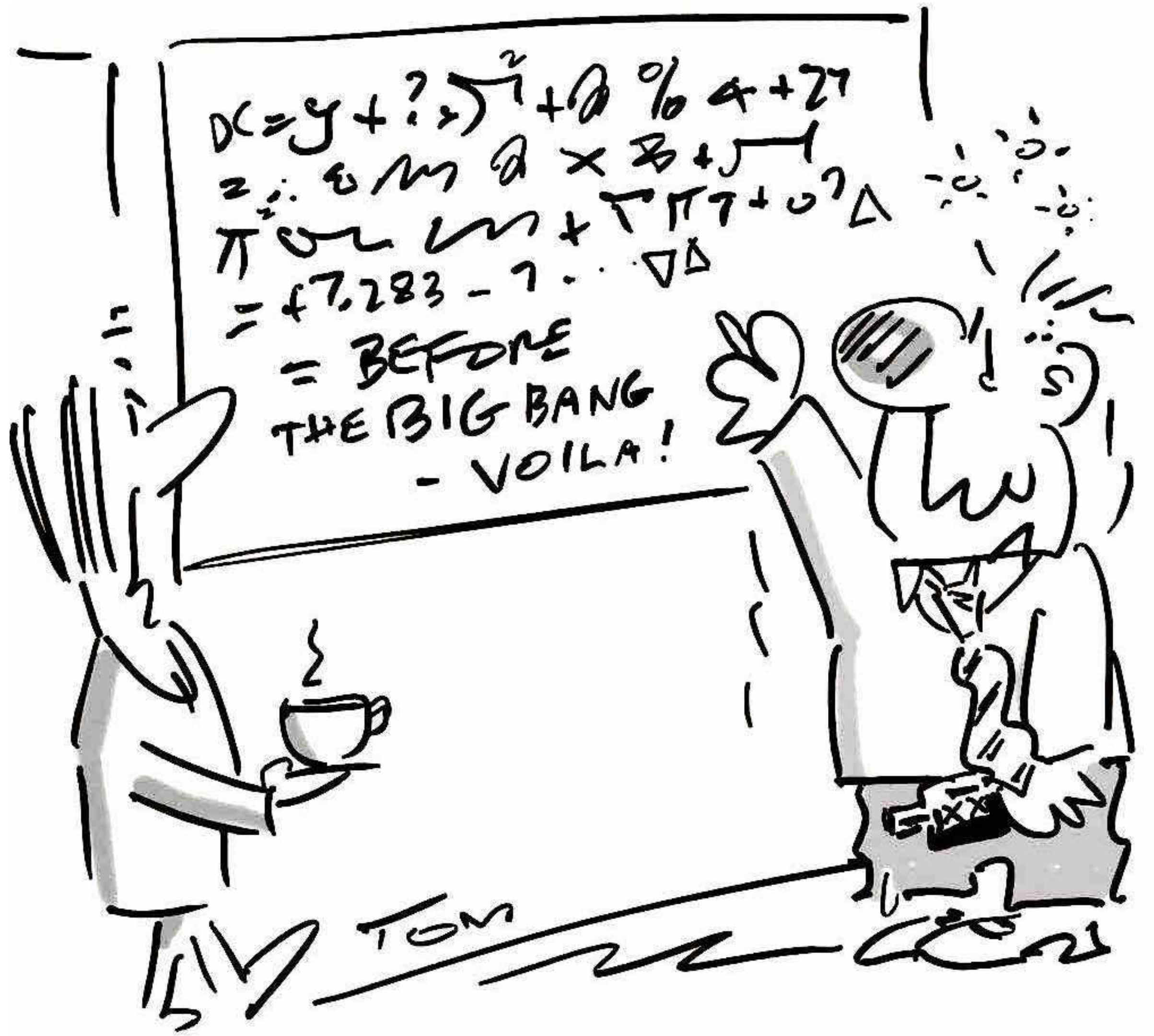
The above Hamiltonian generates the evolution of

conformally invariant degrees of freedom

$$\frac{\partial g_{ij}}{\partial \tau} = \frac{\delta H_{\mathsf{SD}}}{\delta p^{ij}}\,, \qquad \frac{\partial p^{ij}}{\partial \tau} = -\frac{\delta H_{\mathsf{SD}}}{\delta g_{ij}}\,,$$

which determine everything about the gravitational field.

FM, *A Shape Dynamics Tutorial* [arXiv:1409.0105 - to be published by Oxford]



“There’s nothing clever about getting drun... WOW!”

What we've done, in pills

We think we can prove, in fair generality, that **if there is a free scalar field in nature,** we can continue **classical** solutions of GR through the big bang singularity. The two solutions have opposite orientation: **parity reversal.**

We think that setting up a reasonable **measure** at the big bang will result in two universes with a **opposite arrows of time** ('Janus point' explanation).

Barbour–Koslowski–FM, *Identification of a Gravitational Arrow of Time*, [Phys. Rev. Lett. **113**, 181101]

BKL conjecture

Approaching the singularity, each solution to Einstein's equations approaches a solution to the 'Velocity Term Dominated' (VTD) equations, obtained by **neglecting spatial derivatives** (Belinsky–Khalatnikov–Lifshitz 1971).

Lots of numerical evidence in its favour (Garfinkle, Uggla, Elst, Ellis, Wainwright, Curtis, Moncrief, Berger...).

Andersson–Rendall: with a **massless scalar/stiff fluid**, for every solution to the VTD equations there exists a solution to the full field equations that converges to the VTD solution as the singularity is approached [Commun. Math. Phys. **218**, 2001].

The only matter that matters is massless scalar matter

Bianchi I ansatz:

$$ds^2 = -dt^2 + a(t)^2 \left(e^{-2\phi}dx^2 + e^{\phi+\psi}dy^2 + e^{\phi-\psi}dz^2\right) \,,$$

Friedmann equation:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Sigma^2}{a^6} \,,$$

where $\Sigma = \Sigma(\dot{\phi}, \dot{\psi}, \phi, \psi)$ is a constant of motion (anisotropic shear).

Approaching the singularity $a \to 0$ only one kind of source counts:

kind	state equation	behaviour
Λ	$p = -\rho$	$\rho =$ *const.*,
dust	$p = 0$	$\rho \sim a^{-3}$,
radiation	$p = \rho/3$	$\rho \sim a^{-4}$,
stiff fluid / massless scalar	$p = \rho$	$\rho \sim a^{-6}$

How we did it, in pills

BKL: at the singularity the universe tends to a set of decoupled Bianchi models, one per point (or one per Fourier mode).

S^3 topology: **Bianchi IX**. Mixmaster behaviour: gravitational dofs do not admit well-defined limit as $a \to 0$ (go through infinite amount of change).

Bianchi IX + massless scalar: **quiescent cosmology**. After a finite amount of change, evolution stabilizes around a Kasner solution. Andersson–Rendal's result imply we can generate a "non-zero-measure" set of solutions from this.

Our result: we can continue each quiescent-Bianchi IX solution past the big bang, by requiring **continuity of shape degrees of freedom.**

To do that, we identify four **perennials:**

conserved quantities that completely specify the solution.

...but wait a second: Bianchi IX is not integrable! There is chaos!

But **quiescent-Bianchi IX tends to an integrable dynamics** (Bianchi I). So we define **asymptotic perennials** which are conserved only at the singularity. Their asymptotic values completely specify the solution.

The system necessarily reaches degenerate (i.e. lower dimensional) shapes at the singularity. These are the boundary between spatial manifolds with opposite orientation. **Parity change at the big bang.**

Gruesome Details

Homogeneous-but-anisotropic cosmology on S^3

Homogeneity hypothesis:

$$g_{ij} = \sum_{a=1}^{3} C_a \, \sigma_i^a \, \sigma_j^a \,, \qquad p^{ij} = |\det \sigma| \sum_{a=1}^{3} \frac{P^a}{C_a} \chi_a^i \, \chi_a^j \,,$$

$$\begin{aligned} \sigma^x &= \sin\psi \, d\theta - \cos\psi \, \sin\theta \, d\phi \,, & \chi_x &= \cos\psi \, \cot\theta \, \partial_\psi + \sin\psi \, \partial_\theta - \cos\psi \, \csc\theta \, \partial_\phi \,, \\ \sigma^y &= \cos\psi \, d\theta + \sin\psi \, \sin\theta \, d\phi \,, & \chi_y &= -\sin\psi \, \cot\theta \partial_\psi + \csc\theta \left(\cos\psi \partial_\theta + \sin\psi \, \partial_\phi\right) \,, \\ \sigma^z &= -d\psi - \cos\theta \, d\phi \,, & \chi_z &= -\partial_\psi \,. \end{aligned}$$

(the diagonal form of $g_{ij} = q_{ab} \, \sigma_i^a \, \sigma_j^b$ and $p^{ij} = |\det \sigma| \, p^{ab} \, \chi_a^i \, \chi_b^j$

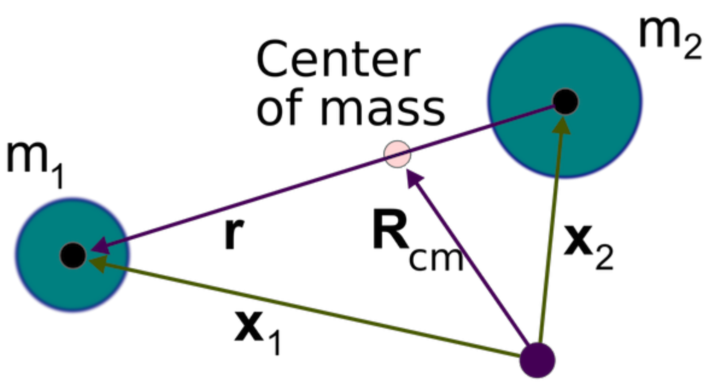
is a gauge choice of the $O(3)$ internal symmetry of frame fields)

$$\Theta = \sum_{a=1}^{3} \frac{P_a \, \dot{C}_a}{C_a} \,, \qquad \mathcal{H} = \sum_a {P_a}^2 - \tfrac{1}{2}(\textstyle\sum_b P_b)^2 + \sum_a {C_a}^2 - \tfrac{1}{2}(\textstyle\sum_b C_b)^2 \,,$$

Separate scale from shape

C_x, C_y, C_z and P_x, P_y, P_z are 6 degrees of freedom. **4 are shape** and their conjugate momenta, **2 are the scale factor** and its conjugate momentum.

Can separate them using Jacobi coordinates for the 3-body problem:



$$\vec{j}_1 = \frac{\vec{x} - \vec{y}}{\sqrt{2}}, \quad \vec{j}_2 = \sqrt{\frac{2}{3}} \left(\vec{z} - \frac{\vec{y} + \vec{x}}{2} \right),$$

$$\vec{r}_{cm} = \frac{\vec{x} + \vec{y} + \vec{z}}{3},$$

(separates centre-of-mass dof from relative separations).

Misner variables

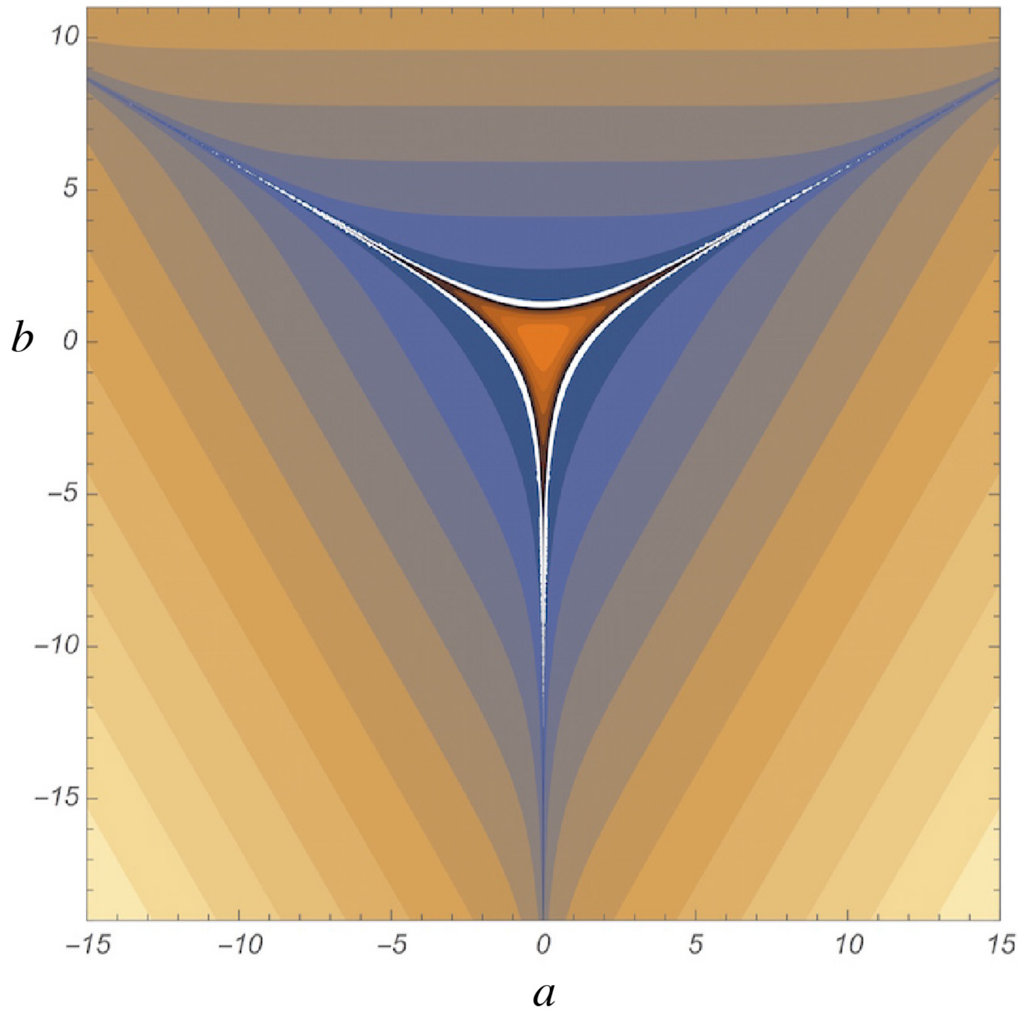
$$k_a = \frac{P_y - P_x}{\sqrt{2}}\,,\;\; k_b = \sqrt{\tfrac{2}{3}}\left(P_z - \frac{P_y + P_x}{2}\right)\,,\;\; D = \frac{P_x + P_y + P_z}{3}\,,$$

$$a = \tfrac{1}{\sqrt{2}} \log\left(\frac{C_y}{C_x}\right)\,,\;\; b = \sqrt{\tfrac{2}{3}} \log\left(C_z/\sqrt{C_y C_x}\right)\,,\;\; v = (C_x C_y C_z)^{\frac{1}{2}}\,,$$

a, b **shape** dofs (anisotropies), $v = \int d^3x \sqrt{g}$ **volume,**

k_a, k_b **shape momenta,** $\tau = \frac{1}{v} \int d^3x (p^{ij} g_{ij})$ **CMC time (York time).**

$$\Theta = k_a\,\dot{a} + k_b\,\dot{b} + \tau\,\dot{v}\,, \qquad \mathcal{H} = \tfrac{3}{8} v^2 \tau^2 - {k_a}^2 - {k_b}^2 + v^{4/3}\, V_{\mathsf{S}}(a, b)\,,$$



$$V_S(a,b) = F(2b)+F(\sqrt{3}a-b)+F(-\sqrt{3}a-b)\,, \qquad F(x) = e^{-\frac{x}{\sqrt{6}}}-\frac{1}{2}e^{2\frac{x}{\sqrt{6}}}$$

York time is monotonic

$$\dot{v} = \frac{\partial \mathcal{H}}{\partial \tau} = \tfrac{3}{4} v^2 \tau \,, \qquad \dot{\tau} = -\frac{\partial \mathcal{H}}{\partial v} \approx -\tfrac{1}{4} v \, \tau^2 - \tfrac{4}{3} \left({k_a}^2 + {k_b}^2 \right) \, v^{-1} < 0 \,,$$

$\dot{\tau} < 0$. So once τ is negative v monotonically goes to zero:

$$(\dot{v^{-1}}) = -\frac{\dot{v}}{v^2} = -\tfrac{3}{4} \tau \,,$$

Isotropic/FRW case: $a = b = k_a = k_b = 0$,

$$\Downarrow$$

$$v \sim t^{-3/2}, \qquad \tau \sim -t^{1/2}.$$

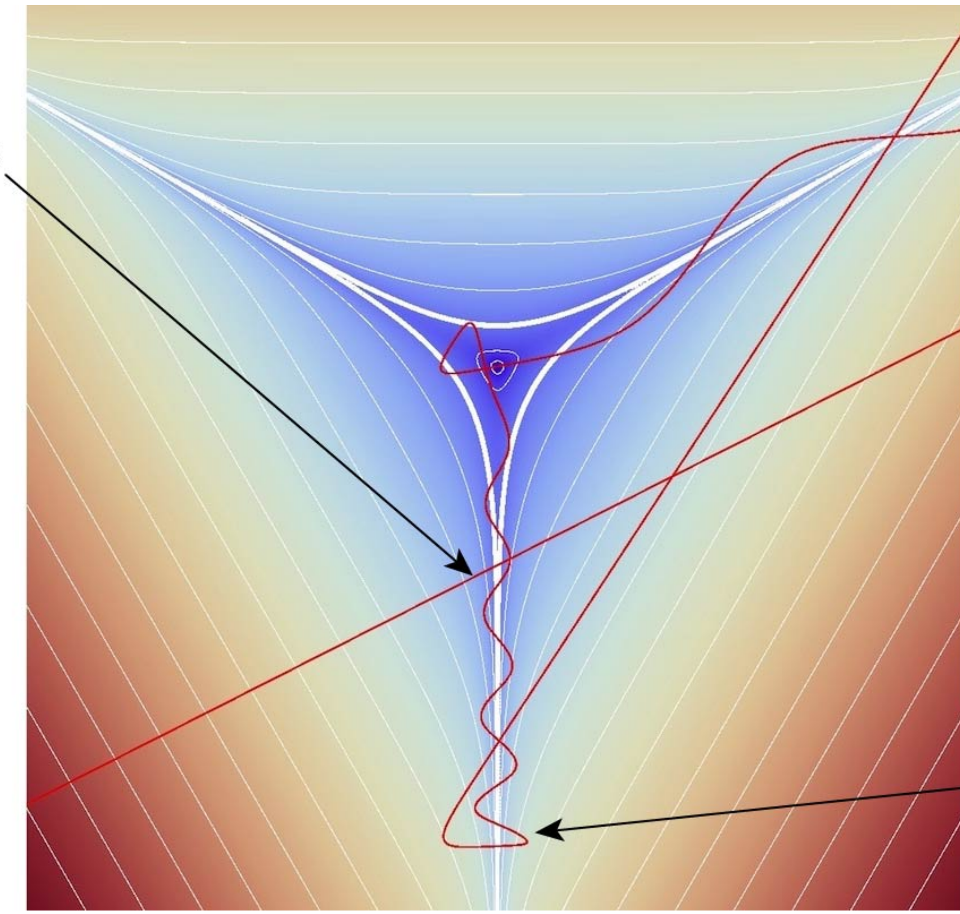
Mixmaster behaviour

$$\mathcal{H} = \tfrac{3}{8} v^2 \tau^2 - {k_a}^2 - {k_b}^2 + v^{4/3}\, V_{\mathsf{S}}(a, b)\,,$$

$$v^{4/3} V_{\mathrm{S}}(a,b) << {k_a}^2 + {k_b}^2$$

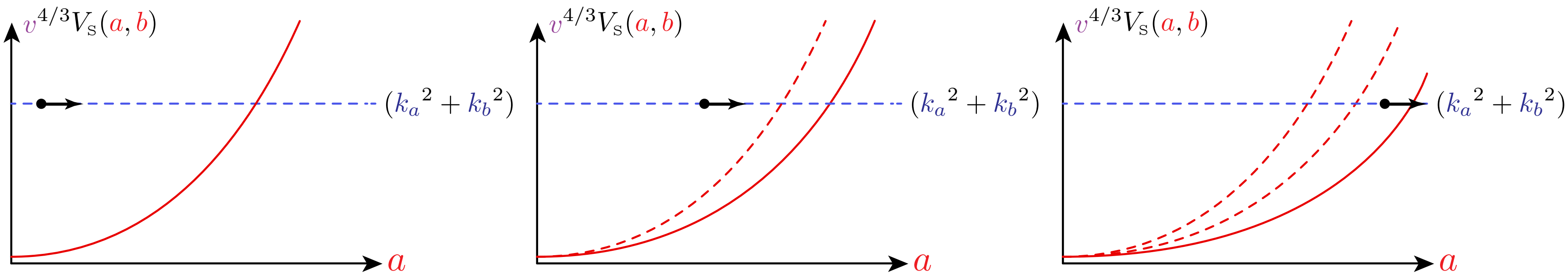
$$\Downarrow$$

$$\mathcal{H} \simeq \tfrac{3}{8} v^2 \tau^2 - {k_a}^2 - {k_b}^2$$



$$v^{4/3} V_{\mathrm{S}}(a,b) \simeq {k_a}^2 + {k_b}^2$$

W/o matter you always catch up with the receding potential



Assume initially $v^{4/3}\,V_S << (k_a{}^2 + k_b{}^2)$. Use $\log v$ as time. Then:

$$\left(\frac{\partial a}{\partial \log v}\right)^2 + \left(\frac{\partial b}{\partial \log v}\right)^2 = \frac{8}{3}\frac{k_a{}^2 + k_b{}^2}{k_a{}^2 + k_b{}^2 - v^{4/3}\,V_S} \sim \frac{8}{3} + \mathcal{O}(v^{4/3}\,V_S)\,,$$

so $(a,b) \sim \sqrt{\frac{8}{3}}(\cos\varphi, \sin\varphi)\;\log v.$

Now recall the form of the potential:

$$V_{\mathsf{S}}(a,b) = F(2b)+F(\sqrt{3}a-b)+F(-\sqrt{3}a-b)\,, \qquad F(x) = e^{-\frac{x}{\sqrt{6}}}-\frac{1}{2}e^{2\frac{x}{\sqrt{6}}}\,,$$

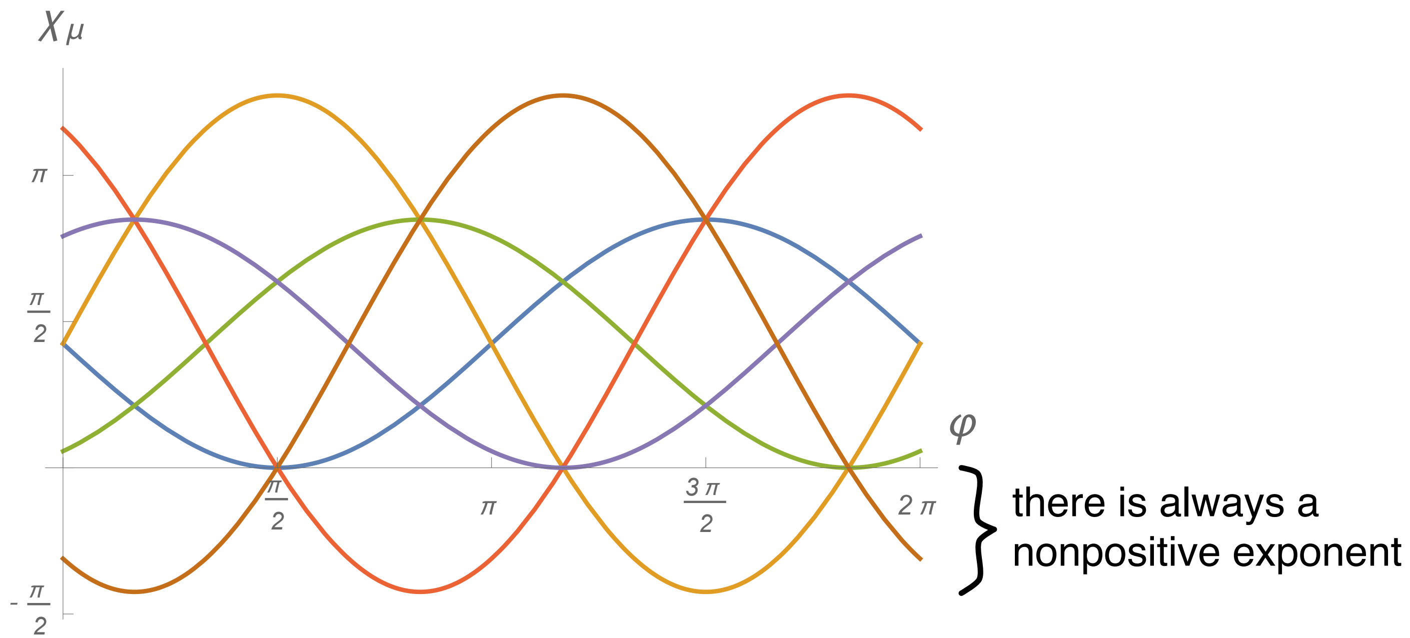
then:

$$v^{4/3}V_{\mathsf{S}} \sim \left(v^{\chi_1} - \tfrac{1}{2}v^{\chi_2} + v^{\chi_3} - \tfrac{1}{2}v^{\chi_4} + v^{\chi_5} - \tfrac{1}{2}v^{\chi_6}\right)\,,$$

where $\chi_\mu = \chi_\mu(\varphi)$. For example:

$$e^{2\sqrt{\frac{2}{3}}b} \sim e^{\frac{8}{3}\sin\varphi\,\log v} = v^{\frac{8}{3}\sin\varphi} = v^{\chi_2(\varphi)}\,, \qquad \text{etc...}$$

Plot of the exponents



one of the $v^{\chi_\mu(\varphi)}$ will **always grow** as $v \to 0$.

You always catch up with the potential.

Quiescent behaviour: add a massless scalar

$$\Theta = k_a\,\dot{a} + k_b\,\dot{b} + \tau\,\dot{v} + \phi\,\dot{\pi_\phi}\,, \;\; \mathcal{H} = \tfrac{3}{8}v^2\tau^2 - {k_a}^2 - {k_b}^2 - \tfrac{1}{2}{\pi_\phi}^2 + v^{4/3}\,V_{\mathsf{S}}(a,b)\,,$$

${\pi_\phi}^2$ is conserved. Now during inertial motion

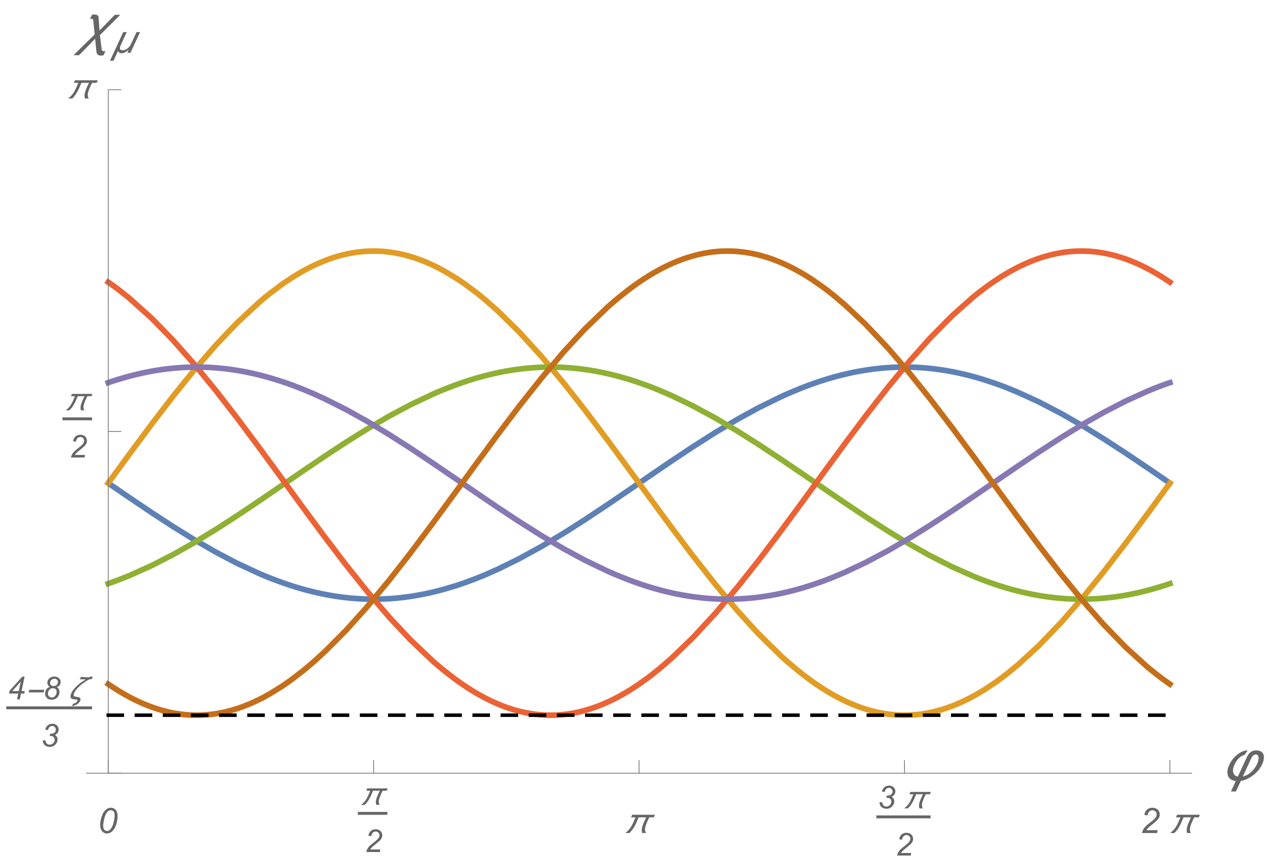
$$\left(\frac{\partial a}{\partial \log v}\right)^2 + \left(\frac{\partial b}{\partial \log v}\right)^2 \sim \frac{8}{3}\frac{{k_a}^2 + {k_b}^2}{{k_a}^2 + {k_b}^2 + \frac{1}{2}{\pi_\phi}^2} < \frac{8}{3}\,,$$

Call $\quad \zeta = \sqrt{\frac{{k_a}^2+{k_b}^2}{{k_a}^2+{k_b}^2+\frac{1}{2}{\pi_\phi}^2}}, \quad 0 < \zeta < 1.$

now $(a, b) \sim \boxed{\zeta}\sqrt{\frac{8}{3}}(\cos\varphi, \sin\varphi)\ \log v$. Again, in the potential:

$$e^{2\sqrt{\frac{2}{3}}b} \sim e^{\frac{8}{3}\boxed{\zeta}\sin\varphi\,\log v} = v^{\frac{8}{3}\boxed{\zeta}\sin\varphi} = v^{\chi_2(\varphi,\zeta)}, \qquad \text{etc...}$$

then $\boxed{\chi_\mu = \chi_\mu(\varphi, \zeta)}$



$\Rightarrow$ If $\boxed{\zeta < \frac{1}{2}}$ all the χ_μ are positive $\forall\varphi$!

But does ζ reach values smaller than $1/2$?

$$\Theta = k_a\,\dot{a} + k_b\,\dot{b} + \tau\,\dot{v} + \phi\,\dot{\pi_\phi}\,, \;\; \mathcal{H} = \tfrac{3}{8}v^2\tau^2 - {k_a}^2 - {k_b}^2 - \tfrac{1}{2}{\pi_\phi}^2 + v^{4/3}\,V_S(a,b)\,,$$

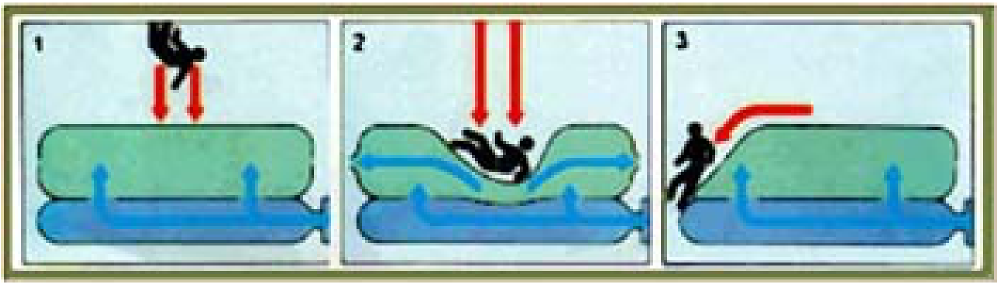
In BKL regime $\nabla_i\phi \simeq 0$: no potential for ϕ. The momentum π_ϕ is conserved.

Observation: the bounces against the potential $v^{4/3}\,V_S(a,b)$

are **inelastic** ($v^{4/3}\,V_S(a,b)$ is time-dependent due to $v^{4/3}$).

After a bounce, the shape kinetic energy ${k_a}^2 + {k_b}^2$ always **decreases**

towards the big bang (because the coupling $v^{4/3}$ decreases monotonically).



For any **nonzero** value of π_ϕ the system will

keep losing shape kinetic energy until eventually

$$\zeta = \sqrt{\frac{{k_a}^2 + {k_b}^2}{{k_a}^2 + {k_b}^2 + \frac{1}{2}{\pi_\phi}^2}} < \frac{1}{2}\,,$$

after that $v^{4/3}\,V_\mathsf{S}$ will be ever-decreasing,

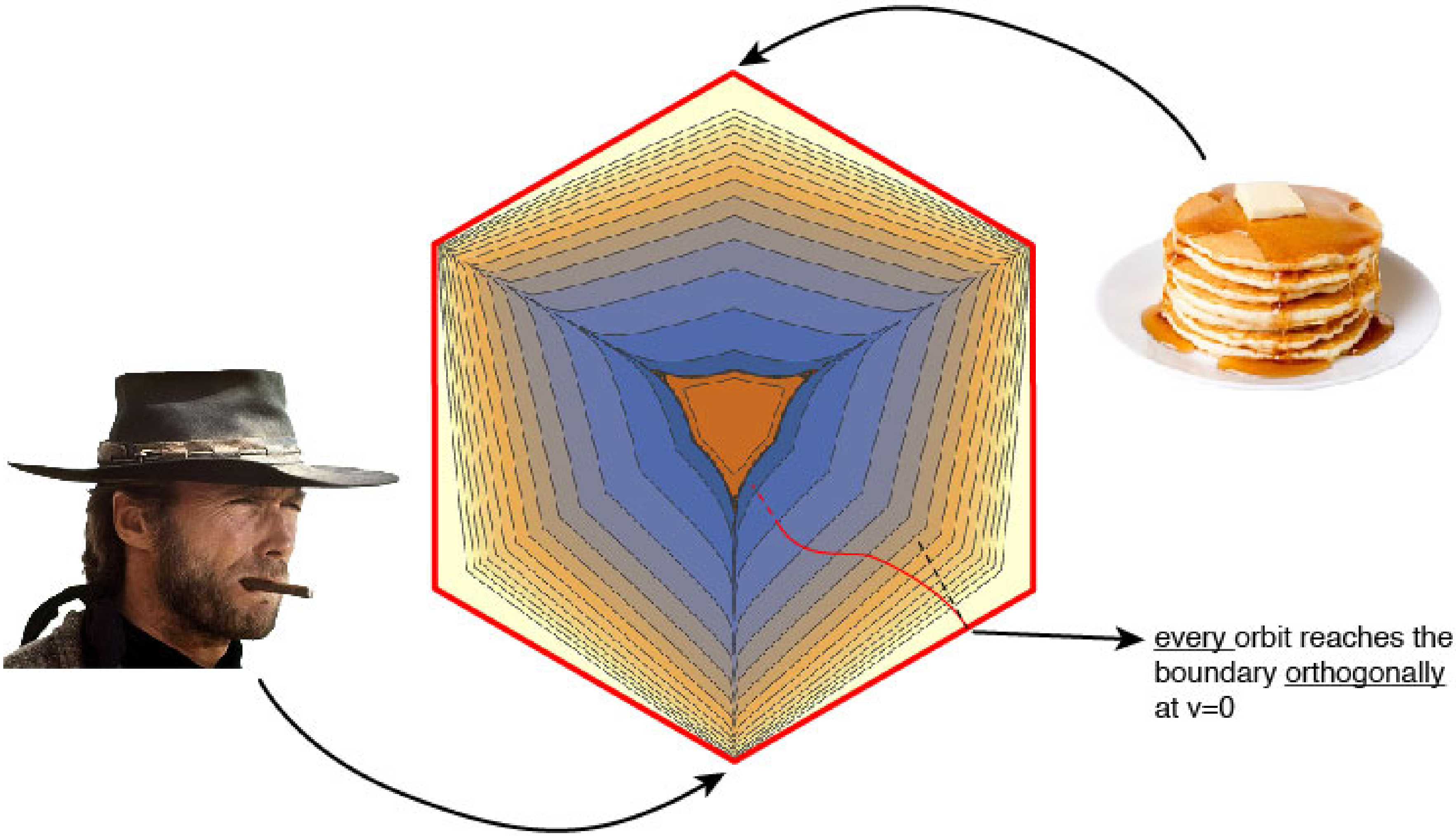
and the solution will stabilize around an inertial solution:

$$a(v) = a_0 + \zeta\ \cos\varphi\ \log v\,, \qquad b(v) = b_0 + \zeta\ \sin\varphi\ \log v\,,$$

THESE SOLUTIONS ALL REACH THE SINGULARITY ($\log v \to -\infty$)

AT THE **BOUNDARY** ($a^2 + b^2 \to \infty$) OF SHAPE SPACE.

Boundary of hexagonally compactified Shape Space



Continuation through the big bang

To prove I can continue through the singularity, I need enough **perennials** (conserved quantities whose values completely fix a solution), built only from shape degrees of freedom.

But Bianchi IX is **not an integrable system** (it is chaotic): we know it doesn't admit enough perennials.

But **quiescent Bianchi IX** asymptotes to an integrable systen (the inertial motion). So I can define **asymptotic perennials** (quantities that are conserved only at $v = 0$).

Conserved quantities

Inertial motion:

$$(a, b)_{\text{inertial}} = (a_0, b_0) + \zeta\,(\cos\varphi, \sin\varphi)\ \log v\,, \quad (k_a, k_b)_{\text{inertial}} = k\ \sin\varphi$$

the above motion conserves $\boxed{\zeta}$ and:

the shape kinetic energy $\boxed{k = \sqrt{{k_a}^2 + {k_b}^2}}$

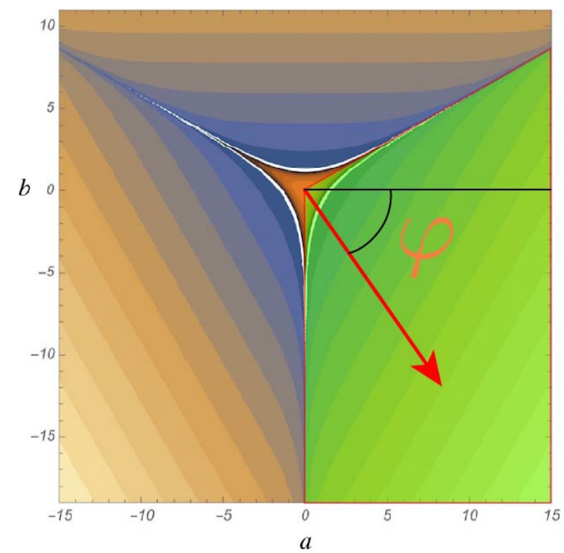
the direction in shape space $\boxed{\varphi = \arctan(k_b/k_a)}$

and a ‘shape angular momentum’ $\boxed{\ell = a\,k_b - b\,k_a}$

But you cannot read all four ζ, k, φ, ℓ off the inertial orbit.

Perturbative expansion around inertial motion

W.l.o.g. assume $-\frac{\pi}{2} < \varphi < \frac{\pi}{6}$,



then, for small v's, the full quiescent admits an expansion around $(a, b)_{\text{inertial}}$

$$a = \left(\frac{\mathcal{D}}{k} + \sqrt{\tfrac{8}{3}}\,\zeta \log v\right)\cos\varphi + \frac{\ell}{k}\sin\varphi + \frac{\zeta^3}{k^2}\left(\sqrt{\tfrac{8}{3}}\cos\varphi - \frac{2\,\zeta\,\cos\varphi\sin\varphi}{1+\zeta\sin\varphi}\right)\frac{v^{\frac{8}{3}+\frac{8}{3}\zeta\sin\varphi}}{\frac{4}{3}+\frac{8}{3}\zeta\sin\varphi} + o(v^{\frac{4-8\zeta}{3}})\,,$$

$$b = \left(\frac{\mathcal{D}}{k} + \sqrt{\tfrac{8}{3}}\,\zeta \log v\right)\sin\varphi - \frac{\ell}{k}\cos\varphi + \frac{\zeta^3}{k^2}\left(\sqrt{\tfrac{8}{3}}\sin\varphi + 8\sqrt{2}\,\zeta\,\frac{\cos^2\varphi + \frac{1}{\zeta^2} - 1}{1+\zeta\sin\varphi}\right)\frac{v^{\frac{8}{3}+\frac{8}{3}\zeta\sin\varphi}}{\frac{4}{3}+\frac{8}{3}\zeta\sin\varphi} + o(v^{\frac{4-8\zeta}{3}})\,,$$

Asymptotic perennials

These are *shape*-phase-space functions that are asymptotically conserved:

$$\arctan(k_b/k_a) \text{ or } \arctan(b/a) \xrightarrow[v\to 0]{} \varphi\,,$$

$$\sqrt{{k_a}^2 + {k_b}^2} \xrightarrow[v\to 0]{} k\,,$$

$$(a\,k_b - b\,k_a) \xrightarrow[v\to 0]{} \ell\,,$$

$$\left(\frac{\sqrt{8/3}\sqrt{a^2+b^2}}{\log[k_b - k_a\,b/a)] - \sqrt{2}\sqrt{a^2+b^2}\,\sin[\arctan(b/a)]}\right) \xrightarrow[v\to 0]{} \zeta\,,$$

Can read them off the perturbative expansion of the full solution!

Quiescent Bianchi IX has 8-dimensional phase space: v, τ, a, b, k_a, k_b, π_ϕ, ϕ

The Hamiltonian constraint reduces it to a 7D hypersurface.

To fix a 1D curve in a 7D hypersurface need to specify 6 conditions.

Two conditions are **unobservable**: one determines the scale of the system (the normalization of v) at one instant, and the other determines $\phi = \phi(v)$ (ϕ is massless so physics is invariant under shifts $\phi \to \phi + \epsilon$. The value of ϕ is as unmeasurable as a global phase shift in the Higgs field).

The remaining four are our asymptotic perennials: ζ, k, φ, ℓ.

Orientation change

Shape space is a quotient:

Frame fields $E_i^a(x)$ know about orientation ($\det E$ has a sign),

$$\Downarrow$$

Quotient wrt internal rotations $E_i^a \to \Omega_b^a E_i^b$,

$$\Downarrow$$

Metric variables g_{ij}, $\det g > 0$ forgot about orientation,

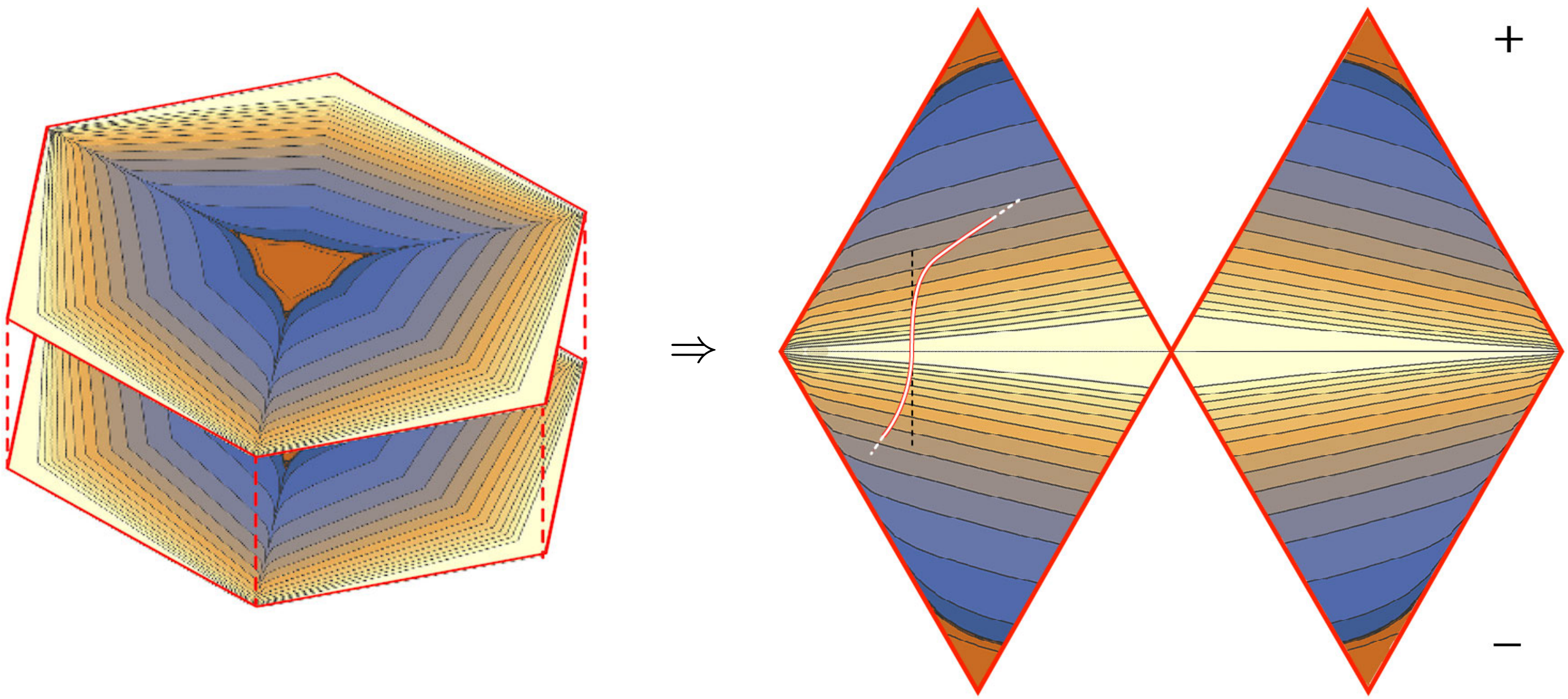
$$\Downarrow$$

Symmetry reduction wrt translations (homogeneity): a, b, v variables.

Quotienting wrt $O(3)$ we discard information about the orientation.

But if we quotient wrt $SO(3)$ we obtain the **double cover** of shape space.

Shape space's **border** is invariant under parity (oriented volume form is zero):



We obtain a unique continuous curve through the boundary of shape space.

Change of orientation through the Big Bang.

Open ends and wild speculations

- We got P, we got T (Janus-point argument)... do we have C?
- Does this imply something for matter-antimatter symmetry? Perhaps most Janus-point condition will have a small asymmetry (and its conjugate on the other side) even if the probability distribution is symmetric...
- Similarly, perhaps the typical initial condition has many patches with differently-oriented Higgs field...
- Need a measure for the values of our perennials at the big bang.
- Need a perturbative evolution out of the big bang, e.g. expanding in Fourier modes and introducing the spatial derivatives as couplings between different modes. IR cutoff: cosmological horizont. UV cutoff: angular resolution of Planck satellite.
- Quantum version of the argument?

Appendix: Conformal method for initial value problem

In a closed spatial hypersurface, in *CMC slicing:*

$$\mathrm{tr}\, p = g_{ij} p^{ij} = \tfrac{3}{2}\tau\sqrt{g} = \textit{const.}\, \sqrt{g}\ ,$$

$\mathcal{H} \approx 0$ and $\mathcal{D}_i \approx 0$ **decouple** and turn into **elliptic equations.**

Start with a reference metric g_{ij}, a tensor density p^{ij} and a real number τ,

1. solve the Lichnerowicz–York equation wrt a scalar field Ω:

$$\frac{\Omega^{-6}}{\sqrt{g}}\left(p^{ij} - \tfrac{1}{3} g^{ij}\mathrm{tr}\, p\right)^2 - \tfrac{3}{8}\sqrt{g}\,\Omega^6\tau^2 - \sqrt{g}\left(R\,\Omega^2 - 8\,\Omega\Delta\Omega\right) = 0\ ,$$

2. solve the transversality condition wrt a vector field χ^i:

$$\nabla_j(\nabla^i\chi^j + \nabla^j\chi^i - \tfrac{2}{3} g^{ij}\nabla_k\chi^k) = \nabla_j(p^{ij} - \tfrac{1}{3}\mathrm{tr}\, p\; g^{ij})\ ,$$

3. then the conformally transformed metric γ_{ij}, and the TT momentum π^{ij}:

$$\gamma_{ij} = \Omega^4 g_{ij}\,, \quad \pi^{ij} = (p^{ij} - \tfrac{1}{3} g^{ij} \mathrm{tr}\, p) - (\nabla^i \chi^j + \nabla^j \chi^i - \tfrac{2}{3} g^{ij} \nabla_k \chi^k) + \tfrac{1}{2} \tau\, \gamma^{ij} \sqrt{\gamma}\,,$$

satisfy all of the ADM constraints **in a CMC slice** $\left(\mathrm{tr}\, \pi = \frac{3}{2} \tau \sqrt{\gamma}\right)$.

The LY equation and the transversality condition are conformally invariant:

$$\left\{ \begin{array}{l} g_{ij} \to \phi^4\, g_{ij} \\ p^{ij} \to \phi^{-4}\, p^{ij} \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \gamma_{ij}[\phi^4 g, \phi^{-4} p] = \gamma_{ij}[g,p] \\ \pi^{ij}[\phi^4 g, \phi^{-4} p] = \pi^{ij}[g,p] \end{array} \right.$$

CMC solutions of GR completely specified by a conformal class of metrics (conformal geometry) and a transverse-traceless tensor + ‘York time’ τ

Shape Dynamics

Reformulate GR as an intrinsically 3-dimensional conformal field theory.

(CMC) time evolution is generated by the conformally-invariant Hamiltonian:

$$\boxed{H_{\mathsf{SD}}[g_{ij}, p^{ij}, \tau] := \int d^3x\, \sqrt{g}\, \Omega^6}$$

where Ω is the solution of the LY equation:

$$\frac{\Omega^{-6}}{\sqrt{g}}\left(p^{ij} - \tfrac{1}{3} g^{ij} \mathrm{tr}\, p\right)^2 - \tfrac{3}{8}\sqrt{g}\, \Omega^6 \tau^2 - \sqrt{g}\left(R\, \Omega^2 - 8\, \Omega \Delta \Omega\right) = 0\,.$$

The above Hamiltonian generates the evolution of

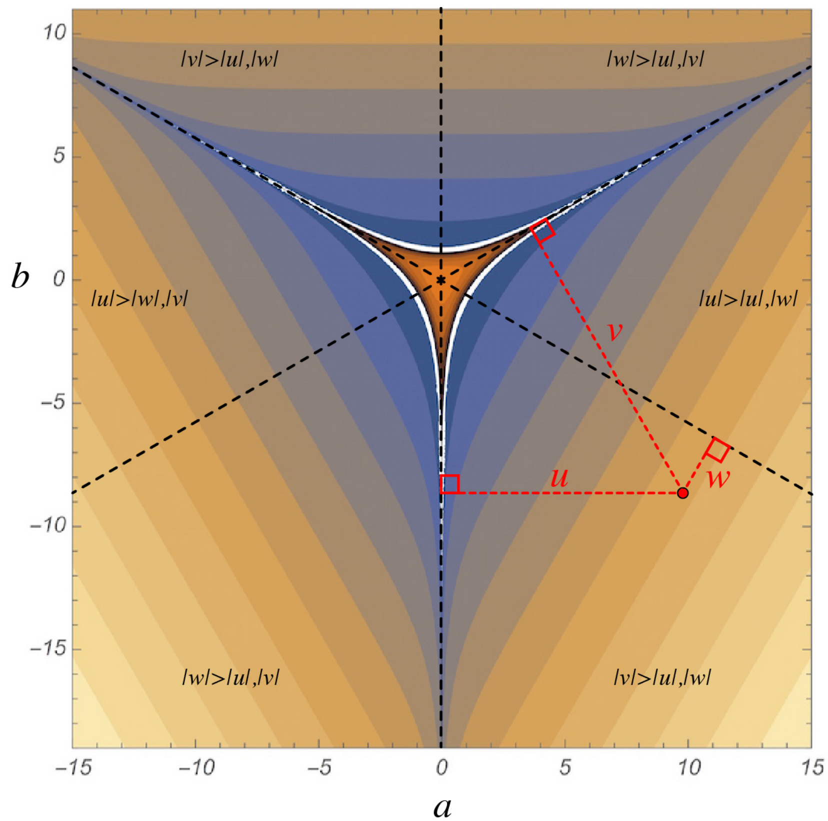
conformally invariant degrees of freedom

$$\frac{\partial g_{ij}}{\partial \tau} = \frac{\delta H_{\mathsf{SD}}}{\delta p^{ij}}\,, \qquad \frac{\partial p^{ij}}{\partial \tau} = -\frac{\delta H_{\mathsf{SD}}}{\delta g_{ij}}\,,$$

which determine everything about the gravitational field.

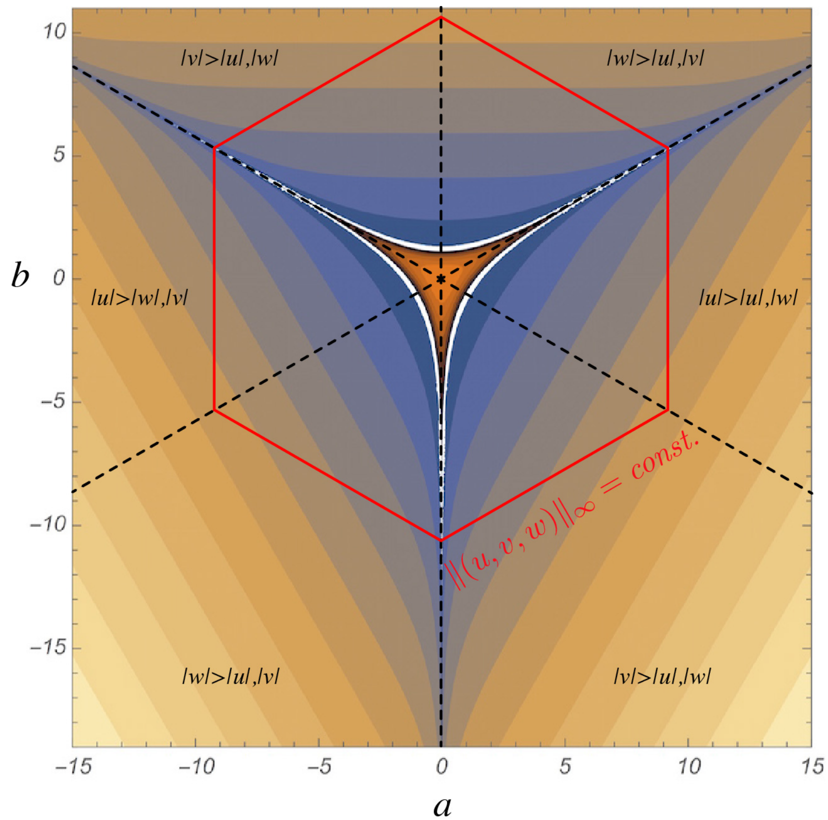
Appendix: nice compactification of shape space

Triangular coordinates in shape space:



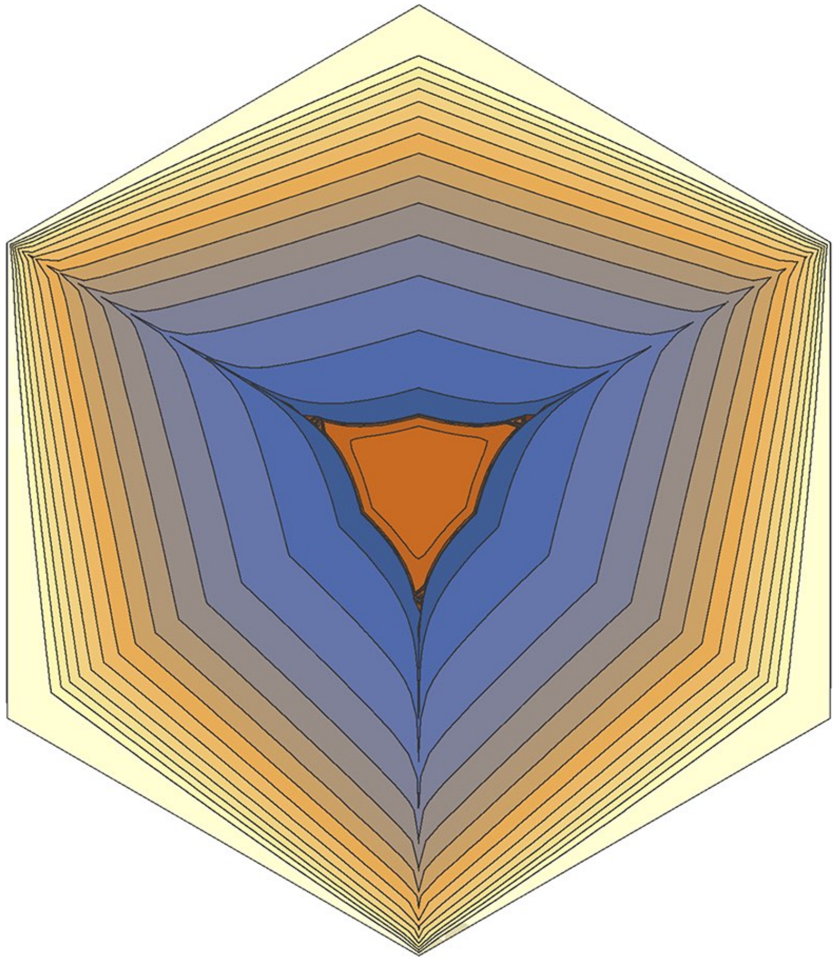
$$u + v + w = 0\,, \qquad V_{\mathsf{S}} = F(w - v) + F(v - u) + F(u - w)\,,$$

Uniform norm:



$$\|(u,v,w)\|_\infty = \max\{|u|,|v|,|w|\}\,,$$

compactification to a hexagon



$$\widehat{u} = \frac{\mathsf{tanh}\,\|(u,v,w)\|_\infty}{\|(u,v,w)\|_\infty}\, u\,, \quad \widehat{v} = \frac{\mathsf{tanh}\,\|(u,v,w)\|_\infty}{\|(u,v,w)\|_\infty}\, v\,, \quad \widehat{w} = \frac{\mathsf{tanh}\,\|(u,v,w)\|_\infty}{\|(u,v,w)\|_\infty}\, w\,,$$

Appendix: quotienting wrt internal rotations

Careful: Riem doesn't know about the orientation of space... for that we need a non-diagonal ansatz $g_{ij} = q_{ab}\,\sigma_i^a\,\sigma_j^b,\;\; p^{ij} = p^{ab}\,\chi_a^i\,\chi_b^j$ then the diffeo constraint does not vanish identically, there is a 3D leftover constraint:

$$V^a = \epsilon^{abc} p^{db} q_{dc} = \tfrac{1}{2}\epsilon^{abc}[p,q]^d{}_c\,,$$

generating infinitesimal $so(3)$ rotations of q_{ab} and p^{ab}.

$V^a \approx 0$ imposes that q_{ab} and p^{ab} commute $\Rightarrow$ simultaneously diagonalizable.

A gauge-fixing of $V^a \approx 0$ is to take q_{ab} (and so p^{ab} also) diagonal.

Appendix: Kasner map

Kasner map (or Bianchi II transition): if w.l.o.g. $P_x > P_y > P_z$,

$$P_x{}^{\mathrm{in}} = P_y + P_z + \sqrt{4P_yP_z - \pi_\phi{}^2}, \qquad P_x{}^{\mathrm{out}} = P_y + P_z - \sqrt{4P_yP_z - \pi_\phi{}^2},$$

while P_y and P_z stay the same. Then:

$$(k_a{}^2 + k_b{}^2)^{\mathrm{in}} - (k_a{}^2 + k_b{}^2)^{\mathrm{out}} = -\tfrac{4}{3}(P_y + P_z)\sqrt{4\,P_yP_z - \pi_\phi{}^2} < 0$$

$$\Downarrow$$

$(k_a{}^2 + k_b{}^2)$ decreases at each bounce until it is comparable with $\pi_\phi{}^2$.