An exactly
solvable
inflationary
model

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Mignemi,
Nicola Pintus

Introduction

Our research

An exactly solvable inflationary model

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Inflationary models are the most successfull attempt to describe the early stages of development of the universe. They are based on the hypothesis that a period of exponential expansion has taken place shortly after the big bang.

Models of gravity minimally coupled to a scalar field with suitable self-interaction potential are usually considered: scalar field starts its evolution from a value that does not minimize the potential (inflation of the scale factor, slow-roll approximation).

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It would be interesting to find exact solutions that describe the transition from an exponential expansion to a later Friedmann-Lemaître behavior in a smooth way.
Skenderis and Townsend showed that cosmological solutions of models of gravity coupled to a scalar can be obtained by analytic continuation to imaginary values of time and radial coordinates of domain-wall solutions of the same model with opposite scalar potential.

The model

I present an application of this observation that leads to an exact solution in which the scale factor expands exponentially at early times and then evolves with a power law. This is based on a model defined by the action

$$I = \int \sqrt{-g} \left[R - 2 \left(\partial \phi \right)^2 - V \left(\phi \right) \right] d^4 x \,,$$

where $V \left(\phi \right) = \dfrac{2 \lambda^2}{3 \gamma} \left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{2\sqrt{3}\phi/\beta} \right)$, $\beta^2 < 1$ e $\gamma = 1 - \beta^2$.
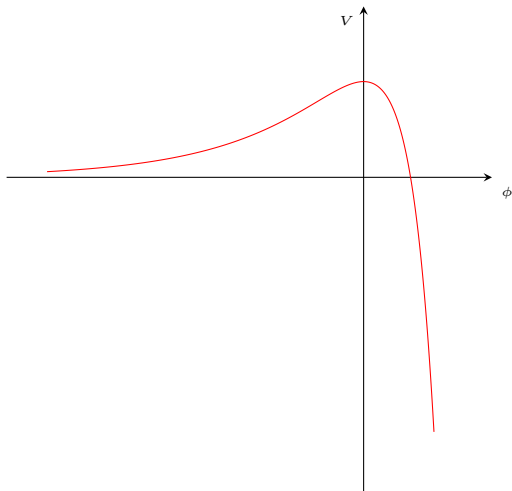
The scalar potential

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The model with $V \leftrightarrow -V$ was considered in the context of AdS/CFT correspondence (Cadoni, Mignemi, Serra): this admits solitonic solutions of the form

$$ds^2 = \hat{R}^{-2/1+3\beta^2} \left(1 + \mu \hat{R}^{-3\gamma/1+3\beta^2}\right)^{2\beta^2/\gamma} d\hat{R}^2 +$$

$$\hat{R}^{2/1+3\beta^2} \left(1 + \mu \hat{R}^{-3\gamma/1+3\beta^2}\right)^{2\beta^2/3\gamma} \left(-d\hat{T}^2 + d\bar{s}_2^2\right)$$

with μ a free parameter. This metric interpolates between AdS for $\hat{R} \to 0$ and domain-wall behavior for $\hat{R} \to \infty$.

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Analitically continuing this solution for $\hat{R} \to iT$, $\hat{T} \to iR$, it is obtained a cosmological solution that behaves as a de Sitter universe for $T \to 0$ and as a Friedmann universe with power-law expansion for $T \to \infty$.

Unfortunately, this is not the most general cosmological solution of the model. Therefore, we have studied the general solutions to understand if it can be describe a viable cosmological model.

We obtain an exactly integrable system if we choose a suitable parametrization of the metric.

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We are interested in the general isotropic and homogeneous cosmological solutions with flat spatial sections, that we parametrize as

$$ds^2 = -e^{2a(\tau)}d\tau^2 + e^{2b(\tau)}d\Omega^2 \,, \quad \phi = \phi(\tau) \,.$$

with τ a time variable. Using this parametrization, the vacuum Einstein equations read

$$3\dot{b}^2 = \dot{\phi}^2 + \frac{V}{2}e^{2a}$$

$$2\ddot{b} + \dot{b}\left(3\dot{b} - 2\dot{a}\right) = -\dot{\phi}^2 + \frac{V}{2}e^{2a}$$

The scalar field obeys the equation

$$\ddot{\phi} + \left(3\dot{b} - \dot{a}\right)\dot{\phi} = -\frac{1}{4}\frac{dV}{d\phi}e^{2a}$$

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In the gauge $b = a/3$, the previous equations become

$$\frac{\dot{a}^2}{3} = \dot{\phi}^2 + \frac{1}{2}Ve^{2a}$$

$$\ddot{a} = \frac{3}{2}Ve^{2a}$$

$$\ddot{\phi} = -\frac{1}{4}\frac{dV}{d\phi}e^{2a}$$

Defining new variables

$$\Psi = a + \sqrt{3}\beta\phi, \qquad \chi = a + \frac{\sqrt{3}}{\beta}\phi.$$

the field equations take the form

$$\ddot{\Psi} = \lambda^2 e^{2\Psi}, \quad \ddot{\chi} = \lambda^2 e^{2\chi}, \quad \dot{\Psi}^2 - \beta^2\dot{\chi}^2 = \lambda^2\left(e^{2\Psi} - \beta^2 e^{2\chi}\right)$$

The first integrals are

$$\dot{\Psi}^2 = \lambda^2 e 2\Psi + Q_1, \quad \dot{\chi}^2 = \lambda^2 e^{2\chi} + Q_2$$

with Q_1 and Q_2 integration constants, which satisfy $Q_1 = \beta^2 Q_2$.

Solutions

If $Q_i = q_i^2 > 0$ $(i = 1, 2)$

$$\lambda^2 e^{2\Psi} = \frac{q_1^2}{\sinh^2 [q_1 (\tau - \tau_1)]} \, , \quad \lambda^2 e^{2\chi} = \frac{q_2^2}{\sinh^2 [q_2 (\tau - \tau_2)]} \, ,$$

with τ_1, τ_2, q_1, q_2 integration constants and $q_1^2 = \beta^2 q_2^2$.
If $Q_i = -q_i^2 < 0$

$$\lambda^2 e^{2\Psi} = \frac{q_1^2}{\sin^2 [q_1 (\tau - \tau_1)]} \, , \quad \lambda^2 e^{2\chi} = \frac{q_2^2}{\sin^2 [q_2 (\tau - \tau_2)]} \, ,$$

If $Q_1 = Q_2 = 0$

$$\lambda^2 e^{2\Psi} = \frac{1}{(\tau - \tau_1)^2} \, , \quad \lambda^2 e^{2\chi} = \frac{1}{(\tau - \tau_2)^2} \, .$$

Solutions

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Defining $q = q_1 = \beta q_2$, the general solutions of the Friedmann equations are:

If $Q_i > 0$,

$$e^{2a} \propto \frac{|\sinh\left[q\left(\tau - \tau_2\right)\right]|^{2\beta^2/\gamma}}{|\sinh\left[\beta q\left(\tau - \tau_1\right)\right]|^{2/\gamma}} \,, \quad e^{2\sqrt{3}\phi/\beta} \propto \left|\frac{\sinh\left[\beta q\left(\tau - \tau_1\right)\right]}{\sinh\left[q\left(\tau - \tau_2\right)\right]}\right|^{2/\gamma} \, ;$$

If $Q_i < 0$,

$$e^{2a} \propto \frac{|\sin\left[q\left(\tau - \tau_2\right)\right]|^{2\beta^2/\gamma}}{|\sin\left[\beta q\left(\tau - \tau_1\right)\right]|^{2/\gamma}} \,, \quad e^{2\sqrt{3}\phi/\beta} \propto \left|\frac{\sin\left[\beta q\left(\tau - \tau_1\right)\right]}{\sin\left[q\left(\tau - \tau_2\right)\right]}\right|^{2/\gamma} \, ;$$
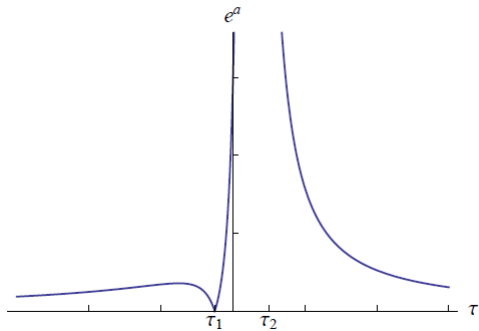
If $Q_i = 0$,

$$e^{2a} \propto \frac{\left|\tau - \tau_2\right|^{2\beta^2/\gamma}}{\left|\tau - \tau_1\right|^{2/\gamma}} \,, \quad e^{2\sqrt{3}\phi/\beta} \propto \left|\frac{\tau - \tau_1}{\tau - \tau_2}\right|^{2/\gamma} \, .$$

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To give a physical interpretation of the solutions, it is useful to define the cosmic time t such that $dt = \pm e^a d\tau$. In this way, we can re-parametrize the line element:

$$ds^2 = -dt^2 + e^{2b(t)} d\Omega^2 .$$

In general, the solutions cannot be written in terms of elementary functions of t. This is only possible when $Q_i = 0$, $\tau_1 = \tau_2$ and therefore

$$\lambda (t - t_0) = \pm \log |\tau - \tau_1|$$

with t_0 an arbitrary integration constant.

Choosing the minus sign, we obtain an expanding universe, with $e^{2b} = e^{2\lambda(t-t_0)/3}$ and $\phi = 0$ (de Sitter spacetime with vanishing scalar field).

In the general case, the solutions have a single acceptable branch if $\tau_1 = \tau_2$, or three qualitatively different if $\tau_1 \neq \tau_2$.
We are only interested in those branches where t is a monotonic function of τ and the universe expands.
Studying their behavior for $\tau \to \tau_{1,2}$ and $\tau \to \pm\infty$, one obtains other physically acceptable solutions:

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- If $Q_i > 0$, $\tau_1 = \tau_2$:

$$e^{2b} \sim t^{2/3}, \, e^{2\sqrt{3}\phi/\beta} \sim t^{2/\beta} \qquad t \to t_0$$

$$e^{2b} \sim e^{2\lambda t/3}, \, e^{2\sqrt{3}\phi/\beta} \sim \text{const} \qquad t \to \infty$$

This solution describes a universe starting at $t = t_0$ with a power-law behavior and presenting an exponential expansion for late times.

- If $Q_i < 0$, $\tau_1 = \tau_2$, we have two branches. The first:

$$e^{2b} \sim e^{2\lambda t/3}, \, e^{2\sqrt{3}\phi/\beta} \sim \text{const} \qquad t \to -\infty$$

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \to 0, \, e^{2\sqrt{3}\phi/\beta} \sim (t - t_0)^{-2} \, t \to t_0$$

The other branch behaves as

$$e^{2b} \sim (t - t_0)^{2\beta^2/3}, \, e^{2\sqrt{3}\phi/\beta} \sim (t - t_0)^{-2}$$

both at $t = t_0$ and at a later finite time, describing an universe that initially expands and then recollapses.

If $\tau_1 \neq \tau_2$, the solutions are more complicated and in general present three branches:

- $Q_i = 0$

$$e^{2b} \sim e^{2\lambda t/3} \, , \, e^{2\sqrt{3}\phi/\beta} \sim \text{const} \qquad t \to -\infty$$

$$e^{2b} \sim t^{2/3\beta^2} \, , \, e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad t \to \infty \, ;$$

another branch behaves as

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \, , \, e^{2\sqrt{3}\phi/\beta} \sim (t - t_0)^{-2} \qquad t \to t_0$$

with

$$e^{2b} \sim t^{2/3\beta^2} \, , \, e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad t \to \infty.$$

In the first case the universe begins with an exponential expansion for $t \to -\infty$ and gradually turns to a power-law behavior for $t \to \infty$. In the second case, for $\beta^2 > 3/2$, we have an initial phase of power-law inflation.

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- If $Q_i > 0$, in the first branch

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \ , \ e^{2\sqrt{3}\phi/\beta} \sim (t - t_0)^{-2} \qquad t \to t_0$$

$$e^{2b} \sim t^{2/3} \ , \ e^{2\sqrt{3}\phi/\beta} \sim t^{2/\beta} \qquad t \to \infty$$

In the second branch:

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \ , \ e^{2\sqrt{3}\phi/\beta} \sim (t - t_0) \qquad t \to t_0$$

$$e^{2b} \sim t^{2/3\beta^2} \ , \ e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad t \to \infty$$

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- If $Q_i < 0$, in the first branch

$$e^{2b} \sim (t - t_0)^{2\beta^2/3} \ , \ e^{2\sqrt{3}\phi/\beta} \sim (t - t_o)^{-2} \qquad t \to t_0$$

$$e^{2b} \sim t^{2/3\beta^2} \ , \ e^{2\sqrt{3}\phi/\beta} \sim t^{-2/\beta^2} \qquad t \to \infty$$

Another branch behaves as

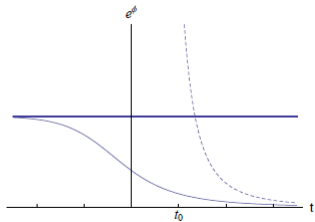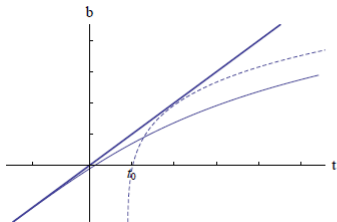$$e^{2b} = (t - t_0)^{2\beta^2/3} \ , \ e^{2\sqrt{3}\phi/\beta} \sim (t - t_0)^{-2}$$

both at $t = t_0$ and at a later finite time.

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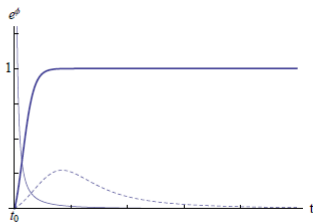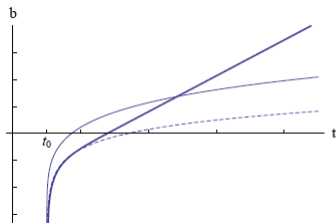Nicola Pintus

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The most interesting solutions are those that behave exponentially for $t \to -\infty$ and as a power-law for $t \to \infty$. These are obtained for $Q_i = 0$ and $\tau_1 \neq \tau_2$.

The exponential expansion lasts until $\tau = \tau_f \sim 1/q\beta$, namely $t - t_0 = t_f - t_0 \sim 1/\lambda$. After this time the acceleration of the expansion becomes negative. At such time the scale factor e^{2b} is of order $\lambda^{-2/3}$.

Denoting t_i the time at which the inflation starts, the scale factor therefore inflates by a factor $e^{-2\lambda(t_i-t_o)/3}$. Choosing $t_i - t_0$ negative, one can then obtain the desired amount of inflation,

Let us consider the effects of ordinary matter on the exact solutions obtained.
We discuss the stability of the solutions by means of methods of the theory of dynamical systems.
Introducing matter in the form of a perfect fluid, Einstein equations become

$$\frac{\dot{a}^2}{3} - \dot{\phi}^2 = \frac{1}{2}Ve^{2a} + \frac{1}{2}\rho e^{2a}$$

$$\ddot{a} = \frac{3}{2}Ve^{2a} + \frac{3}{4}\left(\rho - p\right)e^{2a},$$

while the equation of the scalar field is unchanged.

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Combining those equations, we obtain the continuity equation

$$\dot{\rho} + (3p + \rho)\,\dot{a} = 0$$

which, integrated for the equation of state $p = \omega\rho$ ($\omega \geq -1$), gives

$$\rho = \rho_0 e^{-(1+\omega)a} \, .$$

In terms of the variables Ψ, χ, one has

$$\ddot{\Psi} = \lambda^2 e^{2\Psi} + \frac{3}{4}\rho_0 \left(1 - \omega\right) e^{(1-\omega)\left(\Psi - \beta^2 \chi\right)/\gamma} \, ,$$

$$\ddot{\chi} = \lambda^2 e^{2\chi} + \frac{3}{4}\rho_0 \left(1 - \omega\right) e^{(1-\omega)\left(\Psi - \beta^2 \chi\right)/\gamma}$$

subject to the constraint

$$\dot{\Psi}^2 - \beta^2 \dot{\chi}^2 = \lambda^2 \left(e^{2\Psi} - \beta^2 e^{2\chi}\right) + \frac{3}{2}\rho_0 \gamma e^{(1-\omega)\left(\Psi - \beta^2 \chi\right)/\gamma} \, .$$

Dynamical system

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In general, this system cannot be solved exactly. Therefore, to investigate its properties we put it in the form of a dynamical system, defining

$$X = \frac{1}{2\gamma(1-\omega)}\left(\dot{\psi} - \beta^2\dot{\chi}\right), \ Y = \dot{\chi}, \ Z = \lambda e^{\chi}, \ W = \sqrt{\lambda}e^{(\psi-\beta^2\chi)/2\gamma}$$

The independent equations become

$$\begin{cases} \dot{X} = \frac{\alpha}{2}W^2 - \frac{\beta^2}{2\gamma}Z^2 + \frac{1}{2\gamma}W^{4\gamma(1-\omega)}Z^{2\beta^2} \\ \dot{Y} = Z^2 + \alpha W^2 \\ \dot{Z} = YZ \end{cases}$$

with $\alpha = 3\rho_0/4\lambda$ and W implicitly defined as

$$\frac{2\alpha}{1-\omega}W^2 + \frac{1}{\gamma}Z^{2\beta^2}W^{4\gamma(1-\omega)} = \frac{\beta^2}{\gamma}Z^2 + 4\gamma X^2 + 4\beta^2 XY - \beta^2 Y^2 \ .$$

Stability of the system

The global properties of the solutions of this system can be deduced from the study of their behavior near the critical points of the phase space.

The limit $t \to \pm\infty$ can correspond either to $\tau \to \infty$ or to $\tau \to \tau_0$ for some τ_0 where the funcions Z and W diverge, and hence to critical points at finite distance or at infinity in phase space.

The critical points at finite distance correspond to the limit $\tau \pm \infty$, and lie on two straight lines on the plane $Z = 0$:

$$X_0 = \frac{\pm\beta Y_0}{2\,(1 \pm \beta)}\,, \quad Z_0 = 0\,.$$

The dynamical system can be linearized around its critical points when

$$\frac{1}{2} < \beta^2 < \frac{3 - 4\omega}{4\,(1 - \omega)}$$

and its eigenvalues are $0, 0, Y_0$.

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Taking into account that $e^a = W^2$ and $e^{\sqrt{3}\phi/\beta} = Z/W^2$, one can deduce that the asymptotic behavior of the functions a and ϕ near the critical points with $Y_0 \neq 0$ is

$$a \sim \frac{\pm\beta Y_0}{1 \pm \beta}\tau \sim \log\tau\,, \quad \frac{\sqrt{3}\phi}{\beta} \sim \frac{Y_0}{1 \pm \beta}\tau\,,$$

with $\tau \to \mp\infty$. It follows that the cosmic time t vanishes in this limit and that

$$e^{2b} \sim t^{2/3}\,, \qquad e^{2\sqrt{3}\phi/\beta} \sim t^{2/\beta}\,, \qquad t \to 0\,.$$

If the critical point is the origin:

$$e^a \sim \tau^{-1} \sim e^{\mu t}\,, \quad e^{2b} \sim e^{2\mu t/3} \to 0\,, \quad e^{2\sqrt{3}\phi/\beta} \sim \mathsf{const}\,, t \to \pm\infty\,.$$

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The remaining critical points lie on the surface at infinity of the phase space. They can be studied by defining new variables

$$x = \frac{1}{X} \,,\, y = \frac{Y}{X} \,,\, z = \frac{Z}{X} \,,\, w = \frac{W}{X} \,,\, x \to 0 \,,\, \tau \to \tau_0$$

and the field equations become

$$x' = -\left(\frac{\alpha}{2} w^2 - \frac{\beta^2}{2\gamma} z^2 + \frac{1}{2\gamma} v^2 \right) x$$

$$y' = -\left(\frac{\alpha}{2} w^2 - \frac{\beta^2}{2\gamma} z^2 + \frac{1}{2\gamma} v^2 \right) y + z^2 + \alpha w^2$$

$$z' = -\left(\frac{\alpha}{2} w^2 - \frac{\beta^2}{2\gamma} z^2 + \frac{1}{2\gamma} v^2 \right) z + yz$$

where " ′ " denotes $ud/d\tau$ and $v := z^{\beta^2} w^{2\gamma/(1-\omega)} x^{-\gamma(1+\omega)/(1-\omega)}$. The constraint is

$$\frac{1}{\gamma} v^2 = \frac{\beta^2}{\gamma} z^2 + 4\gamma + 4\beta^2 y - \beta^2 y^2 - \frac{2\alpha}{1-\omega} w^2 \,.$$

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The terms proportional to v^2 in the previous equations must be considered carefully, because the divergence of $x^{-2\gamma(1+\omega)/(1-\omega)}$ can be compensated by zeros of z and w. To have a regular system, these terms must either vanish or go to constant for $u \to \infty$. We find the following critical points ($\eta := \left(\Psi - \beta^2 \chi\right) / \left[2\gamma\left(1 - \omega\right)\right]$):

1. $z = 0$, $y = y_0 = 2\left(1 \pm \beta^{-1}\right)$, with $w = v = 0$ and correspond to the limit $t \to \infty$. Its eigenvalues are $(0, 0, y_0)$. These are the endpoints of the lines containing the critical points at finite distance and are not endpoints of trajectories lying at finite distance;

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② $z = 0$, $y = 2$, with $w^2 = \dfrac{2\,(1 - \omega)}{\alpha}$, $v = 0$, $t \to \infty$ and
eigenvalues $(-1, -1, 1)$. The metric functions near these points
behave as $e^\chi \sim |\tau - \tau_0|^{-2/(1-\omega)}$, $e^\eta \sim |\tau - \tau_0|^{-1/(1-\omega)}$, and
hence $t \sim |\tau - \tau_0|^{-(1+\omega)/(1-\omega)}$, $e^{2b} \sim t^{2/[3(1+\omega)]}$, $e^{2\sqrt{3}\phi/\beta} \sim$
$t^{-2\omega/(1+\omega)}$. This critical point correspond to the limit $t \to \infty$
and it displays the late-time behavior of ordinary cosmological
models, with no scalar coupling ($\lambda = 0$);

③ $z^2 = \dfrac{4\gamma^2}{\beta^4}$, $y = -\dfrac{2\gamma}{\beta^2}$, with $w = v = 0$ and eigenvalues
$\left(\dfrac{2\gamma}{\beta^2}, \dfrac{2\gamma}{\beta^2}, \dfrac{4\gamma}{\beta^2}\right)$. The metric functions near these points behave
as $e^\chi \sim |\tau - \tau_0|^{-1}$, $e^\eta \sim |\tau - \tau_0|^{\beta^2/2\gamma}$, and hence
$t \sim |\tau - \tau_0|^{1/\gamma}$, $e^{2b} \sim t^{2\beta^2/3}$, $e^{2\sqrt{3}\phi/\beta} = t^{-2}$. This point
correspond to the limit $t \to 0$ and hence it can be the origin of
trajectories describing cosmological models.

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4. $z^2 = 4$, $y = 2$, with $w = 0$, $v^2 \to 4$ and eigenvalues $(-2, 2, -4)$. The metric functions near these points behave as $e^\chi \sim |\tau - \tau_0|^{-1}$, $e^\eta \sim |\tau - \tau_0|^{-1/2}$, and hence $t \sim \pm \log |\tau - \tau_0|$, $e^{2b} \sim e^{\pm 2\mu t/3}$, $e^{2\sqrt{3}\phi/\beta} = $ const. This point correspond to $t \to \infty$ and hence to exponential behavior of the metric function.

5. $z = y = 0$, with $w = 0$, $v^2 \to 4\gamma^2$ and eigenvalues $(-2\gamma, -2\gamma, -2\gamma)$. The metric functions near these points behave as $e^\chi \sim$ const, $e^\eta \sim |\tau - \tau_0|^{-1/2\gamma}$, and hence $t \sim |\tau - \tau_0|^{-\beta^2/\gamma}$, $e^{2b} \sim t^{2/3\beta^2}$, $e^{2\sqrt{3}\phi/\beta} = t^{-2/\beta^2}$. This point correspond to possible asymptotic behaviors for $t \to \infty$ and attracts most of the trajectories that do not end at finite distance.

For $\omega = 0$ the solutions $e^\chi \sim |\tau - \tau_0|^{-2/(1-\omega)}$, $e^\eta \sim |\tau - \tau_0|^{-1/(1-\omega)}$ behave asymptotically as $e^{2b} \sim t^{2/3}$, $e^{2\sqrt{3}\phi/\beta} \sim$ const for $t \to \infty$, while for $\omega = 1/3$ they behave as $e^{2b} \sim t^{1/3}$, $e^{2\sqrt{3}\phi/\beta} \sim t^{-1/2}$.

If we consider the potential

$$V(\phi) = -\frac{2\lambda^2}{3\gamma} \left(e^{2\sqrt{3}\beta\phi} - \beta^2 e^{2\sqrt{3}\phi/\beta} \right) ,$$

the properties of the solutions are completely different. However, the field equations are still valid (substituting λ^2 with $-\lambda^2$) and their first integrals become

$$\dot{\Psi}^2 = -\lambda^2 e^{2\Psi} + q_1^2 , \qquad \dot{\chi}^2 = -\lambda^2 e^{2\chi} + q_2^2 ,$$

with positive integration constants satisfying $q_1^2 = \beta^2 q_2^2$. The cosmological solutions are

$$e^{2a} = \frac{q^2}{\lambda^2} \left(\frac{\beta \cosh^{\beta^2} [q(\tau - \tau_2)]}{\cosh [\beta q (\tau - \tau_1)]} \right)^{2/\gamma} , \quad e^{2\sqrt{3}\phi/\beta} = \left(\frac{\cosh [\beta q (\tau - \tau_1)]}{\beta \cosh [q(\tau - \tau_2)]} \right)$$

These solutions are regular everywhere. They represent universes starting with a big bang at $t = 0$ and recontracting after a finite time.

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Thanks for the attention!!