Institut Ruđer Bošković

PEOPLE

MARIE CURIE ACTIONS

NEWFELPRO 2013 2017

Zagreb 4/11/2015

SOME HINTS ON THE RELATION BETWEEN QUANTUM ALGEBRA FORMALISM AND PHENOMENOLOGY OF DEFORMED RELATIVISTIC MODELS

Niccoló Loret

Based on: arXiv:1102.4637, arXiv:1305.5062, arXiv:1404.5093, arXiv:1407.8143

SUMMARY

- HOPF ALGEBRAS AND κ-POINCARÉ FRAMEWORK, WHAT ABOUT PHYSICS ?
- SOME HINTS IN RELATIVE LOCALITY
- INTRODUCING LATESHIFT: PHYSICAL OBSERVATIONS, PHENOMENA AND STUFF...
- LOOKING FOR CURVATURE, LET'S TRY WHITH FINSLER FORMALISM.

HOPF ALGEBRAS

VECTORIAL SPACE $\mathcal{A}$ ON A FIELD F, WHICH IS SIMOULTANEOUSLY AN ALGEBRA AND A COALGEBRA

ALGEBRAIC SECTOR

$$\begin{array}{ll} Product & \mathcal{M}:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A} \\ Unity & \eta: F\to\mathcal{A}, \end{array}$$

COALGEBRAIC SECTOR

$$\begin{array}{ll} Coproduct & \Delta:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A} \\ Counity & \epsilon:\mathcal{A}\to F. \end{array}$$

ANTIPODE

$$\mathcal{S}:\mathcal{A}\to\mathcal{A}$$

κ-MINKOWSKI

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} + i\zeta^\alpha_{\mu\nu} x_\alpha$$

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κ-MINKOWSKI NONCOMMUTATIVE SPACETIME

$$[x_i, x_0] = i\ell x_i \ , \quad [x_i, x_j] = 0$$

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WE LOOK FOR ITS SYMMETRIES

$$[\mathcal{S} \triangleright x_i, \mathcal{S} \triangleright x_0] = i\ell\, \mathcal{S} \triangleright x_i$$

AND WE FIND 10 GENERATORS:

$$P_\mu \qquad R_i \qquad N_j$$

κ-POINCARÉ BICROSSPRODUCT BASIS

ALGEBRAIC SECTOR CHARACTERIZED BY:

$$[N_j, P_0] = iP_j \,, \quad [N_j, P_k] = i\delta_{jk}\left(\frac{1-e^{-2\ell P_0}}{2\ell} + \frac{\ell}{2}|P|^2\right) - i\ell P_j P_k$$

$$[R_j, N_k] = i\epsilon_{jkl} N_l \,, \quad [N_j, N_k] = i\epsilon_{jkl} R_l \,.$$

COALGEBRAIC SECTOR CHARACTERIZED BY:

$$\Delta P_0^r = P_0^r \otimes \mathbb{1} + \mathbb{1} \otimes P_0^r$$

$$\Delta P_j^r = P_j^r \otimes \mathbb{1} + e^{-\ell P_0^r} \otimes P_j^r \,,$$

$$\Delta(N_i^r) = N_i^r \otimes \mathbb{1} + e^{-\ell P_0} \otimes N_i^r + \ell\epsilon_{ikl} P_k^r \otimes R_l$$

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???

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COMPOSITION OF MOMENTA

BAKER-CAMPBELL-HAUSDORF FORMULA

$$e^X e^Y = e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]-\frac{1}{12}[Y,[X,Y]]}$$

WAVES' COMPOSITION IN κ-POINCARÉ:

$$e^{ik_\alpha \hat{x}^\alpha} e^{iq_\beta \hat{x}^\beta} = e^{i(\vec{k}+\vec{q}e^{-\ell k_0})\vec{x}-i(k_0+q_0)x^0}$$

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$$\downarrow$$

$$\boxed{\begin{aligned} E_k \oplus E_q &= E_k + E_q \\ (k \oplus q)_i &= k_i + q_i e^{-\ell k_0} \end{aligned}}$$

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WAIT, WHAT ??

DON'T PANIC...

WE CAN TRY TO LOOK FOR GUIDANCE IN WHAT WE KNOW, FOR EXAMPLE DESITTER ALGEBRA, SINCE

$$[\Pi_i, \Pi_0] = H\, \Pi_i$$

IN WHICH

$$\Pi_i = p_i \ , \ \ \Pi_0 = p_0 - H\, x^i p_i$$

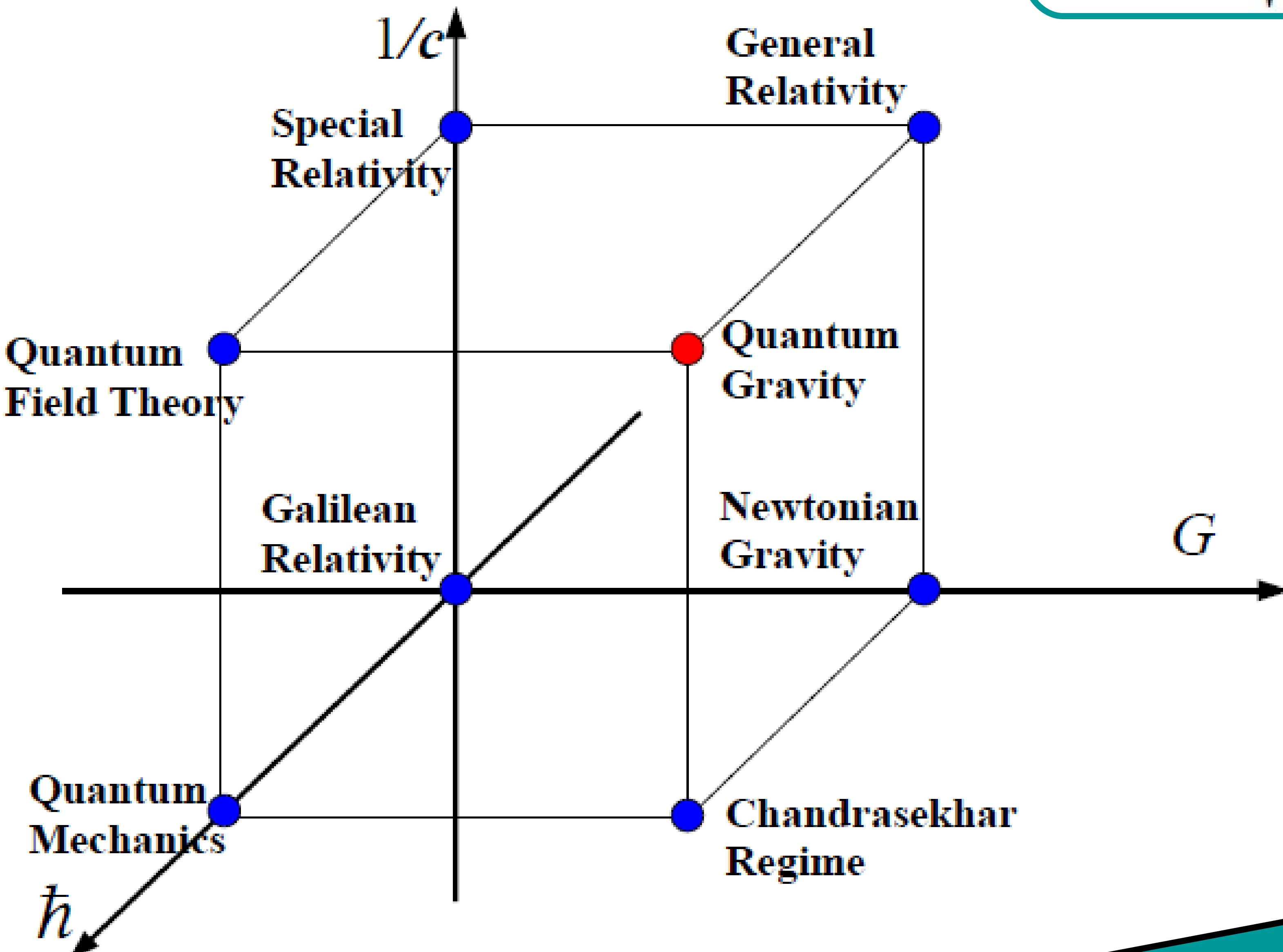
DESITTER ALGEBRA DEPENDS ON ONLY ONE PARAMETER, HOW MANY PARAMETERS HAVE WE GOT ?

THE B-Z-O CUBE

$$E_P \sim \sqrt{\hbar c^5 / G}$$

$$M_P \sim \sqrt{c\hbar / G}$$

$$L_P \sim \sqrt{\frac{\hbar G}{c^3}}$$



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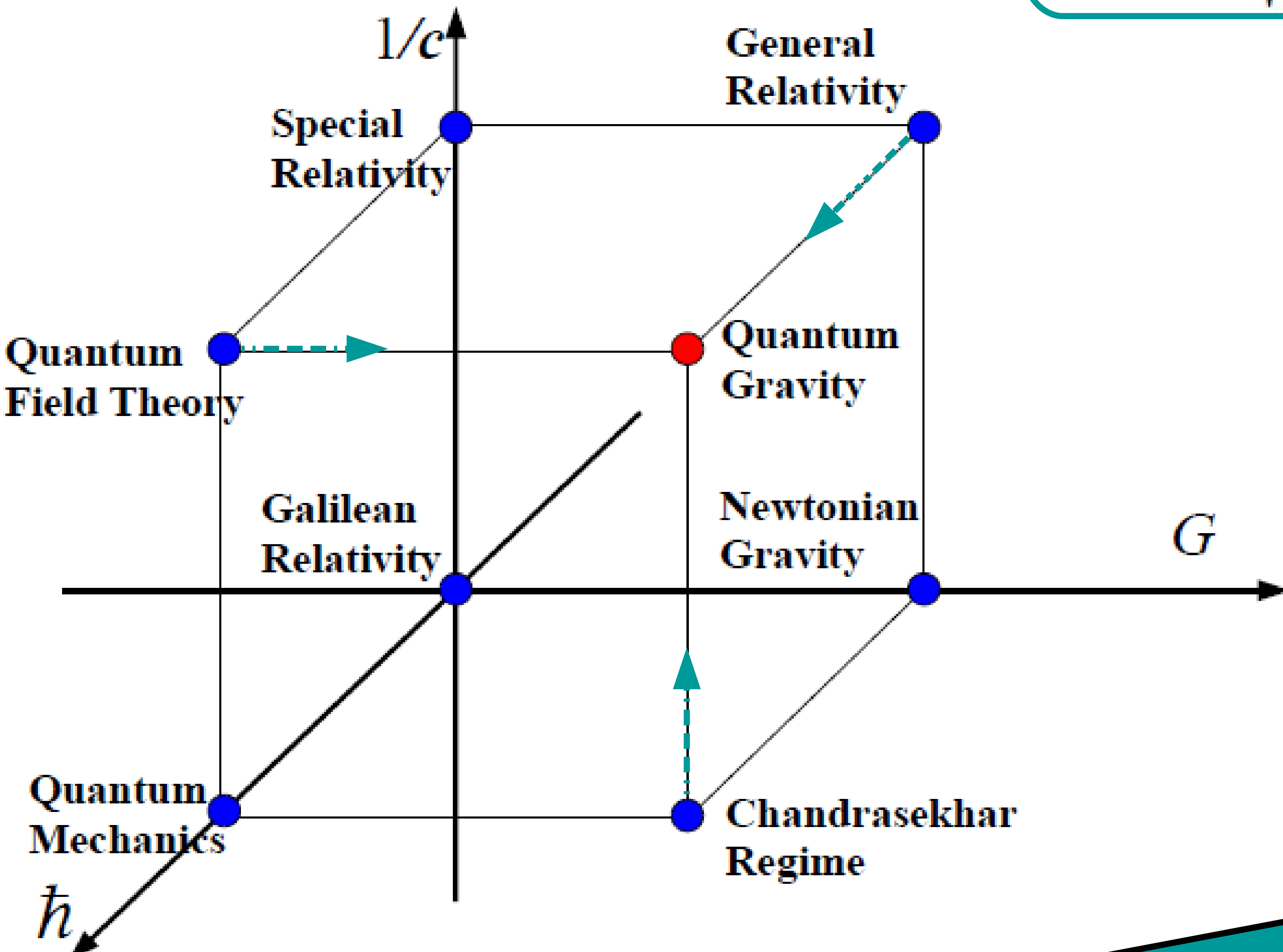
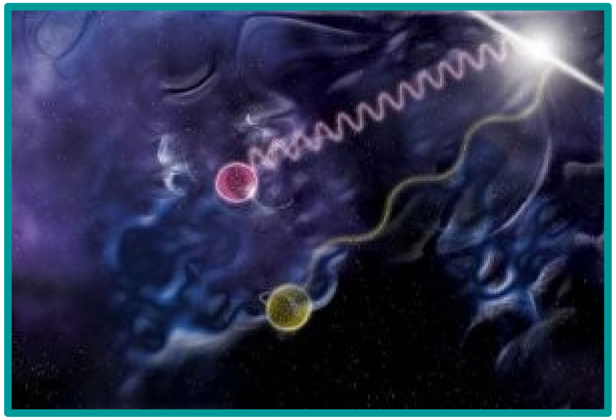


PHENOMENOLOGICAL REALM

IT IS VERY HARD TO TEST QUANTUM GRAVITY EFFECTS ON A GROUND-BASED EXPERIMENT

WHAT WE NEED IS SOME SORT OF MAGNIFICATION OF THE EFFECTS:

- A LOT OF INTERACTIONS
- A LARGE NUMBER OF PARTICLES
- HUGE DISTANCES



PARTICLES' WORDLINES IN FLAT SPACETIME, INSTEAD OF WAVES PROPAGATING IN A CURVED SPACETIME

THE RELATIVE LOCALITY LIMIT

THEN BASICALLY:

$$\boxed{c = 1\ ,\quad G \to 0\ ,\quad \hbar \to 0}$$

$$L_P \sim \sqrt{\frac{\hbar G}{c^3}} \to 0 \qquad \Rightarrow \qquad [x^i, x^0] = 0$$

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IS IT QUANTUM GRAVITY? PROBABLY NOT, HOWEVER:

$$M_P \sim \sqrt{c\hbar/G} \neq 0 \qquad\qquad [A, B] = i\hbar\{A, B\}$$

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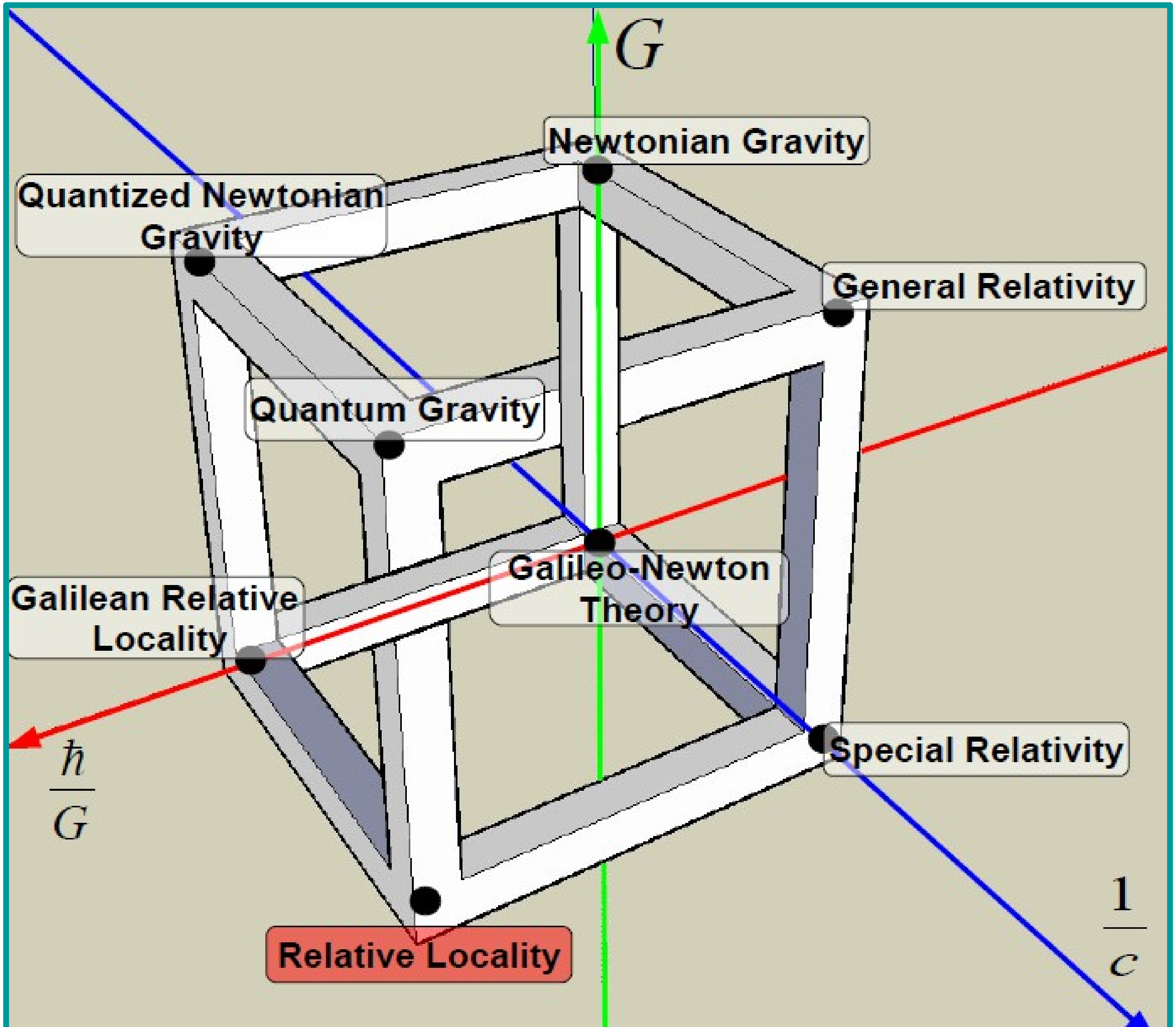
ET VOILÁ !!

$$\boxed{\{\chi^i, \chi^0\} = \ell \chi^i \qquad \ell \sim 1/M_P}$$

OUR CUBE...



G
Newtonian Gravity
Quantized Newtonian Gravity
General Relativity
Quantum Gravity
Galileo-Newton Theory
Galilean Relative Locality
Special Relativity
Relative Locality
$\frac{\hbar}{G}$
$\frac{1}{c}$

QUANTUM GROUP REMNANTS

WHAT REMAINS OF κ-POINCARÉ IN THIS FRAMEWORK IS:

$$\{p_0, p_i\} = 0\,, \quad \{p_i, p_j\} = 0\,, \quad \{\mathcal{N}_{(i)}, \mathcal{R}\} = \epsilon_{ij}\mathcal{N}_{(j)}\,, \quad \{\mathcal{N}_{(i)}, \mathcal{N}_{(j)}\} = \epsilon_{ij}\mathcal{R}\,,$$

$$\{\mathcal{N}_{(i)}, p_0\} = -p_i\,, \quad \{\mathcal{N}_{(i)}, p_j\} = -\delta^i_j \left(\frac{1-e^{-2\ell p_0}}{2\ell} + \frac{\ell}{2}p^2\right) + \ell p_i p_j\,,$$

THE INVARIANT ELEMENT OF THE ALGEBRA IS

$$\mathcal{C}_\ell = \left(\frac{2}{\ell}\sinh\left(\frac{\ell p_0}{2}\right)\right)^2 - e^{\ell p_0} p^2$$

Gubitosi, Mercati, CQG 2013

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$$\Big\downarrow \; m \to 0$$

$$\boxed{p\ (p_0) = \frac{1 - e^{-\ell p_0}}{\ell}}$$

Gubitosi, Mercati, CQG 2013

MOMENTUM-SPACE CURVATURE

$$dk^2 = (dp_0)^2 - e^{2\ell p_0}(dp_1)^2$$

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MOMENTUM-SPACE METRIC

$$\zeta^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\ell p_0} \end{pmatrix}$$

MOMENTUM-SPACE CURVATURE

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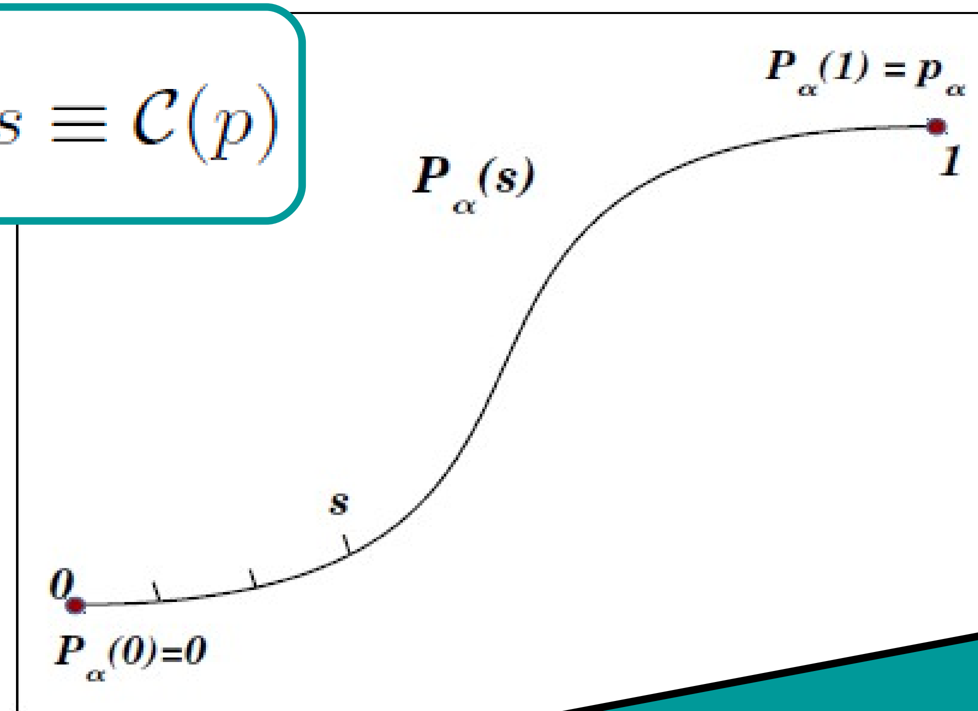
MOMENTUM-SPACE METRIC

$$\zeta^{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & -e^{2\ell p_0} \end{pmatrix}$$

$$\boxed{m^2 = \int_0^1 \zeta^{\alpha\beta}(P)\dot{P}_\alpha \dot{P}_\beta \, ds \equiv \mathcal{C}(p)}$$



COSMOLOGICAL EVENT HORIZON

CONFORMAL COORDINATES

$$r = e^{Ht}x \,, \quad \tau = t - \frac{1}{2H}\ln\left(x^2 e^{2Ht} - \frac{1}{H^2}\right)$$

$$\boxed{ds^2 = (1 - H^2 r^2)d\tau^2 - \frac{1}{1 - H^2 r^2}dr^2}$$

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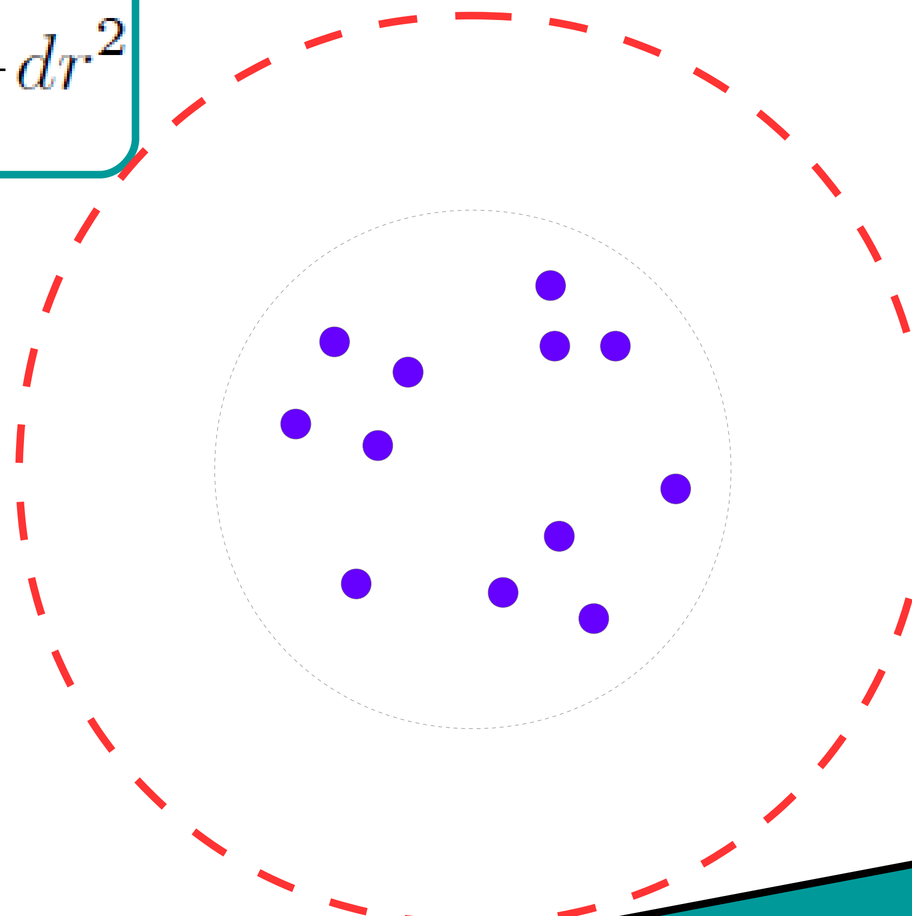
- I NEVER SEE OBJECTS REACH THE HORIZON



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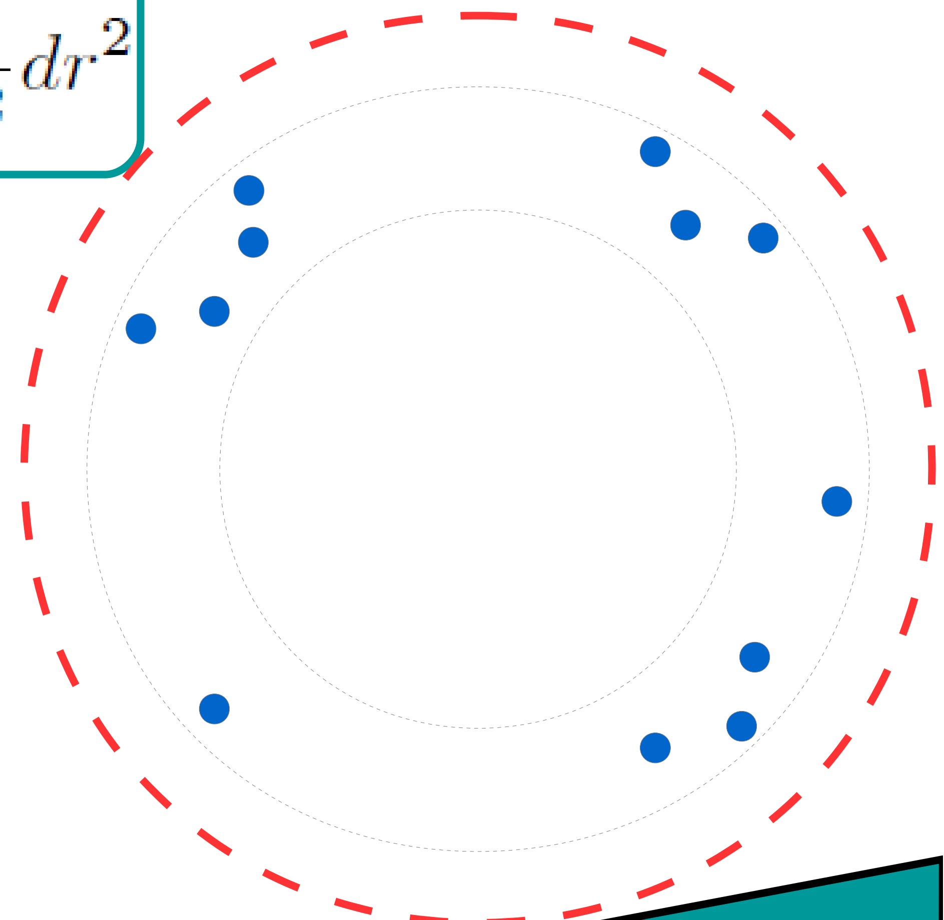

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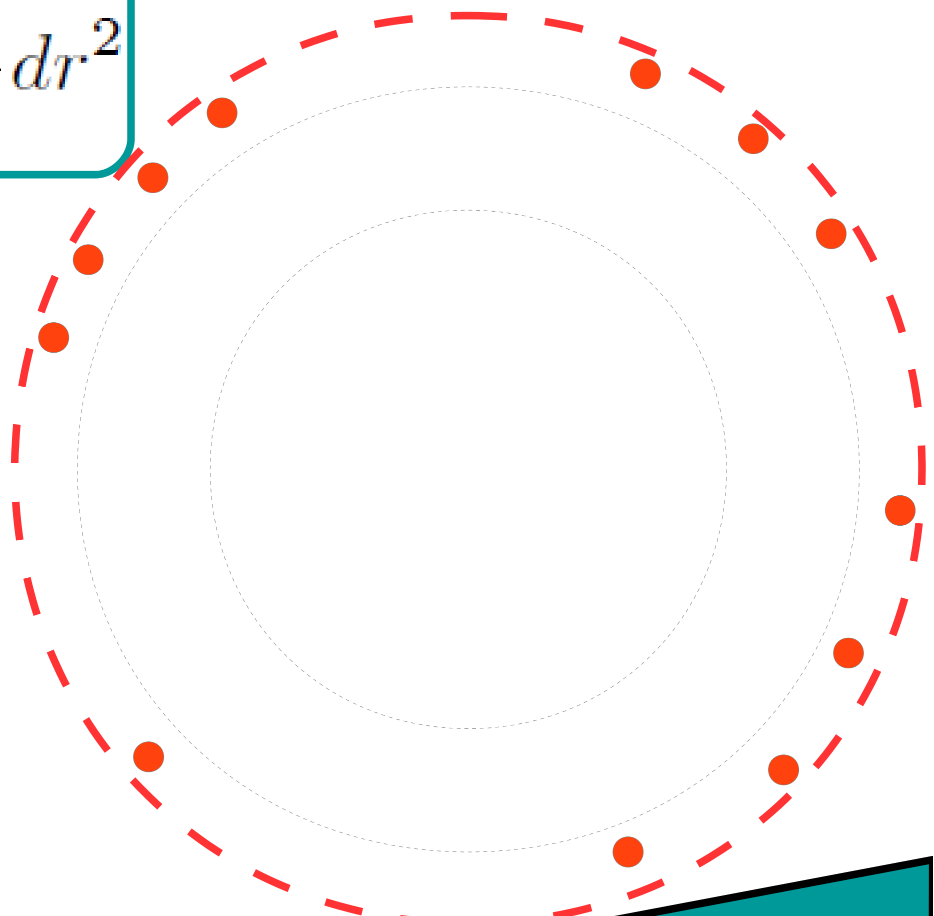
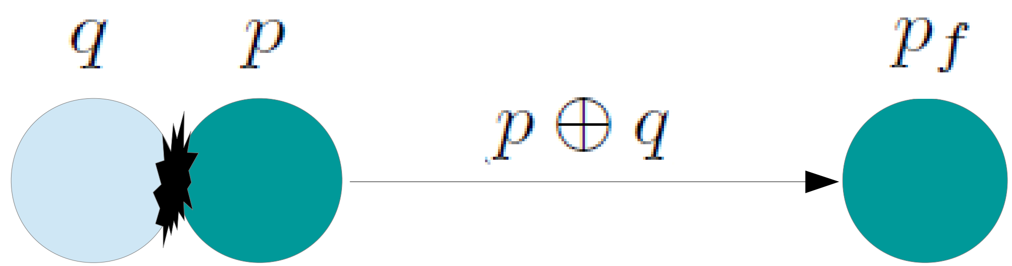


- I NEVER SEE OBJECTS REACH THE HORIZON
- WHILE TIME PASSESS
- STEPS BECOME MORE AND MORE TINY

HORIZON IN MOMENTUM-SPACE



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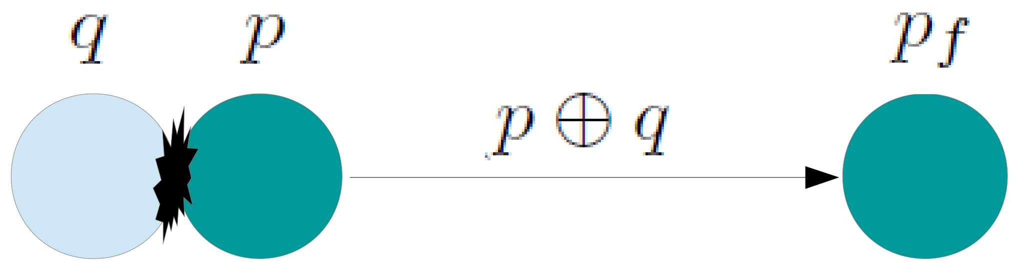
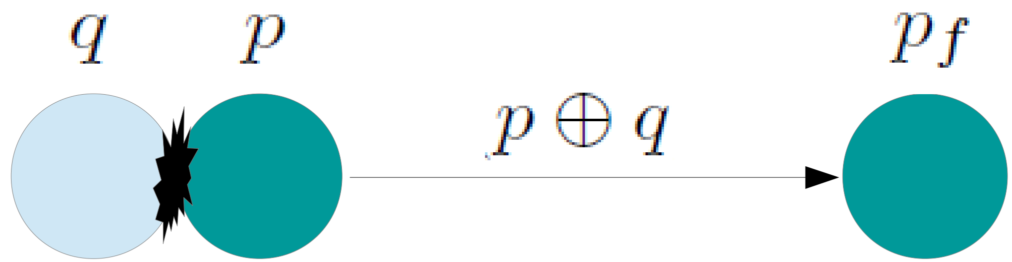


$$p_f = p + qe^{-\ell p_0} + qe^{-\ell(p_0+q_0)} + \ldots = p + qe^{-\ell p_0} \sum_{n=0}^{N-1} e^{-n\ell q_0}$$

HORIZON IN MOMENTUM-SPACE



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USING κ-POINCARÉ MASSLESS PARTICLES MODIFIED DISPERSION RELATION

$$p_f = p + qe^{-\ell p_0} \frac{1 - e^{-N\ell q_0}}{1 - e^{-\ell q_0}} = p + (1 - \ell p) \frac{1 - e^{-N\ell q_0}}{\ell} \xrightarrow[N\to\infty]{} \frac{1}{\ell}$$

$$\boxed{p_f \sim M_P}$$

LATESHIFT

THE CASIMIR GENERATES THE EVOLUTION OVER τ

$$\frac{d\chi^\mu}{d\tau} \equiv \dot{\chi}^\mu = \{\mathcal{C}_\ell, \chi^\mu\}$$

$$\dot{x}^0 = \{\mathcal{C}_\ell, x^0\} = \frac{1}{\ell}\left(e^{\ell p_0} - e^{-\ell p_0}\right) - \ell p_1^2 e^{\ell p_0}$$

$$\dot{x}^1 = \{\mathcal{C}_\ell, x^1\} = 2\, p_1\, e^{\ell p 0}.$$

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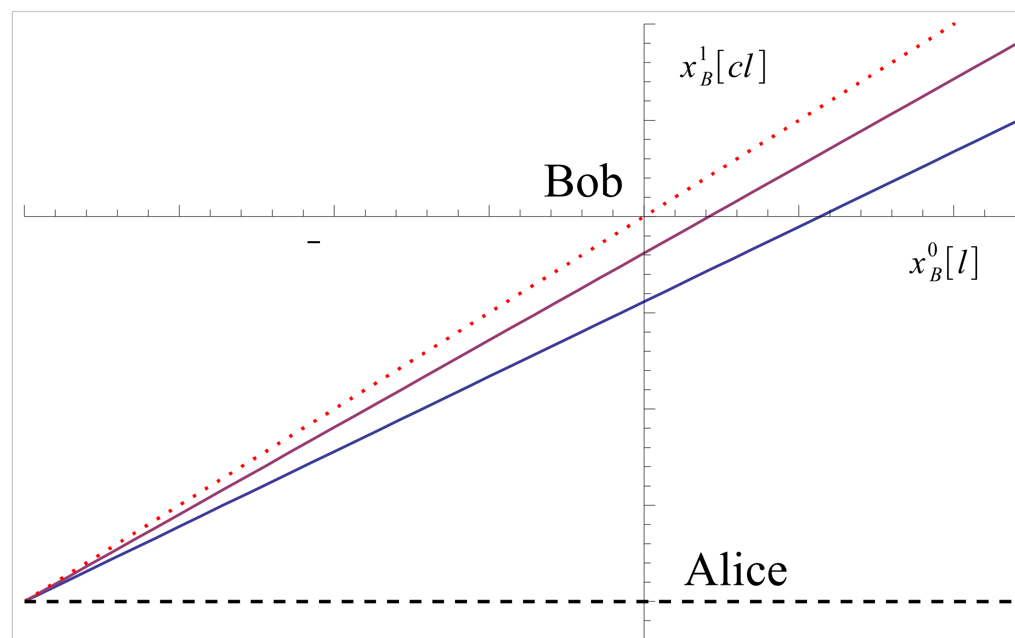
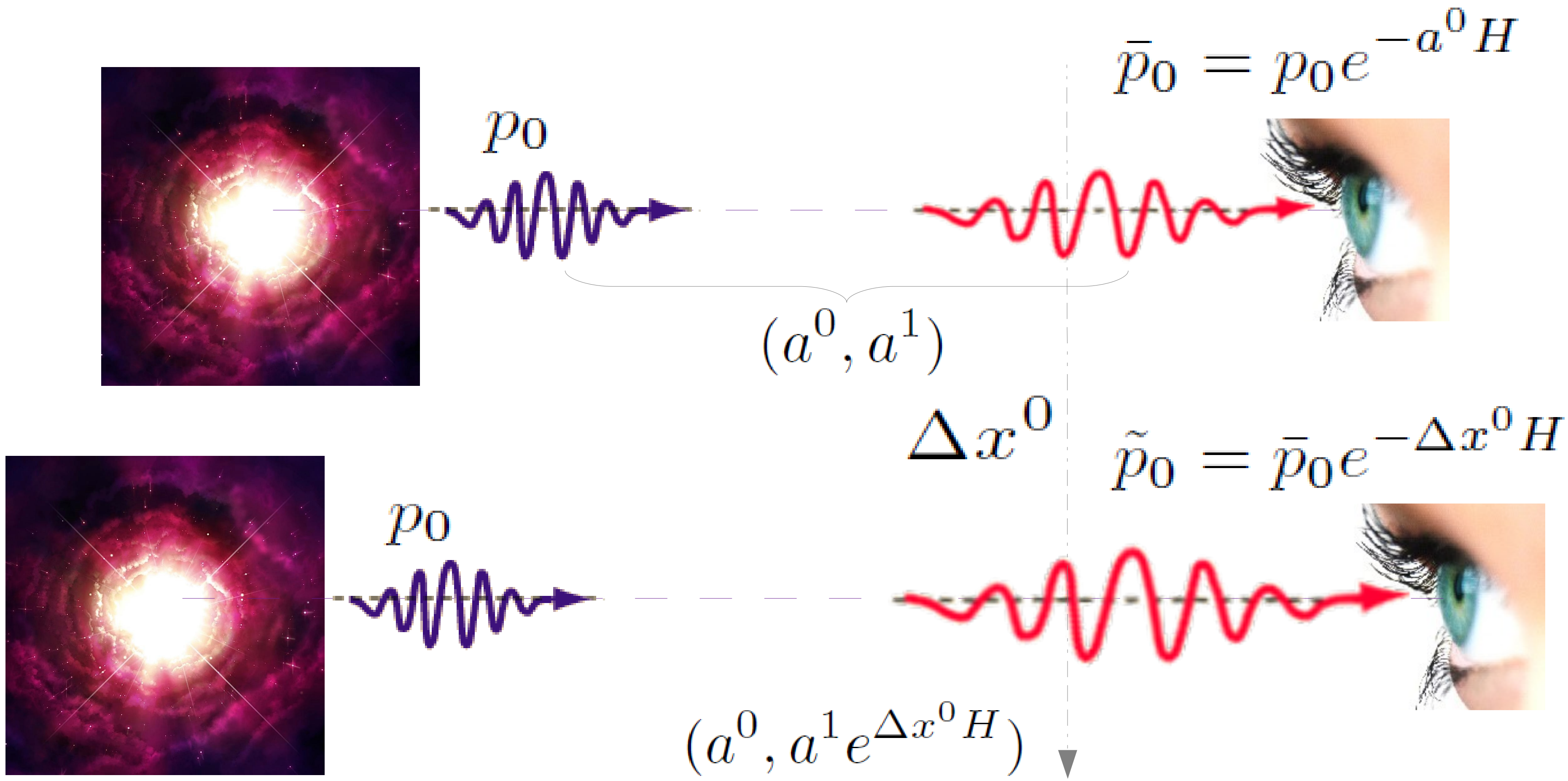
MOMENTUM-DEPENDENT VELOCITY !!

$$\boxed{x^1 - \bar{x}^1 = \quad e^{\ell p 0}\left(x^0 - \bar{x}^0\right)}$$



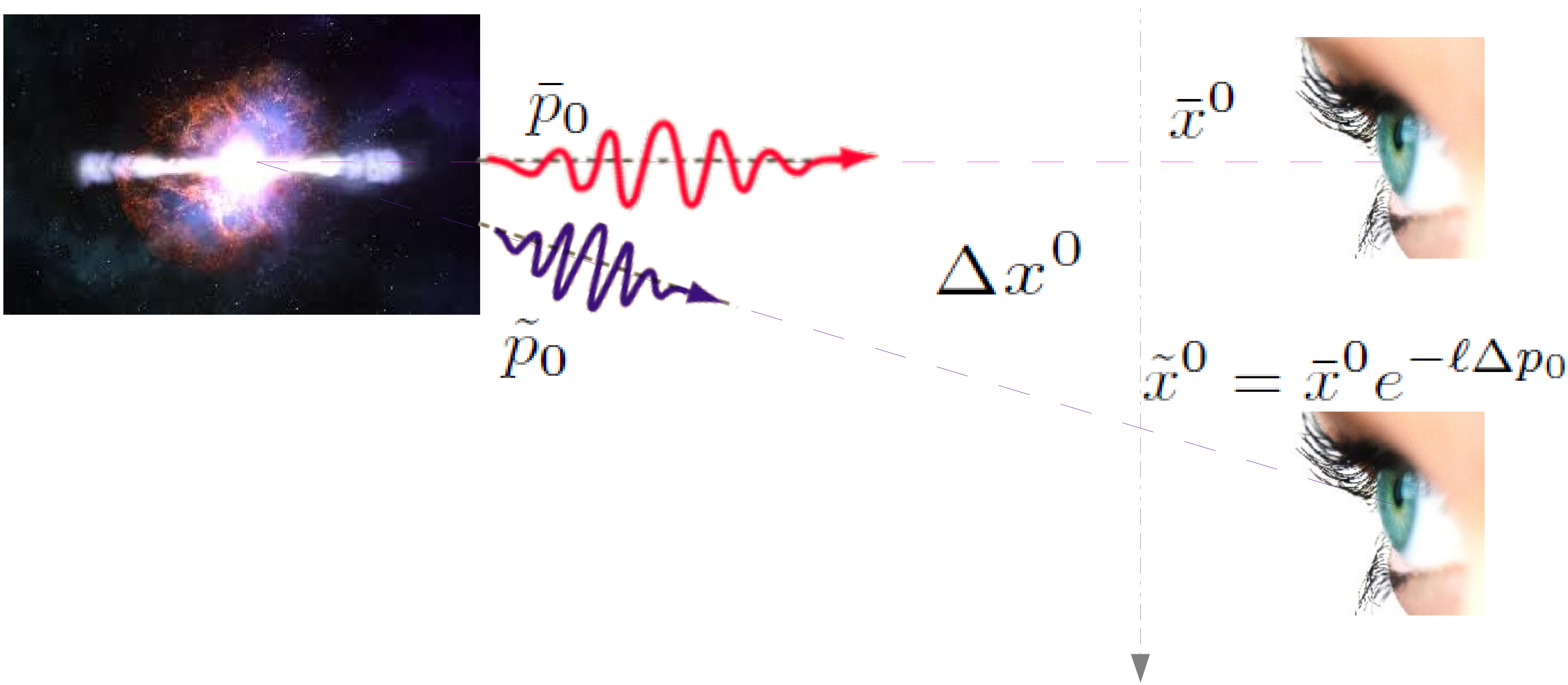
SUMMARISING REDSHIFT



$$\boxed{\frac{\Delta p_0}{\bar{p}_0} = 1 - e^{-\Delta x^0 H}}$$

SUMMARISING LATESHIFT

$$x^1 = x^0 e^{\ell p_0}$$



$$\boxed{\frac{\Delta x^0}{\bar{x}^0} = 1 - e^{-\ell \Delta p_0}}$$

WE STILL NEED CURVATURE

RAINBOW METRICS

$$ds^2 = -\frac{\tilde{F}(\tilde{r})}{f^2(E)}d\tilde{t}^2 + \frac{\tilde{H}(\tilde{r})}{g^2(E)}d\tilde{r}^2 + \frac{\tilde{r}^2}{g^2(E)}d\Omega^2$$

- EURISTIC APPROACH
- ONLY LIV !!

Magueijo, Smolin, CQG 2004

MOMENTUM-DEPENDENT TETRADS

$$E_a^\alpha(p(\tau))$$

- NEEDS FURTHER EXPLORATIONS

Cianfrani, Kowalski-Glikman, Rosati, PRD 2014

SLICING APPROACH

$$x^{B_N}\left(t^{B_N}\right)_n = x_{O_A}^{B_N} + \sum_{k=1}^{n-1}\int_{t_{O_{k-1}}^{B_N}}^{t_{O_k}^{B_N}} dt\, v_k^{B_N} + \int_{t_{O_{n-1}}^{B_N}}^{t^{B_N}} dt\, v_n^{B_N}$$

- VERY COMPLEX

Amelino-Camelia, Rosati, Marcianó, Matassa, arXiv:1507.02056

WHAT'S FINSLER GEOMETRY ?

FINSLER NORM

$$F(x, \dot{x}) = \sqrt{g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}$$

$$x \in M$$
$$\dot{x} \in T_x M$$

- POSITIVE FUNCTION IN THE TANGENT BUNDLE
- HOMOGENEOUS OF DEGREE ONE IN $\dot{x}$

$$\begin{cases} F(\dot{x}) \neq 0 \quad \text{if} \quad \dot{x} \neq 0 \\ F(\epsilon \dot{x}) = |\epsilon| F(\dot{x}) \end{cases}$$

VELOCITY-DEPENDENT GENERALIZATION OF RIEMANNIAN METRIC

$$g_{\mu\nu}(x, \dot{x}) = \frac{1}{2} \frac{\partial^2 F^2(x, \dot{x})}{\partial \dot{x}^\mu \partial \dot{x}^\nu}$$

Rund 1959

FINSLER AND DSR

IF $\mathcal{M}(p)$ IS A MODIFIED DISPERSION RELATION

$$I = \int \left(\dot{x}^\mu p_\mu - \lambda \left(\mathcal{M}(p) - m^2\right)\right) d\tau$$

Girelli, Liberati, Sindoni, PRD 2007

FINSLER AND DSR

IF $\mathcal{M}(p)$ IS A MODIFIED DISPERSION RELATION

$$I = \int \left(\dot{x}^\mu p_\mu - \lambda\left(\mathcal{M}(p) - m^2\right)\right) d\tau$$

$$\downarrow$$

$$\dot{x}^\mu = \lambda \frac{\partial \mathcal{M}}{\partial p_\mu} \quad \Rightarrow \quad \boxed{p_\mu(\dot{x}, \lambda)}$$

$$\downarrow$$

$$I = \int \mathcal{L}(\dot{x}, \lambda(\dot{x})) d\tau \qquad \mathcal{L}(\dot{x}, \lambda(\dot{x})) \equiv mF(\dot{x})$$

Girelli, Liberati, Sindoni, PRD 2007

FINSLER AND DSR

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$$\downarrow$$

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$$\downarrow$$

$$I = \int \mathcal{L}(\dot{x}, \lambda(\dot{x})) d\tau \qquad \mathcal{L}(\dot{x}, \lambda(\dot{x})) \equiv mF(\dot{x})$$

$$\downarrow$$

$$\boxed{I = m \int F(\dot{x}) d\tau = m \int \sqrt{g_{\mu\nu}(\dot{x}) \dot{x}^\mu \dot{x}^\nu}}$$

Girelli, Liberati, Sindoni, PRD 2007

κ-POINCARÉ-INSPIRED FINSLER

WHITH κ-POINCARÉ DISPERSION RELATION

$$m^2 = \mathcal{C}_\ell(p)$$

$$F(\dot{x}) = \left(\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2} + \frac{\ell}{2} m \frac{\dot{x}^0 (\dot{x}^1)^2}{(\dot{x}^0)^2 - (\dot{x}^1)^2} \right)$$

κ-POINCARÉ-INSPIRED FINSLER

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FROM THE ON-SHELL RELATION WE OBTAIN THE LIGHT-CONE STRUCTURE

$$F(\dot{x}) = 1 - \ell \frac{m \dot{x}^0 (\dot{x}^1)^2}{\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2}^{\,3}}$$

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FROM THE ON-SHELL RELATION WE OBTAIN THE LIGHT-CONE STRUCTURE

$$F(\dot{x}) = 1 - \ell \frac{m \dot{x}^0 (\dot{x}^1)^2}{\sqrt{(\dot{x}^0)^2 - (\dot{x}^1)^2}}$$

$$x^1 - \bar{x}^1 = \frac{\sqrt{p_0^2 - m^2}}{p_0} \left(1 + \ell \frac{(2p_0^2 - m^2)}{2p_0} \right) (x^0 - \bar{x}^0)$$

INVARIANT LINE-ELEMENT

- THE ON-SHELL RELATION CAN ALSO BE EXPRESSED AS

$$\zeta_{\alpha\beta}(\dot{x})\dot{x}^{\alpha}\dot{x}^{\beta} = 1$$

- AND THE LIGHT-CONE STRUCTURE

$$\mathcal{C}(p) = m^2\zeta_{\alpha\beta}(\dot{x})\dot{x}^{\alpha}\dot{x}^{\beta}$$

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INVARIANT MOMENTUM-DEPENDENT LINE-ELEMENT

$$\boxed{ds^2 = \zeta_{\mu\nu}(p)dx^{\mu}dx^{\nu}}$$

THIS WOULD ALLOW US TO SATISFY THE CONTRACTION REQUIRMENT

$$\boxed{\zeta_{\mu\alpha}\zeta^{\beta\mu} = \delta^{\beta}_{\alpha}}$$

CONCLUSIONS

- RELATIVE LOCALITY IS A RELIABLE AND COHERENT FORMALISM TO DEVELOP A PHENOMENOLOGY OF NON-COMMUTATIVE GEOMETRIES
- WE ARE WORKING ON ITS GENERALIZATION IN SCENARIOS WITH SPACETIME CURVATURE (USING FINSLER GEOMETRY)
- TO BE DEVELOPPED IS ALSO A PHENOMENOLOGY OF PLANCK-SCALE EFFECTS FOR COSMOLOGY (RELATIVE LOCALITY IN FRW)
- MANY OPEN ISSUES, FROM STATISTICAL MECHANICS TO A GENERALIZED EINSTEIN EQUATION