The Epstein–Glaser method in scalar QFT models, with emphasis on the Compton effect

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The beginning of trouble

In almost every book of quantum field theory (QFT) one finds the neat formula for the scattering matrix:

$$\mathbf{S} = \mathrm{T}\exp\left(i \int \mathcal{L}(x)\, d^4x\right),$$

with T the time-ordering operator and $\mathcal{L}$ an interaction Lagrangian (OVD).

However, this expression is not really defined: the (in)famous ultraviolet divergences of QFT originate in this fact.

The virtue of Epstein–Glaser (EG) renormalization is that it deals with the problem by the methods of distribution theory, so that all quantities appearing in the calculations are mathematically well defined.

The pedigree

The idea that a (as much as possible) *rigorously defined* $\mathbf{S}$-matrix can be reached through systematic use of causality goes back to Stückelberg and Rivier. It was retaken by Bogoliubov and coworkers and led to fruition by Epstein and Glaser.

The procedures by Epstein and Glaser, however, have remained somewhat outside the mainstream of physics, despite efforts by Scharf's Zurich school, Stora and more recently Fredenhagen's Hamburg school, with Brunetti and Dütsch as his main collaborators.

Symbolically,

$$\mathbf{S} = 1 + \sum_{1}^{\infty} \frac{i^n}{\hbar^n \, n!} \int_{\mathbb{M}_4} d^4x_1 \cdots \int_{\mathbb{M}_4} d^4x_n \; \mathrm{T}[\mathcal{L}(x_1) \cdots \mathcal{L}(x_n)].$$

The knot of a problem

The trouble is that the OVD do not like to be multiplied by Heaviside functions. For instance, one would put

$$T_2(x_1,x_2) = \Theta(t_1-t_2)\mathcal{L}(x_1)\mathcal{L}(x_2) + \Theta(t_2-t_1)\mathcal{L}(x_2)\mathcal{L}(x_1),$$

where $\Theta(t) = 1$ for $t > 0$ and $\Theta(t) = 0$ for $t < 0$; but already this product is undefined.

The EG method consists of: (1) Making an "adiabatic switching", replacing $\mathcal{L}(x)$ by $g(x)\mathcal{L}(x)$, so $\mathbf{S}$ becomes a functional on the Fock space of the free fields. (2) Postulating a perturbative expansion of $\mathbf{S}[g]$ of the form:

$$\mathbf{S}[g] = \sum_0^\infty \frac{i^n}{\hbar^n\, n!} \int_{\mathbb{M}_4} d^4x_1 \cdots \int_{\mathbb{M}_4} d^4x_n\, T_n(x_1,\dots,x_n) g(x_1)\cdots g(x_n).$$

The T_n are called time-ordered products; $T_0 = 1$. (3) Trying to determine these T_n by some natural conditions.

Causality as a principle

Those postulates are most intuitively formulated in terms of $\mathbf{S}[g]$. Let $\Gamma_{\mp}(x)$ be the closed backward/forward lightcone of the point x. When $x \notin \Gamma_{-}(y)$ we write $x \gtrsim y$; this notation extends to sets of points in the obvious way. Besides unitarity in the sense of formal power series, the foremost condition is causality:

$$\mathbf{S}[g_1+g_2+g_3] = \mathbf{S}[g_1+g_2]\mathbf{S}^{-1}[g_2]\mathbf{S}[g_2+g_3] \quad \text{if} \quad \operatorname{supp} g_1 \gtrsim \operatorname{supp} g_3.$$

When $\operatorname{supp} g_1, \operatorname{supp} g_3$ are compact, the relation $\operatorname{supp} g_1 \gtrsim \operatorname{supp} g_3$ means that a spacelike surface passes between those supports.

The last equation implies the factorization

$$\mathbf{S}[g_1+g_2] = \mathbf{S}[g_1]\mathbf{S}[g_2] \quad \text{if} \quad \operatorname{supp} g_1 \gtrsim \operatorname{supp} g_2.$$

This is Bogoliubov's original formulation of causality.

The time-ordered products

In practice one has to construct the T_n by their own system of postulates, so as to verify those for the $\mathbf{S}$-matrix.

- Induction start: $T_1(x_1) = \mathcal{L}_1(x_1)$.
- Symmetry: the T_n are symmetric in their arguments.
- Causality: if $I \gtrsim I^c$, then $T_n(N) = T_{|I|}(I)T_{n-|I|}(I^c)$. This implies spacelike commutativity:

$$\Big[T_{|I|}(I), T_{n-|I|}(I^c)\Big] = 0 \quad \text{when} \quad I \gtrsim I^c,\ I^c \gtrsim I.$$

- Wick ordering: the formula for multiplication of Wick polynomials remains valid for the T_n. So, in particular, their construction reduces to a problem of extending the vacuum expectation values, which are numerical distributions.
- Scaling: the extensions of those distributions must keep their (generalized) homogeneity degree.

On the technique I

Many discussions of the EG method turn around prescriptions for extending the numerical distributions. We say that a Feynman amplitude f associated to a graph Γ is primitively divergent when f is not locally integrable, but is integrable away from zero. A tempered distribution $\tilde{f} \in \mathcal{S}'(\mathbb{R}^d)$ is an extension (renormalization) of f if $\langle \tilde{f}, \phi \rangle =: \int_{\mathbb{R}^d} f(x)\phi(x)\, d^d x$ holds whenever ϕ belongs to $\mathcal{S}(\mathbb{R}^d \setminus \{\mathbf{0}\})$.

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Let $f(x) = O(|x|^{-a})$ as $x \to \mathbf{0}$, with a an integer, and let $k = a - d \geq 0$. Then $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^d)$. But f can be regarded as a well-defined functional on the space of Schwartz functions vanishing at $\mathbf{0}$ of order $k + 1$.

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Thus the simplest way to get an extension of f appears to be standard Taylor series surgery: to throw away the k-jet of ϕ at the origin, in order to define $\tilde{f}$ by transposition.

On the technique II

Denote the corresponding Taylor remainder by $R_0^k\phi$. By that definition,

$$\langle \tilde{f}, \phi \rangle = \langle f, R_0^k \phi \rangle.$$

Use Lagrange's integral formula for the remainder:

$$R_0^k\phi(x) = (k+1) \sum_{|\beta|=k+1} \frac{x^\beta}{\beta!} \int_0^1 dt\,(1-t)^k\,\partial^\beta\phi(tx),$$

with the usual multi-index notation. By integration by parts, an explicit integral formula for $\tilde{f}$ follows:

$$\tilde{f}(x) = (-)^{k+1}(k+1) \sum_{|\beta|=k+1} \partial^\beta \left[\frac{x^\beta}{\beta!} \int_0^1 dt\, \frac{(1-t)^k}{t^{k+d+1}} f\left(\frac{x}{t}\right) \right]. \qquad (1)$$

The last expression brings home the importance of dilations. Remember: the scaling degree is just a generalized homogeneity degree.

On the technique III

For (1) to work without trouble, the infrared behaviour of f must be good; otherwise, because the remainder is not a test function, we end up with an undefined integral. In the massless theory f is homogeneous with an algebraic singularity, and the infrared behaviour is pretty bad.

On the technique III

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A way to avoid the infrared problem is to weight the Taylor subtraction. Epstein and Glaser introduced weight functions w ("Glaser's cat") with the properties $w(0) = 1$, $w^{(\alpha)}(0) = 0$ for $0 < |\alpha| \leq k$, and projector maps given by

$$W_w\phi(x) := \phi(x) - w(x)\, j_0^k \phi(x).$$

The previous ordinary Taylor surgery corresponds to $w \equiv 1$.

I do prefer the definition:

$$T_w\phi(x) := \phi(x) - j_0^{k-1}(\phi)(x) - w(x) \sum_{|\alpha|=k} \frac{x^\alpha}{\alpha!}\, \phi^{(\alpha)}(0).$$

The *splitting method* of Epstein and Glaser

It will be enough to go to second order in the $\mathbf{S}$-matrix. We have

$$T_2(x_1, x_2) = \Theta(t_1 - t_2) T_1(x_1) T_1(x_2) + \Theta(t_2 - t_1) T_1(x_2) T_1(x_1),$$
$$R_2(x_1, x_2) = \Theta(t_1 - t_2)[T_1(x_1) T_1(x_2) - T_1(x_2) T_1(x_1)],$$
$$A_2(x_1, x_2) = \Theta(t_2 - t_1)[T_1(x_2) T_1(x_1) - T_1(x_1) T_1(x_2)],$$

respectively for the retarded and advanced products —remark that $R_2 = 0$ unless $x_2 \in \Gamma_-(x_1)$ and $A_2 = 0$ unless $x_1 \in \Gamma_-(x_2)$.

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Neither of these is well defined, by the mentioned ultraviolet problem. Note however that there is good definition of

$$D_2(x_1,x_2) := R_2(x_1,x_2) - A_2(x_1,x_2) = [T_1(x_1), T_1(x_2)]$$

Thus, if D_2 satisfactorily *splits* into a retarded and an advanced part —demanding $\operatorname{supp} D_2(x_1,x_2) \subset \Gamma_+(x_2) \cup \Gamma_-(x_2)$— the three problems are solved. For then in particular

$$T_2(x_1,x_2) = R_2(x_1,x_2) + T_1(x_1)T_1(x_2).$$

Going to physics: *scalar* QED

To formulate models of quantum electrodynamics, one thinks of a Lagrangian of the form $A(x)j(x)$, where $A(x)$ stands for the vector potential of the EM field and j is a conserved current.

For a scalar charged field φ, there is the conserved current $j_\mu = \varphi^\dagger \overleftrightarrow{\partial_\mu} \varphi$, that is, we have an interaction of the form

$$\mathcal{L}_{1,\mathrm{int}} = eA^\mu{:}\varphi^\dagger \overleftrightarrow{\partial_\mu} \varphi{:} = T_1.$$

This is rather like spinor QED. Normally people use instead

$$\mathcal{L}_{\mathrm{int}} = e\,A^\mu{:}\varphi^\dagger \overleftrightarrow{\partial_\mu} \varphi{:} + e^2\,{:}\varphi^\dagger \varphi{:}{:}(AA){:},$$

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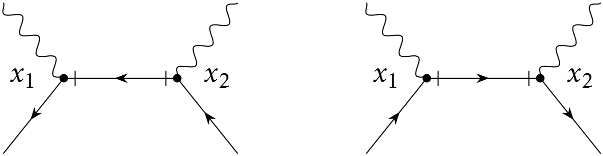
I am going to show that and how the second-order coupling above is *generated* in the process of constructing the time-ordered products!

Going to physics: the basic graphs



These describe the ordinary s-channel and u-channel for Compton scattering. Actually, the second order causal distribution D_2 for Compton scattering in scalar QED sports eight graphs instead of two. The reason: the internal line may contain no derivatives, or one on each side, or two derivatives.

If we think of these graphs as representing D_2, then the internal propagator is the Jordan–Pauli "function" Δ or derivatives of it. Splitting of the terms with at most one derivative of Δ is trivial, and leads to the Feynman propagators $\Delta_{\mathrm{F}} = \Delta_{\mathrm{ret}} - \Delta^{-}$.

The *Schwerpunkt*

The outcome is a splitting of the form

$$D_2(x_1,x_2) = ie^2 :A^\mu(x_1)A^\nu(x_2): \Big[-:\varphi^\dagger(x_2)\varphi(x_1): \partial_\mu\partial_\nu\Delta_{\mathrm{F}}(x_1-x_2) + C_{\mathrm{Compton}}\, g_{\mu\nu}\,\delta(x_1-x_2) - :\varphi^\dagger(x_1)\varphi(x_2): \partial_\mu\partial_\nu\Delta_{\mathrm{F}}(x_1-x_2) - C_{\mathrm{Compton}}\, g_{\mu\nu}\,\delta(x_1-x_2)\Big]$$

plus terms obtained by replacing Δ by Δ_{F} in the initial D_2 expression for terms with fewer than two derivatives. (We admit that charge-conjugation invariance allows us to use the same C_{Compton} in both terms here.)

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$$+ C_{\rm Compton}\, g_{\mu\nu}\,\delta(x_1-x_2) - :\varphi^\dagger(x_1)\varphi(x_2): \partial_\mu\partial_\nu\Delta_{\rm F}(x_1-x_2)$$
$$- C_{\rm Compton}\, g_{\mu\nu}\,\delta(x_1-x_2)\Big]$$

plus terms obtained by replacing Δ by $\Delta_{\rm F}$ in the initial D_2 expression for terms with fewer than two derivatives. (We admit that charge-conjugation invariance allows us to use the same $C_{\rm Compton}$ in both terms here.)

The constant $C_{\rm Compton}$ remains to be determined. It so happens that invariance under $A(x) \mapsto A(x) + d\Lambda(x)$ demands precisely $C_{\rm Compton} = -1$ [Dütsch, Krahe, Scharf 1993]. This guarantees that the divergence of the above expression between square brackets with respect to x_1 and x_2 vanishes on-shell.

Seagulls flying

With $C_{\mathrm{Compton}} = -1$, one finally sees appearing in $\frac{1}{2}T_2(x_1, x_2)$ a term of the form

$$ie^2 A^\mu(x_1) A_\mu(x_1) {:}\varphi^\dagger(x_1)\varphi(x_1){:},$$

giving rise to the **seagull** graph:

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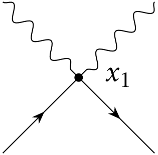

The same argument generates the quartic couplings at second order in non-Abelian Yang–Mills models from the cubic ones. The EG paradigm detects renormalization ambiguities in tree graphs, as well as in loop graphs.

Towards the cross sections I

Now we gear up to compute physical quantities. Recall two-body covariant kinematics: a scattering process $p + q \to p' + q'$, with $p^2 = m_1^2$, $p'^2 = m_3^2$, $q^2 = m_2^2$, $q'^2 = m_4^2$ is described with the help of the Mandelstam invariants:

$$s := (p+q)^2 = (p'+q')^2; \quad t := (p'-p)^2 = (q-q')^2;$$
$$u := (p'-q)^2 = (q'-p)^2.$$

For Compton scattering, it is natural to use the laboratory frame, in which the initial "selectron" is at rest. Thus

$$p = (m, 0), \quad q = (\omega, \boldsymbol{q}), \quad p' = (E', \boldsymbol{p}'), \quad q = (\omega', \boldsymbol{q}'),$$

where $\omega = |\boldsymbol{q}|$ and $\omega' = |\boldsymbol{q}'|$. Here

$$s = m^2 + 2m\omega, \quad u = m^2 - 2m\omega', \quad \text{thus } s - u = 2m(\omega + \omega').$$

Towards the cross sections II

On the other hand,

$$t := (p' - p)^2 = 2m(\omega' - \omega), \quad \text{while}$$
$$t := (q' - q)^2 = -2\omega\omega'(1 - \cos\vartheta) = -4\omega\omega' \sin^2(\vartheta/2),$$

where ϑ is the angle between incident and outgoing photons in the lab frame, giving rise to the famous Compton formula:

$$\frac{\omega'}{\omega} = 1 - \frac{2\omega'}{m} \sin^2(\vartheta/2).$$

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By going to momentum space one obtains the amplitude:

$$\mathbf{S}_{fi} = i(2\pi)^4\delta(p + p' - q - q')\mathbf{T}_{fi} = i(2\pi)^4\delta(P_{fi})\bar{\varepsilon}^\mu(q')M_{\mu\nu}\varepsilon^\nu(q)$$
$$= i(2\pi)^4\delta(P_{fi})\bar{\varepsilon}^\mu(q')(-e^2)\bigg[\frac{(2p+q)_\nu(2p'+q')_\mu}{s - m^2}$$
$$+ \frac{(2p'-q)_\nu(2p-q')_\mu}{u - m^2} - 2g_{\mu\nu}\bigg]\varepsilon^\nu(q).$$

The really dirty trick

Here transversality of photons implies, even off-shell: $M_{\mu\nu}\, q'^{\mu} = M_{\mu\nu}\, q^{\nu} = 0$, whose verification is a simple exercise.

Now, working in the lab frame, the Coulomb gauge adapted to it: $\varepsilon^0 = 0$, $(\varepsilon' q') = 0 = (\varepsilon q)$ imposes itself, with the result that the s-channel and the u-channel poles both vanish. All that remains is the seagull contribution $2e^2(\bar{\varepsilon}'\varepsilon)$!

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Introducing the kinematic factors, the differential cross section for unpolarized photons finally turns out to be

$$\frac{d\sigma^{\text{unpol}}}{d\Omega} = \frac{\omega'^2}{2\omega^2}\,\frac{\alpha^2}{m^2}(1+\cos^2\vartheta) = \frac{\omega'^2}{2\omega^2}\, r_0^2(1+\cos^2\vartheta),$$

where α is the fine structure constant and r_0 is the classical electron radius.