# Geometric origin of scaling in large traffic networks

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Large scale traffic networks are an indispensable part of contemporary human mobility and international trade. Networks of airport travel or cargo ships movements are invaluable for the understanding of human mobility patterns[1], epidemic spreading[2], global trade[3] and spread of invasive species[4]. Universal features of such networks are necessary ingredients of their description and can point to important mechanisms of their formation. Different studies[5] point to the universal character of some of the exponents measured in such networks. Here we show that exponents which relate i) the strength of nodes to their degree and ii) weights of links to degrees of nodes that they connect have a geometric origin. We present a simple robust model which exhibits the observed power laws and relates exponents to the dimensionality of 2D space in which traffic networks are embedded. The model is studied both analytically and in simulations and the conditions which result with previously reported exponents are clearly explained. We show that the relation between weight strength and degree is $s(k) \sim k^{3/2}$, the relation between distance strength and degree is $s^d(k) \sim k^{3/2}$ and the relation between weight of link and degrees of linked nodes is $w_{ij} \sim (k_i k_j)^{1/2}$ on the plane 2D surface. We further analyse the influence of spherical geometry, relevant for the whole planet, on exact values of these exponents. Our model predicts that these exponents should be found in future studies of port networks and impose constraints on more refined models of port networks.

| Study | $\beta$ | $\beta^d$ | $\alpha$ |
|---|---|---|---|
| Barrat *et al.*[6] World Airport Network (WAN) | $1.5 \pm 0.1$ | | $0.5 \pm 0.1$ |
| Barrat *et al.*[7] North American Airport Network | 1.7 | 1.4 | |
| Bagler [8] Airport Network of India | $1.43 \pm 0.06$ | | |
| Zhang *et al.*[9] Passenger Airport Network of China | 1.58 | | |
| Zhang *et al.*[9] Cargo Airport Network of China | 2.2 | | |
| O'Danleyman *et al.*[10] (WAN) | 1.33 | | |
| Kaluza *et al.*[11] Network of Global Cargo Ship Movements (NGCSM) | $1.46 \pm 0.1$ | | |
| Hu and Zhu[12] World Wide Maritime Transportation Network | 1.3 | | |
| O'Danleyman *et al.*[10] (NGCSM) | 1.33 | | |

TABLE I. Measured exponents in different studies. Except for the exponent $\beta$ of Cargo Airport Network of China, all the reported exponents fit well to the predictions of our model. Li and Cai[13] also suggested the value of $\alpha = 1/2$ for the Chinese Airport Network as an ansatz for their data, although without clear fitting procedure. Most of the studies did not report error intervals for the measured exponents but, based on those that did, we can assume errors of around 0.1.

An intense empirical research of traffic networks of aeroplanes[1, 6–10, 13, 14] and cargo ships[11, 12, 15] points to the fact that there are exponents of power laws universally present in these types of networks. In general, nodes in such traffic networks represent ports. Weighted links of these networks represent fluxes of some units - people or goods - between these ports[6]. In general these fluxes are directed, but it is also common to represent overall fluxes between ports as undirected, which is the approach that we follow in this study. The weighted link $w_{ij}$ found in empirical networks therefore represents some average number of transported objects back and forth, during some time interval between ports $i$ and $j$. The strength $s_i$ of the port $i$ is calculated as a sum of fluxes $w_{ij}$ of units through the port i.e. $s_i = \sum_j w_{ij}$. The distance strength of the port $s_i^d$ is a sum of distances to the neighbouring ports i.e. $s_i^d = \sum_{j \in n(i)} d_{ij}$, where $n(i)$ represents the set of ports connected to the port $i$. Different studies [1, 6, 7, 11–15] observed interesting relations between aforementioned properties and topological variables. Namely, the weighted strength of the port $i$ depends on its number of neighbours $k_i$ as $s_i \sim k_i^{\beta}$ [1, 6, 11–15]. The distance strength of the port $i$ scales with its number of neighbours $k_i$ as $s_i^d \sim k_i^{\beta_d}$ [7]. Finally the flux between two ports of degrees $k_i$ and $k_j$ scales as $w_{ij} \sim (k_i k_j)^{\alpha}$ [6, 13]. Values of these exponents in different studies are presented in Table I. Exponents in airport networks have been much more studied than exponents of cargo ship networks and we do not know for any attempt of measuring $\alpha$ in cargo ship networks. Nevertheless, agreement between findings in cargo ship networks and predictions of our model lead us to believe that our model could capture the behaviour of cargo ship networks as well.

Although a number of models coupling topological and traffic properties of such spatial networks has been proposed, none of these models managed to explain all three observed exponents. In[16] authors proposed a model based on spatial preferential attachment[17]. They found a relation between exponents, which explains only one or the other observed exponent, but not all of them. Further, in[7, 18] authors studied the model of weight driven attachment which was able to explain some of the observed properties of transport networks, but failed to reproduce exponents which relate degree properties with weight properties. Their model yields the value of $\beta = 1$, and the authors proposed that the measured exponents are a consequence of some non-linear process lying behind the attachment model they have used. The most complete analysis of coupling between topology and weights is laid out in[19], in which the author proposed two models - a weighted attachment model with the addition of weights and a fitness based model[20]. These models can reproduce any possible value of $\beta$, including $\beta = 3/2$, and a number of other network properties, but are sensitive to tunable model parameters, and exponents $\beta^d$ and $\alpha$ were not reproduced.

We observe that values of measured exponents point to the possibility that they are rational numbers. Traditional statistical mechanics[21] suggest that rational exponents can often be attributed to the dimensionality of space. If the values of measured exponents are really of geometric origin, then a fairly general model should be able to reproduce these exponents. Such a model should produce relatively stable exponents with respect to other model parameters and possible additional refinements of the model.

We propose a simple and robust mathematical model of port networks. This model is based on three assumptions. First, we assume that the probability for an individual unit to travel from one port to the other is determined by some hidden variables assigned to all ports. These variables depend on properties of the port such as the size of population, its economic power, number of international companies, etc., but could also depend on the traffic network funcionality to include transient ports etc. The predictions of the model do not depend on the precise definition of these variables or the precise mechanism of their assignement to ports. These hidden variables are modelled as the fitness of every port - a random number $x$, which is attached to every port and is drawn from some probability distribution $\rho(x)$[20]. We will later show that exponents are very robust to the choice of probability distribution and to possible spatial correlations between different fitness variables. The second assumption is that the probability of intended travel between two ports is proportional to the product of their fitness variables $p(i \to j) \sim x_i x_j$. This assumption is related to the usual model of random mixing in networks[22]. The fitness variables represent some units which travel back and forth to interact with some other units. The probability that some unit from port $i$ will travel to interact with some unit from port $j$ is, in its simplest instance, proportional to the amount of units in port $j$. This assumption means that the expected flux $w_{ij}$ of units from port $i$ to port $j$ and from port $j$ to port $i$ is proportional to the product of fitness variables $w_{ij} \sim x_i x_j$. The third assumption is related to the economic and geometric properties of traffic networks. We assume that the link between ports $i$ and $j$ will exist only if the expected volume of traffic will cover the costs of distant travel. The costs of travel $c_{ij}$ from port $i$ to port $j$ are modelled as a linearly increasing function of their distance $r_{ij}$. This assumption is related to the fact that costs of travel are, among other things, determined by the amount of fuel one has to spend travelling from one place to another and by the costs of crews which are also proportional to the distance via the number of travels a crew can manage in the unit of time. We also present the data on the travel fares between different U.S. airports and their mutual distances[23]. The relationship between distance and fares has Pearson's correlation $r = 0.64$ (Supplementary Figure 1), which justifies our assumption of linearity in the model. Notice that this data state real commercial fares which are not equal to the lower bound of transport profitability, which is relevant for this model. Nevertheless, the linear relationship between fares and distance is still clearly present. We expect that this relationship would be even more pronounced in the data on pure costs of travel between different ports.

The proposed model is studied both analytically and in simulations. Let us first consider an idealized case in which ports are randomly spread on an infinite 2-dimensional plane. We assume that the distribution of ports is relatively homogeneous and model their spatial distribution as a spatial Poisson process[25]. To every existing port $i$ a random number $X(i)$ drawn from some probability distribution $\rho(x)$ is assigned. Then a relation between the expected degree $\langle k(X) \rangle$ of the port and its fitness variable $X$ can be written as

$$\langle k(X) \rangle = \sigma \int dA \int_0^\infty \Theta\left(f(xX) - c(r)\right) \rho(x) dx. \tag{1}$$

Here $\sigma$ is a surface density of ports, $dA$ is a differential element of the surface, $\Theta$ is a Heaviside step function, $f$ is function of earnings generated from the flux of travelling units, and $c$ is a cost function of travel which grows with distance. We assume that these functions are monotonously increasing. In the rest of this paper we assume $f(u) = c(u) = u$ as the simplest linear relationship. In Supplementary Information 1 we give a short analysis of some other possibilities for functions $f$ and $c$ which lead to the same exponents. In it, we also show that the relation $w_{ij} = x_i x_j$ needs to be satisfied on the average only to produce the same exponents as the exact version of the model.

Similarly to equation (1) the expected strength and expected distance strength of the port can be related to its fitness $X$ as

$$\langle s(X) \rangle = \sigma \int dA \int_0^\infty xX \Theta\left(xX - r\right) \rho(x) dx. \tag{2}$$

$$\langle s^d(X) \rangle = \sigma \int dA \int_0^\infty r \Theta\left(xX - r\right) \rho(x) dx. \tag{3}$$

The integrations in these equations are easily performed to obtain $\langle k(X) \rangle = \pi\sigma M_2(\rho) X^2$, $\langle s(X) \rangle = \pi\sigma M_3(\rho) X^3$ and $\langle s^d(X) \rangle = \frac{2}{3}\pi\sigma M_3(\rho) X^3$. Here $M_{2,3}(\rho)$ represent the second and the third moment of the distribution $\rho(x)$. From these relations immediately follows that $\langle s(k) \rangle \sim k^{3/2}$, $\langle s^d(k) \rangle \sim k^{3/2}$ and $\langle w(k_i, k_j) \rangle \sim (\langle k_i \rangle \langle k_j \rangle)^{1/2}$. For simplicity we refer to these exponents as infinite exponents in the remaining text. One can also understand this relationship qualitatively. Port $i$ will be connected with some port $j$ up to certain radius $r^c_{ij}$ given by fitness values of these ports $X(i)$ and $X(j)$. Some port $i'$ which has the double fitness of the port $i$, $X(i') = 2X(i)$ will connect with port $j$ on double distance $r^c_{i'j} = 2r^c_{ij}$. Since we also assume the homogeneous distribution of ports, the number of ports to which port $i'$ is connected grows with the surface i.e. $k \sim r^2 \sim X^2$. Weights are assumed to be proportional to the fitness $X$ and to the number of connections $k$ which yields $s \sim X r^2 \sim X^3$. The average distance of the port is also proportional to the distance and to the typical number of connections $s^d \sim rk \sim X^3$. Having this in mind, it is clear that the relationship between the cut off distance $r^c$ and the fitness $X$ has to be linear in order to reproduce measured exponents.

Although distribution $\rho(x)$ can, in principle, be freely chosen on the infinite 2D plane, it is constrained for the applications by the observed degree distributions $p_o(k_o)$ extracted from the data. If we assume that $k_o(x)$ is well approximated by $\langle k(x)\rangle$ and since the relationship between expected degree $\langle k(x)\rangle$ and the fitness variable $x$ is a monotonic function $k = \mathcal{F}(x) \sim x^2$, then the relation $\rho(x) \sim |\frac{d\mathcal{F}(x)}{dx}| p_o(\mathcal{F}(x))$ follows. In the case of scale free distributions $p_o(k) \sim k^{-\gamma}$, the relation yields $\rho(x) \sim x^{1-2\gamma}$. Relevant studies[6, 8, 12, 15] have found power law distributions of degrees in port networks and in validations of our model we investigate power law distributions of fitnesses for exponents $\gamma \in \langle 1, 5]$. Another constraint on the distribution is the requirement that every port should be reachable from every other port. In other words, we demand the existence of a giant component in the network[24]. Yet another constraint is that the network should be sparse enough as the real port networks are. This condition can be controlled by monitoring the average degree of the network. All these factors have to be taken in account to choose model parameters, since not every distribution of fitnesses will satisfy all constraints for all densities or for all functions $f$ and $c$. These constraints were accounted for by considering the minimal and maximal values of fitness variables.

In Figure 1 we show dependences of weighted properties on topological properties for finite 2D space depending on the exponent $\gamma$ of the power law distribution $\rho(x)$. For a broad range of power law exponents simulated exponents $\alpha$, $\beta$ and $\beta^d$ are close to theoretically predicted values, as can be seen in Figure 1. The increase of the observed exponents as power law is becoming steeper which is related to the process of network sparsification, as shown in Figure 2. In the limiting case of a network consisting of disconnected subgraphs it is hard to talk about exponents at all. The degree distribution and the aforementioned exponents are clearly not the only properties of port networks previously reported. In Supplementary Information 2, for the reasons of completeness, we also present an analytical calculation for the clustering coefficient in infinite 2D plane. For thelarge $k$ we observe behavior close to $C(k) \sim k^{-1}$, a feature previously reported by other researchers [6, 7].

In the Supplementary Information 3 we present an analytical treatment of the model for the finite 2D plane and for the sphere of radius $R$. The case of the sphere is particularly important since it represents the real geometry in which port networks operate. The expected degree $\langle k(X)\rangle$, the strength $\langle s(X)\rangle$ and the distance strength $\langle s^d(X)\rangle$ of the port with fitness $X$ on a sphere can be computed from relations

$$\langle k(X)\rangle = \sigma R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_{R\theta/X}^{\infty} \rho(x) dx, \tag{4}$$

$$\langle s(X)\rangle = \sigma R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_{R\theta/X}^{\infty} X x \rho(x) dx, \tag{5}$$

and

$$\langle s^d(X)\rangle = \sigma R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_{R\theta/X}^{\infty} \theta R \rho(x) dx. \tag{6}$$

The change of the geometry influences the dependences of these variable on the fitness $X$, and we find deviations from the infinite exponents. Nevertheless, the average distances between ports reported in the previous studies, point to the fact that the majority of the traffic takes place on the locally almost flat elements of a sphere. The average distance $\langle d\rangle$ of travels in the case of the North American airport network is around 1000 km[7], which means that the relative difference between $\sin(\langle d\rangle/R)$ and $\langle d\rangle/R$ is only around 0.4%. We believe that this is the main reason why the monitored exponents of port networks really resemble theoretical values calculated for the infinite 2D plane.

In real traffic networks, ports are not homogeneously distributed. In the previous text we dealt with density of ports which are scattered over the space as Possionian random variables[25]. To check if the heterogeneities which necessarily exist among the real ports influence the behaviour of exponents, we gathered the data on geographical locations in the U.S.[26]. In Figure 2 we present simulations of our model on the geographical locations of American airports.

In Figure 3, we present three different sets of data points taken from the database[26] and construct a network among them. The exponents do not differ much among these different spatial port distributions, as long as the distribution of the fitness variable fulfils the aforementioned constraints. The robustness of this model to the heterogeneity of spatial port distribution is clear. It is also clear that many ports in the world are used just as transit airports for a significant amount of traffic and the fitness variables used in the model do not capture such behaviour. On the other hand, such usage of airports also influences the realistic possible values of fitnesses and we expect that in reality they have a certain spatial hierarchical distribution.

In conclusion, in this paper we presented a very robust model which can be altered in many conceivable ways, which generally reproduce observed exponents, which relate weighted properties to topological properties. Even further it can be shown that in the case of general D-dimensional flat space, the exponents would be $\alpha = 1/D$ and $\beta = \beta^d = (D+1)/D$ (see Supplementary Information 4), which could (apart from intellectual curiosity) possibly be interesting for the case of interstellar traffic with D=3. Similar exponents of geometric origin have been found in the case of allometric scaling in biological systems [27], and at present it is not clear if the presented model is related to these phenomena. Furthermore, we believe that some other modes of transportation like, for instance, intercity buses, could also be well described with this model, but at present we are not aware of any additional research in that direction. Since a convincing simple relationship between fitness variables proposed in this paper and measurable socio-economic quantities is still lacking, further research in this direction is clearly needed.

## I. METHODS

All simulations of the model were performed in Fortran90. The locations of the ports were drawn randomly from uniform distribution defined on the circle of area 10 in arbitrary units in the case of a finite 2D plane. We used different truncated power-law distribution for fitness variables. The parameters of the distribution were: minimal value of fitness $X_{min} = 0.1$, maximal value of fitness $X_{max} = 1.6$ and exponent $1 < \gamma < 5$. For every set of points or different parameters we have performed 100 different simulations with stochastic variables drawn from power-law distributions. In the case of the sphere with radius $R = 1$ we have used exactly the same parameters. The locations of ports on the real maps were acquired from geographical data. We have used geolocations of of airports, hospitals and capes in the U.S. [26] in order to encapsulate as many of real spatial nonpoissonian heterogeneous point distributions as possible. For the Geolocations we have used $X_{min} = 0.01$ or $X_{min} = 0.02$ and $X_{max} = 0.65$. We have also used 20 realizations, due to a large number of data points. The exponents were calculated by the least square fitting procedure of the logarithms of related variables. Reported errors are standard deviations of fitted exponents in the simulation sample.

## ACKNOWLEDGEMENTS

H.S. and V.Z. have been funded through Croatian ministry of education, science and sport project No. 098-0352828-2863. V.Z. acknowledges partial support from EU FET Open Project FOC nr 255987. Authors would like to thank G. Caldarelli and A. Gabrielli for useful comments and to D. Glavan on a very useful comment.

## CONTRIBUTIONS

M.P. developed analytical model and performed all of the simulations and most of the analytical calculations. H.S. contributed to the development of the model and made a contribution to analytical treatment. V.Z. conceived the problem, proposed the model, made a minor contribution to analytical description, created figures and was principal writer of the paper. All of the authors contributed in the design of the research and interpretation of the results.

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## SUPPLEMENTARY INFORMATION 1

The cost function is generally defined as $f(x_1 x_2) - c(r_{12})$. The expected degree of node with fitness $X$ is

$$\begin{aligned}\langle k(X)\rangle &= \sigma \int dA \int_0^\infty \Theta\left(f(xX) - c(r)\right)\rho(x)dx \\ &= 2\pi\sigma \int_0^\infty \rho(x)dx \int_0^{c^{-1}(f(xX))} rdr \\ &= \pi\sigma \int_0^\infty [c^{-1}(f(xX))]^2 \rho(x)dx. \end{aligned} \tag{7}$$

Similarly the expected strength of the node with fitness $X$ is

$$\langle s(X)\rangle = \pi\sigma \int_0^\infty [c^{-1}(f(xX))]^2 (xX)\rho(x)dx. \tag{8}$$

Equations (7) and (8) again give the same exponent behaviour if we choose $f = c$

$$\begin{aligned}\langle k(X)\rangle &= \pi\sigma \int_0^\infty [f^{-1}(f(xX))]^2 \rho(x)dx \\ &= \pi\sigma X^2 M_2(\rho). \end{aligned} \tag{9}$$

$$\begin{aligned}\langle s(X)\rangle &= \pi\sigma \int_0^\infty [f^{-1}(f(xX))]^2 (xX)\rho(x)dx \\ &= \pi\sigma X^3 M_3(\rho). \end{aligned} \tag{10}$$

In the case of different functions we can again easily reconstruct proposed exponents. For instance let $f(x) = fx$, and $c(x) = c_1 x + c_2$. Then equations for the $s$ and $k$ are:

$$\begin{aligned} k(X) &= \pi\sigma \left[\frac{f^2 X^2 M_2(\rho)}{c_1^2} - 2\frac{f c_2 X M_1(\rho)}{c_1^2} + \frac{c_2^2}{c_1^2}\right], \\ s(X) &= \pi\sigma \left[\frac{f^2 X^3 M_3(\rho)}{c_1^2} - 2\frac{f c_2 X^2 M_2(\rho)}{c_1^2} + \frac{c_2^2 X M_1(\rho)}{c_1^2}\right]. \end{aligned} \tag{11}$$

Proposed exponents will be reconstructed as long as $\left|\frac{f^2 X^2 M_2(\rho)}{2fc_2 X M_1(\rho) - c_2^2}\right| \ll 1$ and $\left|\frac{f^2 X^3 M_3(\rho)}{2fc_2 X^2 M_2(\rho) - c_2^2 X M_1(\rho)}\right| \ll 1$. In practice it will workj allways when $fX\langle X\rangle \gg c_2$.

Let us further consider the case in which flow $w_{ij}$ is not necessarily exactly equal to the product $x_i x_j$. Realistically, one can expect that some ports are better connected than expected by chance, while some others are not so well connected. For example, airports of two neighbouring states which do not have diplomatic relationships because of political reasons can have much lower traffic than one would expect from fits of their overall traffic. This additional preference of the links can be modelled with a new random variable $\eta$ taken from some distribution $p(\eta)$. Let us assume that the average value $\langle\eta\rangle = 1$. Traffic flows can be written as $w_{ij} = \eta_{ij} x_i x_j$. If $\eta$ and $x$ are independent random numbers, the equations for degree, strength and strength distance for infinite 2D plane are $\langle k(X)\rangle = \pi\sigma M_2(\rho) M_2(p) X^2$, $\langle s(X)\rangle = \pi\sigma M_3(\rho) M_3(p) X^3$ and $\langle s(X)^d\rangle = 2\pi\sigma M_3(\rho) M_3(p) X^3/3$. Obviously, infinite exponents are preserved even if we allow some fluctuations around expected values of flows.

## SUPPLEMENTRAY INFORMATION 2

The expected degree dependent clustering $\langle C(k)\rangle$ coefficient of some port on an infinite 2D plane can be calculated as:

$$\langle C(X)\rangle = \frac{\sigma^2 \int_0^\infty r_1 dr_1 \int_0^\infty r_2 dr_2 \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \int_0^\infty \rho(x_1)dx_1 \int_0^\infty \rho(x_2)dx_2 \Theta(Xx_1 - r_1)\Theta(Xx_2 - r_2)\Theta(x_1x_2 - r_{12})}{\sigma^2 \int_0^\infty r_1 dr_1 \int_0^\infty r_2 dr_2 \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \int_0^\infty \rho(x_1)dx_1 \int_0^\infty \rho(x_2)dx_2 \Theta(Xx_1 - r_1)\Theta(Xx_2 - r_2)} \tag{12}$$

where $r_{12} = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos(\varphi_1 - \varphi_2)}$
The denominator $D$ is equal to:

$$D = 4\sigma^2\pi^2 \int_0^\infty \rho(x_1)dx_1 \int_0^{Xx_1} r_1 dp_1 \int_0^\infty \rho(x_2)dx_2 \int_0^{Xx_2} r_2 dr_2 \tag{13}$$

$$= 4\sigma^2\pi^2 \left[\frac{1}{2}X^2 M_2(\rho)\right]^2 \tag{14}$$

and therefore

$$D = \sigma^2\pi^2 X^4 \left[M_2(\rho)\right]^2 \tag{15}$$

The numerator $A$ is equal to

$$A = \sigma^2 \int_0^\infty \rho(x_1)dx_1 \int_0^{Xx_1} r_1 dr_1 \int_0^\infty \rho(x_2)dx_2 \int_0^{Xx_2} r_2 dr_2 \int_0^{2\pi} d\varphi_1 \int_0^{2\pi} d\varphi_2 \Theta(x_1x_2 - r_{12}). \tag{16}$$

We can make a convenient substitution

$$\varphi = \frac{1}{2}(\varphi_2 - \varphi_1) \tag{17}$$

$$\varphi' = \frac{1}{2}(\varphi_2 + \varphi_1). \tag{18}$$

Separating the integral domain in two parts we can write integrals over new angles:

$$\int_{-\pi}^{0} d\varphi \int_{-\varphi}^{2\pi+\varphi} d\varphi' \frac{\partial(\varphi_2, \varphi_1)}{\partial(\varphi, \varphi')} \Theta(x_1x_2 - r_{12}) + \int_0^\pi d\varphi \int_\varphi^{2\pi-\varphi} d\varphi' \frac{\partial(\varphi_2, \varphi_1)}{\partial(\varphi, \varphi')} \Theta(x_1x_2 - r_{12}). \tag{19}$$

Using Eq. (19)the numerator can be written as

$$A = \sigma^2 \int_0^\infty \rho(x_1)dx_1 \int_0^{Xx_1} r_1 dr_1 \int_0^\infty \rho(x_2)dx_2 \int_0^{Xx_2} r_2 dr_2 2 \int_0^{2\pi} d\varphi (2\pi - \varphi)\Theta(x_1x_2 - r_{12}) \tag{20}$$

The random variables are positive semidefinite and thus

$$\Theta(x_1x_2 - \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\varphi}) = \Theta(\cos\varphi - \frac{r_1^2 + r_2^2 - (x_1x_2)^2}{2r_1r_2}). \tag{21}$$

We can write

$$A = 4\pi^2 \int_0^\infty \rho(x_1)dx_1 \int_0^{Xx_1} r_1 dr_1 \int_0^\infty \rho(x_2)dx_2 \int_0^{Xx_2} r_2 dr_2 \;\; \chi\left(\frac{r_1^2 + r_2^2 - (x_1x_2)^2}{2r_1r_2}\right). \tag{22}$$

Function $\chi(x)$ is defined as

$$\chi(x) = \begin{cases} 1 & x < -1 \\ \frac{1}{\pi}\arccos x & -1 < x < 1 \\ 0 & x > 1 \end{cases} \tag{23}$$

Finally the clustering coefficient is

$$\langle C(X)\rangle = \frac{4}{X^4\left[M_2(\rho)\right]^2}\int_0^\infty \rho(x_1)dx_1 \int_0^{Xx_1} r_1 dr_1 \int_0^\infty \rho(x_2)dx_2 \int_0^{Xx_2} r_2 dr_2 \;\; \chi\left(\frac{r_1^2+r_2^2-(x_1x_2)^2}{2r_1r_2}\right) \tag{24}$$

Behavior of the clustering coefficient with respect to $k$ is presented in supplementary figure 2.

**SUPPLEMENTARY INFORMATION 3**

Let us consider a point on the pole of the sphere of radius $R$. The sphere is covered with random points at locations $\theta$ and $\phi$. The density of these points is given with $\sigma = \sum_i \int_{\mathcal{S}} \delta(\vec{r}-\vec{r}_i)d\vec{r}/S$. The distance of points from the pole is given with $R\theta$, where $R$ is the radius of the sphere and $\theta$ is the inclination angle. We assume that the point on the pole is a port with the variable $X$ and that other points represent different ports whose fitnesses are drawn from some unspecified distribution $\rho(x)$. The expected degree of the port on the pole is then

$$\langle k(X)\rangle = \sigma R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_{R\theta/X}^{\infty} \rho(x)dx. \tag{25}$$

Similarly the strength $\langle s\rangle$ and distance strength $\langle s^d\rangle$ of the vertex can be calculated from:

$$\langle s(X)\rangle = \sigma R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_{R\theta/X}^{\infty} Xx\rho(x)dx, \tag{26}$$

$$\langle s^d(X)\rangle = \sigma R^2 \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta d\theta \int_{R\theta/X}^{\infty} \theta R\rho(x)dx. \tag{27}$$

The expected degree can be written as two terms equation as

$$\begin{aligned} k(X) &= 2\pi\sigma R^2 \int_0^{\pi} \sin\theta d\theta \int_{\theta R/X}^{\infty} \rho(x)dx, \\ &= 2\pi\sigma R^2 \left[\int_0^{\pi R/X} \left(1-\cos\frac{Xx}{R}\right)\rho(x)dx + 2\int_{\pi R/X}^{\infty}\rho(x)dx\right], \\ &= 2\pi\sigma R^2 \left[\int_0^{\infty} \left(1-\cos\frac{Xx}{R}\right)\rho(x)dx + \int_{\pi R/X}^{\infty}\left(1+\cos\frac{Xx}{R}\right)\rho(x)dx\right], \end{aligned}$$

and similar equations can be written for strength and distance strength also. The second integral can be neglected for distributions whose tail falls fast enough. In the case of our simulations ($X_{max} = 1.6$, $R = 1$), the second integral is exactly 0 since it lower limit is $\pi/1.6 \approx 2$ while the distribution $\rho(x > 1.6) = 0$. After the calculation of space integrals we make an expansion of trigonometric functions up to the second order:

$$\langle k(X)\rangle = \pi\sigma X^2 M_2(\rho) - \frac{\pi\sigma}{12R^2}X^4 M_4(\rho) = \pi\sigma X^2 M_2(\rho)\left(1+c_k\right), \tag{28}$$

$$\langle s(X)\rangle = \pi\sigma X^3 M_3(\rho) - \frac{\pi\sigma}{12R^2}X^5 M_5(\rho) = \pi\sigma X^3 M_3(\rho)\left(1+c_s\right), \tag{29}$$

$$\langle s^d(X)\rangle = \frac{2}{3}\pi\sigma X^3 M_3(\rho) - \frac{\pi\sigma}{15R^2}X^5 M_5(\rho) = \frac{2}{3}\pi\sigma X^3 M_3(\rho)\left(1+c_{s^d}\right). \tag{30}$$

We can now calculate:

$$\frac{d\langle k(X)\rangle}{dX} = 2\pi\sigma X M_2(\rho) - \frac{\pi\sigma}{3R^2}X^3 M_4(\rho) \tag{31}$$

$$\frac{d\langle s(X)\rangle}{dX} = 3\pi\sigma X^2 M_3(\rho) - \frac{5\pi\sigma}{12R^2}X^4 M_5(\rho) \tag{32}$$

Thus

$$\frac{d\langle s\rangle}{d\langle k\rangle}=\left(3\pi\sigma X^2M_3(\rho)-\frac{5\pi\sigma}{12R^2}X^4M_5(\rho)\right)\frac{1}{2\pi\sigma XM_2(\rho)}\frac{1}{1-\frac{M_4(\rho)X^2}{6R^2M_2(\rho)}} \tag{33}$$

With the assumption that corrections are small $\frac{X^2M_5(\rho)}{12R^2M_3(\rho)}\ll 1$ and $\frac{X^2M_4(\rho)}{12R^2M_2(\rho)}\ll 1$ we get:

$$\frac{d\langle s\rangle}{d\langle k\rangle}=\frac{3}{2}\frac{M_3(\rho)}{M_2(\rho)}X-\frac{M_4(\rho)}{4R^2M_2(\rho)}\left[\frac{5}{6}\frac{M_5(\rho)}{M_4(\rho)}-\frac{M_3(\rho)}{M_2(\rho)}\right]X^3 \tag{34}$$

Solving (28) for $X$ and expanding in series:

$$\frac{d\langle s\rangle}{d\langle k\rangle}=\frac{3}{2}\frac{M_3(\rho)}{\sqrt{\pi\sigma}M_2^{3/2}(\rho)}\langle k\rangle^{1/2}+\frac{M_4(\rho)}{48R^2M_2^{5/2}(\rho)(\pi\sigma)^{3/2}}\left[15\frac{M_3(\rho)}{M_2(\rho)}-10\frac{M_5(\rho)}{M_4(\rho)}\right]\langle k\rangle^{3/2} \tag{35}$$

Integrating with condition that $\langle s\rangle=0$ when $\langle k\rangle=0$ we get:

$$\langle s\rangle=\frac{M_3(\rho)}{\sqrt{\pi\sigma}M_2^{3/2}(\rho)}\langle k\rangle^{3/2}+\frac{M_4(\rho)}{24R^2M_2^{5/2}(\rho)(\pi\sigma)^{3/2}}\left[3\frac{M_3(\rho)}{M_2(\rho)}-2\frac{M_5(\rho)}{M_4(\rho)}\right]\langle k\rangle^{5/2} \tag{36}$$

Similarly

$$\frac{d\langle s^d\rangle}{d\langle k\rangle}=\frac{M_3(\rho)}{\sqrt{\pi\sigma}M_2^{3/2}(\rho)}\langle k\rangle^{1/2}+\frac{M_4(\rho)}{24R^2M_2^{5/2}(\rho)(\pi\sigma)^{3/2}}\left[5\frac{M_3(\rho)}{M_2(\rho)}-4\frac{M_5(\rho)}{M_4(\rho)}\right]\langle k\rangle^{3/2} \tag{37}$$

$$\langle s^d\rangle=\frac{2}{3}\frac{M_3(\rho)}{\sqrt{\pi\sigma}M_2^{3/2}(\rho)}\langle k\rangle^{3/2}+\frac{M_4(\rho)}{60R^2M_2^{5/2}(\rho)(\pi\sigma)^{3/2}}\left[5\frac{M_3(\rho)}{M_2(\rho)}-4\frac{M_5(\rho)}{M_4(\rho)}\right]\langle k\rangle^{5/2} \tag{38}$$

Leading order corrections to degree $c_k$ and strength $c_s$ are negligible if $\frac{X^2M_4(\rho)}{12R^2M_2(\rho)}\ll 1$ and $\frac{X^2M_5(\rho)}{12R^2M_3(\rho)}\ll 1$. We calculated l.h.s. of this relations for the simulations described in this paper. These contributions were computed for the minimal and maximal exponent of $\rho(x)$ and for mutual couplings between ports with maximal fitness value as well as couplings between ports with maximal fitness value and ports with typical fitness value.

The influence of the spherical geometry in the case of the sphere with homegnous distribution of ports can be quantified via typical values of the flows i.e. products of fitnesses: $X_{min}=0.1$ $X_{max}=1.6$:

- $X^2=X_{max}^2$, $\gamma=1:$ $c_k=0.274$, $c_s=0.328$
- $X^2=X_{max}\langle X\rangle$, $\gamma=1:$ $c_k=0.0927$, $c_s=0.1106$
- $X^2=X_{max}^2$, $\gamma=2.6:$ $c_k=0.136$, $c_s=0.2296$
- $X^2=X_{max}\langle X\rangle$, $\gamma=2.6:$ $c_k=0.0185$, $c_s=0.0314$

United states Airports: $X_{min}=0.02$, $X_{max}=0.65$:

- $X^2=X_{max}^2$, $\gamma=1:$ $c_k=7.445\cdot 10^{-3}$, $c_s=8.926\cdot 10^{-3}$
- $X^2=X_{max}\langle X\rangle$, $\gamma=1:$ $c_k=2.073\cdot 10^{-3}$, $c_s=2.485\cdot 10^{-3}$
- $X^2=X_{max}^2$, $\gamma=2.6:$ $c_k=3.298\cdot 10^{-3}$, $c_s=6.172\cdot 10^{-3}$
- $X^2=X_{max}\langle X\rangle$, $\gamma=2.6:$ $c_k=2.380\cdot 10^{-4}$, $c_s=4.454\cdot 10^{-4}$

As shown, the influence of spherical geometry is practically non-existent for typical situations and can have some effect only on ports with fitnesses drawn from the tail of the distribution. Spherical geometry clearly influences the relation between strength $s$, distance strength $s^d$ and degree $k$. In general we can not expect the infinite exponents on spherical geometries. Nevertheless, in cases in which most of the links between ports are below a certain threshold, the geometry is effectively almost flat and the predicted exponents are expected to be found.

Here we calculate the expected values of degree and strength for given $X$ in the case of finite flat geometry. Now the domain of integration is finite and, if we assume the convexity of the domain, we can write the equation for the expected degree of the port with fitness $X$ as:

$$k(X) = \int_0^{2\pi} d\varphi \int_0^{\Gamma(\varphi)} r dr \int_{r/X}^{\infty} \rho(x) dx, \tag{39}$$

where $\Gamma(\varphi)$ describes the distance of the domain edge to the origin with respect to angle $\varphi$. This integral can be separated in two parts as:

$$k(X) = \sigma \int_0^{2\pi} d\varphi \left[ \frac{1}{2} X^2 \int_0^{\Gamma(\varphi)/X} x^2 \rho(x) dx + \frac{1}{2} \Gamma^2(\varphi) \int_{\Gamma(\varphi)/X}^{\infty} \rho(x) dx \right], \tag{40}$$

$$= \pi \sigma X^2 \int_0^{\infty} x^2 \rho(x) dx - \frac{1}{2} \sigma X^2 \int_0^{2\pi} d\varphi \int_{\Gamma(\varphi)/X}^{\infty} \left[ 1 - \left( \frac{\Gamma(\varphi)}{Xx} \right)^2 \right] x^2 \rho(x) dx. \tag{41}$$

For the strength the equation is very similar:

$$s(X) = \pi \sigma X^3 \int_0^{\infty} x^3 \rho(x) dx - \frac{1}{2} \sigma X^3 \int_0^{2\pi} d\varphi \int_{\Gamma(\varphi)/X}^{\infty} \left[ 1 - \left( \frac{\Gamma(\varphi)}{Xx} \right)^2 \right] x^3 \rho(x) dx. \tag{42}$$

These expressions can be rewritten in a more transparent form as:

$$k(X) = \pi \sigma X^2 \int_0^{\infty} x^2 \rho(x) dx \left[ 1 - \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{\int_{\Gamma(\varphi)/X}^{\infty} \left[ 1 - \left( \frac{\Gamma(\varphi)}{Xx} \right)^2 \right] x^2 \rho(x) dx}{\int_0^{\infty} x^2 \rho(x) dx} \right] \tag{43}$$

$$= \pi \sigma X^2 \int_0^{\infty} x^2 \rho(x) dx \left[ 1 - \left\langle \epsilon_k \left( \frac{\Gamma(\varphi)}{X} \right) \right\rangle_{\varphi} \right], \tag{44}$$

and

$$s(X) = \pi \sigma X^3 \int_0^{\infty} x^3 \rho(x) dx \left[ 1 - \frac{1}{2\pi} \int_0^{2\pi} d\varphi \frac{\int_{\Gamma(\varphi)/X}^{\infty} \left[ 1 - \left( \frac{\Gamma(\varphi)}{Xx} \right)^2 \right] x^3 \rho(x) dx}{\int_0^{\infty} x^3 \rho(x) dx} \right] \tag{45}$$

$$= \pi \sigma X^3 \int_0^{\infty} x^3 \rho(x) dx \left[ 1 - \left\langle \epsilon_s \left( \frac{\Gamma(\varphi)}{X} \right) \right\rangle_{\varphi} \right]. \tag{46}$$

If we assume an upper cut-off of the distribution $X_{max}$, an assumption which is very realistic for any imaginable port system, then if $\forall \varphi : \Gamma(\varphi) > X X_{max}$ both $\epsilon$ functions are equal to zero. If this condition is not fulfilled, a weaker condition will still preserve infinite limit exponents. Namely, a factor $1 - \left( \frac{\Gamma(\varphi)}{Xx} \right)^2$ can achieve values from 0 for $x = \frac{\Gamma(\varphi)}{X}$ up to 1 for large $X$ which reduces the contribution of the second term. Finally, by averaging over the angle $\varphi$ with the assumption of a convex domain the contribution of terms different than zero is reduced by factor 2 at worst. Finally, if we assume $X_{max} \leq \sqrt{\Gamma}$ where $\Gamma$ represents the characteristic linear dimension of the domain, we can easily expect the infinite limit exponents recovered too.

## SUPPLEMENTARY INFORMATION 4

In the hypothetical case of a D-dimensional infinite plane, our model is integrated over D-dimensional volume with the differential element $dV_D = r^{D-1} C_D dr$. From this trivially follows

$$\langle k(X)\rangle = \sigma C_D \int_0^\infty r^{D-1} dr \int_0^\infty \Theta\left(xX - r\right) \rho(x) dx, \tag{47}$$

$$\langle s(X)\rangle = \sigma C_D \int_0^\infty r^{D-1} dr \int_0^\infty xX\Theta\left(xX - r\right) \rho(x) dx, \tag{48}$$

$$\langle s^d(X)\rangle = \sigma C_D \int_0^\infty r^{D-1} dr \int_0^\infty r\Theta\left(xX - r\right) \rho(x) dx. \tag{49}$$

$$\tag{50}$$

One can see that the exponents $\beta = \beta^d = (D+1)/D$ and $\alpha = 1/D$

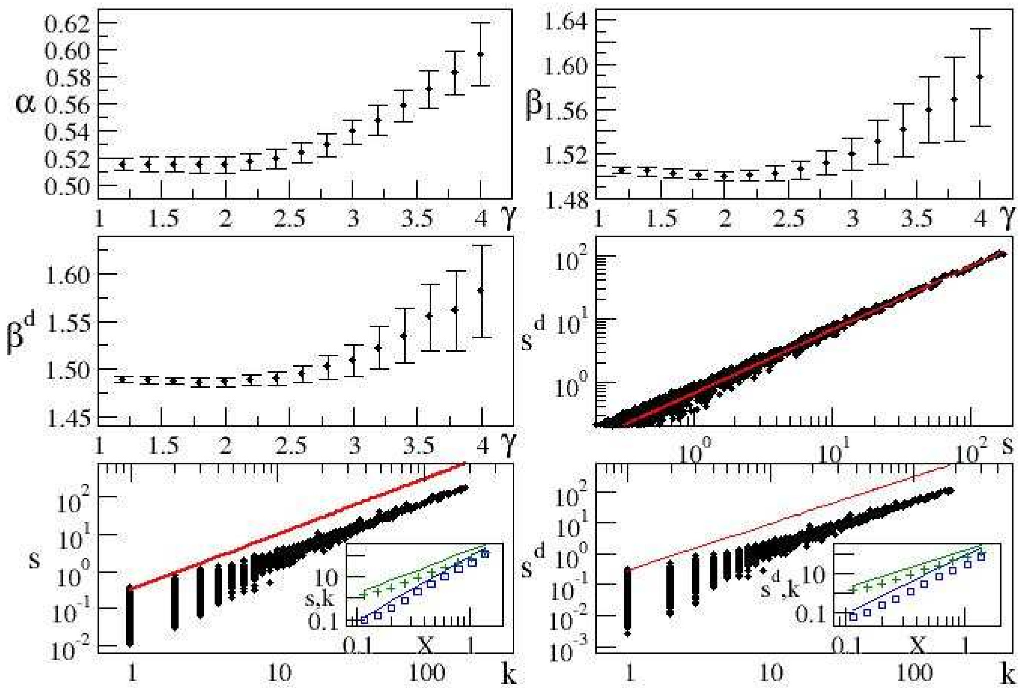


FIG. 1. Simulations for a finite 2D plane are represented in these figures. The upper left figure represents the $\alpha$ dependence on the power law exponent $\gamma$, the upper right figure represents the $\beta$ dependence on the power law exponent $\gamma$, and the middle left figure represent the $\beta^d$ dependence on the power law exponent $\gamma$. It is clear that infinite exponents are well reproduced in finite flat space as well. The middle right figure represents the relation between strength and distance strength in the simulated data. The red line is the theoretically predicted result of $s = 2/3s^d$. The lower left figure and the lower right figure represent the strength $s$ and degree strength $s^d$ dependence on degree $k$. The data points are simulations and the red line has an exponent 3/2 and is drawn for comparison. Strength $s$, distance strength $s^d$ and degree $k$ as a function of fitness $X$ are in the insets. The data in insets are exponentially binned to improve visibility and lines with exponents 3 and 2 are drawn for comparison. The networks for this simulation are realized on a finite flat space as described in the Methods section. The exponents are well reproduced for a broad range of distribution parameters.

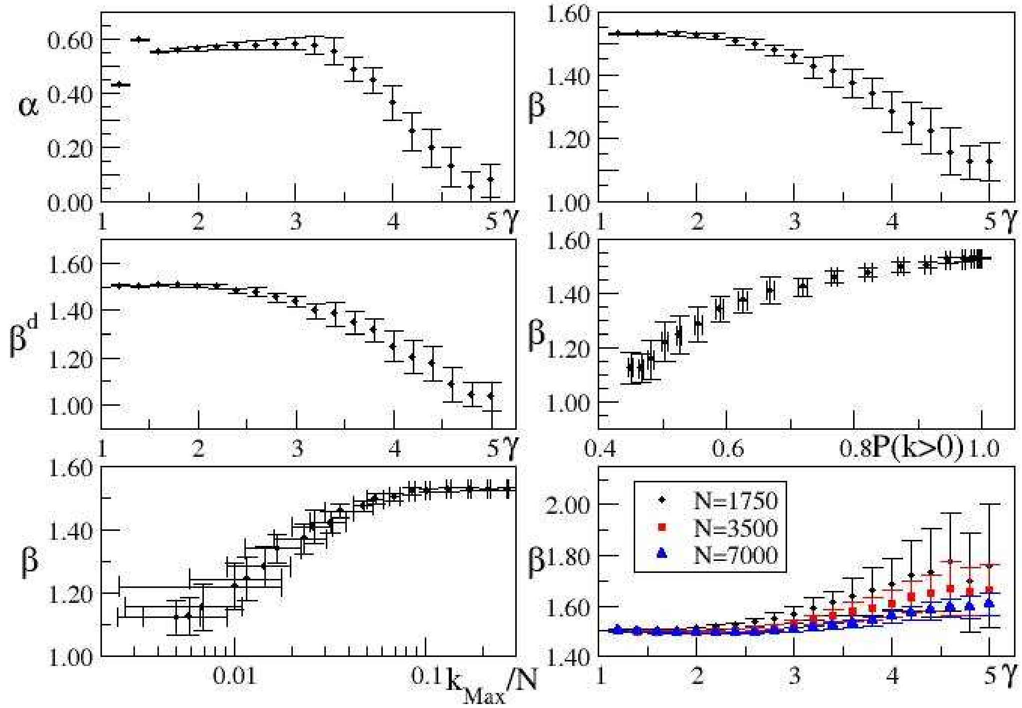


FIG. 2. The upper left figure represents the $\alpha$ dependence on the power law exponent $\gamma$, the upper right figure represents the $\beta$ dependence on the power law exponent $\gamma$, and the middle left figure represent the $\beta^d$ dependence on the power law exponent $\gamma$. Locations of points are locations of 18745[26] American airports and fitness variables are randomly drawn from the power law distributions described with exponent $\gamma$, minimal value of fitness $x_{min} = 0.02$ and maximal value of fitness $x_{max} = 0.65$. It is clear that infinite exponents are well reproduced in spherical geometry for $\gamma < 3$. The middle right figure represents the dependence of $\beta$ on the fraction of airports $P(k > 0)$ which have at least one neighbour. It is clear that the simulated exponents differ from infinite ones when the network starts to disconnect due to a big number of airports with small fitness parameters. The lower left figure represents the relationship between $\beta$ and the maximal degree of the airport in the network. Again, when the maximal degree is too small, networks effectively disconnect and simulated exponents deviate from the infinite case. The lower right figure demonstrates that higher density of points reproduces the infinite exponents better. The relationship between $\beta$ and $\gamma$ for three cases of the different density of points in the case of a sphere of radius $R = 1$ is presented. $N$ is the number of points on the sphere and the density is $\sigma = N/4\pi$.

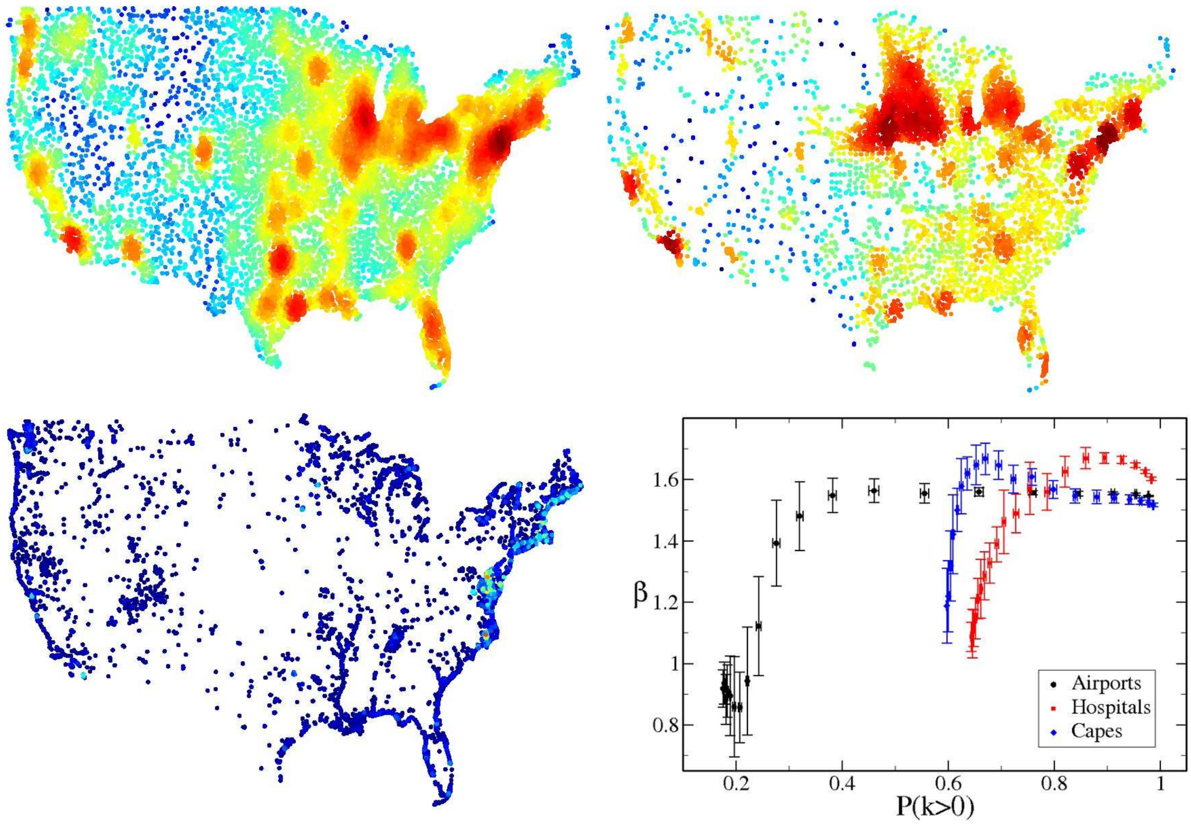


FIG. 3. The upper left figure represents locations of 18745 American airports, the upper right figure represents locations of 13385 American hospitals and the lower left figure represents locations of 11938 American capes. Colours represent different local densities of locations in order to emphasize differences among heterogeneities of this data. The relationship between $\beta$ and $P(k > 0)$ for these sets of ports is shown in the lower right figure. The minimal value of fitness is $x_{min} = 0.01$ and maximal value of fitness is $x_{max} = 0.65$. The infinite exponent is again reproduced for all three sets of spatial points, as long as $\rho(x)$ is sufficiently broad i.e. a network is connected.

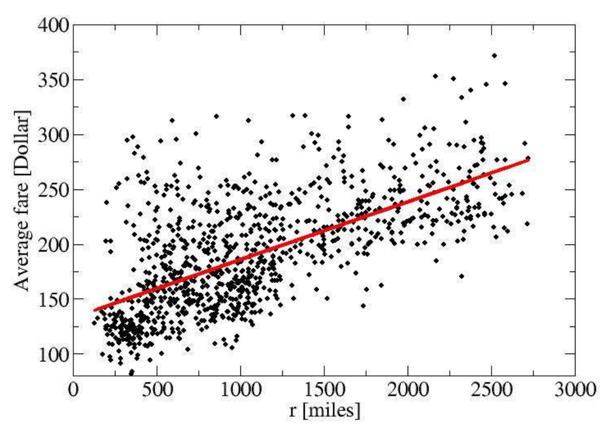


FIG. 4. The average fares of travel among 200 largest american airports are placed on the y-axis, and on the x-axis is the mutual distance of airports. Data points represent the average fare of travel in the third quarter of 2010 between two airports at a given distance. The data was downloaded from [23]. The straight line represents linear regression obtained via least squares. Pearson's correlation of the data is 0.64. It is visible that the smallest fares at a given distance also have a linear like behaviour, and we can expect that the smallest fares are a better representation of distance costs then the average fares.

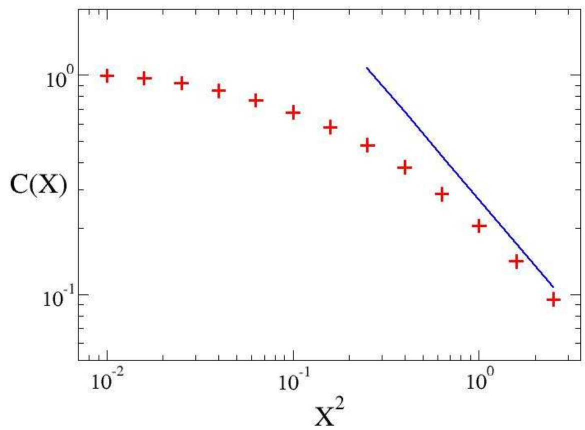


FIG. 5. An average value of the clustering coefficient $C$ is placed Onon the y-axis, and the square of the fitness $X^2$ is placed on the x-axis. The data points (pluses) represent the expected clustering dependent on the square of $x^2 \sim k$. The full line represents the power law with exponent $-1$ and such assymptotic relationship is found in real networks. The minimal and maximal value of $x$ are again 0.1 and 1.6, which were used in simulations as well.