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ABSTRACT

We consider the Yang algebras isomorphic to $o(1,5), o(2,4), o(3,3)$ and derive dual κ-Minkowski and κ-Poincaré algebras in terms of a metric g. The corresponding Weyl realization is presented and coproduct, star product, and twist are computed in terms of the metric g. Finally, we construct reduced κ-Minkowski and κ-Poincaré algebras as special cases.

I. INTRODUCTION

It is well known that the principles of quantum theory and general relativity are incompatible and a more general theory reconciling them is necessary. This will presumably be based on new assumptions on the foundations of the two theories. A debated possibility is that the quantum phase space must be modified, for example deforming the canonical Heisenberg relations. This of course would induce observable effects, as a generalized uncertainty principle (GUP).[1] However, it is evident on dimensional grounds that such effects are related to the Planck scale and therefore are besides the present experimental reach.

By definition, a deformation of the commutation relations of position operators induces a noncommutativity of spacetime[2] that can be related to the curvature of momentum space.[3] On the other hand, it is well known that the curvature of spacetime gives rise to a deformation of the commutation relations of momentum operators. In general, both deformations could be considered simultaneously.

This idea was first advanced by Yang in Ref. 4, who, building on an earlier proposal of Snyder,[5] proposed an algebra isomorphic to the fifteen-dimensional orthogonal algebra $o(1,5)$, consisting in a combination of the Heisenberg algebra of a deformed phase space with noncommuting coordinates $\hat{x}_\mu$ and $\hat{p}_\mu$ and the Lorentz algebra generated by $M_{\mu\nu}$, together with a further scalar generator $\hat{h}$, necessary for the closure. The Yang algebra depends on two deformation parameters M and R that fix the scale of the curvature of momentum and position spaces, and are often identified with the Planck mass and the de Sitter radius. The Yang model enjoys the remarkable property of being invariant under a generalized Born duality,[6] which interchanges positions and momenta. Moreover, taking suitable limits, it can be reduced to the Snyder algebra or its dual de Sitter algebra.

After some time, the study of the Yang model was resumed in recent years. Some generalizations were presented by Khruschev and Leznov in Ref. 7, while in Ref. 8 an extension of the model that includes also the related triply special relativity (TSR) theory[9,10] was introduced. Further investigations concerning in particular its realizations on a canonical phase space have been recently performed in Refs. 11–13, using the methods introduced in Refs. 14–16 for the study of Snyder space. Other contributions to the study of the Yang model are given in

Refs. 17–19, while different models of noncommutative geometry in curved spaces can be found in Refs. 20–23. Also, relations of the Yang model with other theories, as for example conformal field theory or fuzzy gravity, have been investigated in Ref. 24.

Using the methods presented in Refs. 14 and 15 for Snyder space, in Refs. 25–28 for κ-deformed Snyder spaces, and in Ref. 29 for isomorphism between the Yang model and orthogonal algebras, the Hopf algebra structure, coproduct, twist, and the associative star product were obtained for the Yang model.[30]

Recently in Refs. 30 and 31, it was shown that the Yang algebra can be subjected to a further deformation of κ type,[32] depending on two different κ parameters. This doubly κ-deformed Yang model is still isomorphic to the $o(1,5)$ algebra. This proposal was investigated in more detail in Ref. 30 where the change of basis necessary to pass from the original Yang algebra to the new one was explicitly worked out, and the Weyl realization of such algebra in extended phase space was given. Later, in Ref. 33 a new relativistic doubly κ-deformed quantum phase space was considered. In this way, a generalized κ-Poincaré algebra is obtained, that displays a duality of Born type between position and momentum deformations. We call the associated spacetimes dual κ-Minkowski spaces.

In this paper, we analyze the properties of the doubly κ-Poincaré algebra and the associated κ-Minkowski space in the cases of $o(1,5)$, $o(2,4)$, and $o(3,3)$ algebras, in a way analogous to that used in Ref. 30 for the doubly κ-deformed Yang models. The results are written in terms of a metric g; dual κ-Minkowski and dual κ-Poincaré algebras are obtained as special cases for $g_{44} = g_{55} = 0$. Weyl realization, coproduct, star product and twist are presented in terms of the metric g. Finally, reduced dual κ-Minkowski and dual κ-Poincaré algebras, where the scalar generator $\hat{h}$ is taken as dependent,[34] are constructed.

II. GENERALIZED YANG MODELS ISOMORPHIC TO $o(1,5,g)$

There are infinitely many Born-dual κ-Minkowski spaces among non linear algebras satisfying the Jacobi identities.[33] In this section we review and generalize the construction of a class of Lie algebra-type spaces containing dual κ-Minkowski spaces that can be obtained from the Yang model[30] with a suitable choice of the parameters.

Let us start with the Yang algebra[4]

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i\epsilon_1}{M^2} M_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = \frac{i\epsilon_2}{R^2} M_{\mu\nu}, \tag{2.1}$$

$$[M_{\mu\nu}, \hat{x}_\lambda] = i\big(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu\big), \quad [M_{\mu\nu}, \hat{p}_\lambda] = i\big(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\nu\lambda}\hat{p}_\mu\big), \tag{2.2}$$

$$[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}\hat{h}, \quad \big[\hat{h}, \hat{x}_\mu\big] = \frac{i\epsilon_1}{M^2}\hat{p}_\mu, \quad \big[\hat{h}, \hat{p}_\mu\big] = -\frac{i\epsilon_2}{R^2}\hat{x}_\mu, \tag{2.3}$$

$$\big[M_{\mu\nu}, \hat{h}\big] = 0, \tag{2.4}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}), \tag{2.5}$$

where $\epsilon_1^2 = \epsilon_2^2 = 1$ and $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

In this section we shall consider the case $\epsilon_1 = \epsilon_2 = 1$ corresponding to the $o(1,5)$ algebra.

Let us define the most general new generators linear in $\hat{x}_\mu, \hat{p}_\mu$ and $M_{\mu\nu}$ introducing parameters κ and $\tilde{\kappa}$ generalizing the results of Ref. 30,

$$\tilde{X}_\mu = u\left(\cos\,\varphi\hat{x}_\mu + \frac{R}{M}\sin\,\varphi\hat{p}_\mu\right) + \frac{1}{\kappa}a_\rho M_{\mu\rho}, \tag{2.6}$$

$$\tilde{P}_\mu = v\left(\cos\,\psi\hat{p}_\mu + \frac{M}{R}\sin\,\psi\hat{x}_\mu\right) + \frac{1}{\tilde{\kappa}}b_\rho M_{\mu\rho}, \tag{2.7}$$

with the Lorentz generators $\tilde{M}_{\mu\nu} = M_{\mu\nu}$ unchanged. The scalar parameters u, v, φ, ψ and the four-vectors a_μ, b_μ are dimensionless with $uv \neq 0$.

The inverse transformations are

$$\hat{x}_\mu = \frac{u^{-1}\frac{1}{R}\cos\,\psi\big(\tilde{X}_\mu - \frac{1}{\kappa}a_\rho M_{\mu\rho}\big) - v^{-1}\frac{1}{M}\sin\,\varphi\big(\tilde{P}_\mu - \frac{1}{\tilde{\kappa}}b_\rho M_{\mu\rho}\big)}{\frac{1}{R}\cos\,(\varphi+\psi)}, \tag{2.8}$$

$$\hat{p}_\mu = \frac{v^{-1}\frac{1}{M}\cos\,\varphi\big(\tilde{P}_\mu - \frac{1}{\tilde{\kappa}}b_\rho M_{\mu\rho}\big) - u^{-1}\frac{1}{R}\sin\,\psi\big(\tilde{X}_\mu - \frac{1}{\kappa}a_\rho M_{\mu\rho}\big)}{\frac{1}{M}\cos\,(\varphi+\psi)}. \tag{2.9}$$

and we define

$$\tilde{H} = \hat{h}uv\,\cos\,(\varphi+\psi) + \frac{1}{\kappa}a\tilde{P} - \frac{1}{\tilde{\kappa}}b\tilde{X} - \frac{1}{\kappa\tilde{\kappa}}a_\rho b_\sigma M_{\rho\sigma}. \tag{2.10}$$

The generators $\tilde{X}_\mu, \tilde{P}_\mu, M_{\mu\nu}$ and $\tilde{H}$ generate a new class of Lie algebras isomorphic to the initial Yang algebra. These algebras are defined by

$$\big[\tilde{X}_\mu, \tilde{X}_\nu\big] = i\left(\frac{1}{M^2}\left(u^2 + a^2\frac{M^2}{\kappa^2}\right)M_{\mu\nu} + \frac{1}{\kappa}\big(a_\mu\tilde{X}_\nu - a_\nu\tilde{X}_\mu\big)\right), \tag{2.11}$$

$$\left[\tilde{P}_\mu, \tilde{P}_\nu\right] = i\left(\frac{1}{R^2}\left(v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}\right)M_{\mu\nu} + \frac{1}{\tilde{\kappa}}\left(b_\mu\tilde{P}_\nu - b_\nu\tilde{P}_\mu\right)\right), \tag{2.12}$$

$$\left[\tilde{X}_\mu, \tilde{P}_\nu\right] = i\left(\eta_{\mu\nu}\tilde{H} + \frac{1}{\tilde{\kappa}}b_\mu\tilde{X}_\nu - \frac{1}{\kappa}a_\nu\tilde{P}_\mu + \frac{1}{MR}\left(uv\,\sin\left(\varphi+\psi\right) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)M_{\mu\nu}\right), \tag{2.13}$$

$$\left[M_{\mu\nu}, \tilde{X}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{X}_\nu - \eta_{\nu\lambda}\tilde{X}_\mu + \frac{1}{\kappa}\left(a_\mu M_{\lambda\nu} - a_\nu M_{\lambda\mu}\right)\right), \tag{2.14}$$

$$\left[M_{\mu\nu}, \tilde{P}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{P}_\nu - \eta_{\nu\lambda}\tilde{P}_\mu + \frac{1}{\tilde{\kappa}}\left(b_\mu M_{\lambda\nu} - b_\nu M_{\lambda\mu}\right)\right), \tag{2.15}$$

where

$$\left[M_{\mu\nu}, \tilde{H}\right] = i\left(\frac{1}{\tilde{\kappa}}\left(b_\nu\tilde{X}_\mu - b_\mu\tilde{X}_\nu\right) - \frac{1}{\kappa}\left(a_\nu\tilde{P}_\mu - a_\mu\tilde{P}_\nu\right)\right), \tag{2.16}$$

$$\left[\tilde{H}, \tilde{X}_\mu\right] = i\left(\frac{1}{M^2}\left(u^2 + a^2\frac{M^2}{\kappa^2}\right)\tilde{P}_\mu - \frac{1}{MR}\left(uv\,\sin\left(\varphi+\psi\right) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)\tilde{X}_\mu - \frac{1}{\kappa}a_\mu\tilde{H}\right), \tag{2.17}$$

$$\left[\tilde{H}, \tilde{P}_\mu\right] = -i\left(\frac{1}{R^2}\left(v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}\right)\tilde{X}_\mu - \frac{1}{MR}\left(uv\,\sin\left(\varphi+\psi\right) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)\tilde{P}_\mu + \frac{1}{\tilde{\kappa}}b_\mu\tilde{H}\right). \tag{2.18}$$

The generalized Born duality[6] still holds for $M \leftrightarrow R, \kappa \leftrightarrow \tilde{\kappa}, a_\mu \to -b_\mu, b_\mu \to a_\mu, u \leftrightarrow v, \varphi \to -\psi, \psi \to -\varphi, \tilde{X}_\mu \to -\tilde{P}_\mu, \tilde{P}_\mu \to \tilde{X}_\mu, M_{\mu\nu} \leftrightarrow M_{\mu\nu}, \tilde{H} \leftrightarrow \tilde{H}, \epsilon_1 \leftrightarrow \epsilon_2$. These commutation relations are similar to those of Refs. 30 and 33. If $\frac{a_\mu}{\kappa} = 0$ and $\frac{b_\mu}{\tilde{\kappa}} = 0$, then the above algebra reduces to that of the Khruschev-Leznov model,[7] with $\rho = uv\sin(\varphi+\psi)$ proportional to their parameter $1/H$ (cf. Ref. 30).

If we define generators

$$M_{\mu 4} = M\hat{x}_\mu, M_{\mu 5} = R\hat{p}_\mu, M_{45} = MR\hat{h} \tag{2.19}$$

then the Yang algebra (2.1)–(2.5) takes the form of an orthogonal algebra $o(1,5)$ with

$$\left[M_{AB}, M_{CD}\right] = i(\eta_{AC}M_{BD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC} - \eta_{BC}M_{AD}), \tag{2.20}$$

where $A, B, C, D = 0, 1, 2, 3, 4, 5, \eta_{AB} = \mathrm{diag}(-1, 1, 1, 1, 1, 1)$.

Furthermore, if we define

$$\tilde{M}_{\mu 4} = M\tilde{X}_\mu, \tilde{M}_{\mu 5} = R\tilde{P}_\mu, \tilde{M}_{45} = MR\tilde{H} \tag{2.21}$$

then the algebra (2.11)–(2.18) becomes the $o(1,5,g)$ algebra with

$$\left[\tilde{M}_{AB}, \tilde{M}_{CD}\right] = i\left(g_{AC}\tilde{M}_{BD} - g_{AD}\tilde{M}_{BC} + g_{BD}\tilde{M}_{AC} - g_{BC}\tilde{M}_{AD}\right), \tag{2.22}$$

where the metric g_{AB} is given by a symmetric matrix defined as

$$\begin{gathered} g_{\mu\nu} = \eta_{\mu\nu}, g_{\mu 4} = \frac{M}{\kappa}a_\mu, g_{\mu 5} = \frac{R}{\tilde{\kappa}}b_\mu, g_{44} = u^2 + a^2\frac{M^2}{\kappa^2}, g_{55} = v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}, \\ g_{45} = g_{54} = uv\,\sin\left(\varphi+\psi\right) + ab\frac{MR}{\kappa\tilde{\kappa}}. \end{gathered} \tag{2.23}$$

It follows that

$$\begin{gathered} \frac{a_\mu}{\kappa} = \frac{1}{M}g_{\mu 4}, \frac{b_\mu}{\tilde{\kappa}} = \frac{1}{R}g_{\mu 5}, u^2 = g_{44} - g_{\mu 4}g_{\mu 4}, v^2 = g_{55} - g_{\mu 5}g_{\mu 5} \quad \text{and} \\ uv\,\sin\left(\varphi+\psi\right) = g_{45} - g_{\mu 4}g_{\mu 5}, \quad \text{which implies that } \sin\left(\varphi+\psi\right) = \frac{1}{uv}\left(g_{45} - g_{\mu 4}g_{\mu 5}\right). \end{gathered} \tag{2.24}$$

Note that

$$\det g = \left(g_{45} - g_{\mu 4}g_{\mu 5}\right)^2 - \left(g_{44} - g_{\mu 4}g_{\mu 4}\right)\left(g_{55} - g_{\nu 5}g_{\nu 5}\right) < 0, \tag{2.25}$$

i.e., using the above formulae for g_{AB} (2.23) we get $\det g = -u^2v^2\cos^2(\varphi+\psi)$ and $g_{AB} = \left(S\eta S^T\right)_{AB}, \det S^2 = -\det g$ with

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{M}{\kappa}a_0 & \frac{M}{\kappa}a_1 & \frac{M}{\kappa}a_2 & \frac{M}{\kappa}a_3 & \sigma & 0 \\ -\frac{R}{\tilde{\kappa}}b_0 & \frac{R}{\tilde{\kappa}}b_1 & \frac{R}{\tilde{\kappa}}b_2 & \frac{R}{\tilde{\kappa}}b_3 & \upsilon & \tau \end{bmatrix}, \tag{2.26}$$

where $\sigma = u, v = v\sin(\varphi + \psi)$ and $\tau = v\cos(\varphi + \psi)$, see (12) in Ref. 30.

The relations between $\tilde{M}_{AB}$ and M_{AB} can be obtained from the relations for $\tilde{X}_\mu$ and $\hat{x}_\mu$, $\tilde{P}_\mu$ and $\hat{p}_\mu$, $\tilde{H}$ and $\hat{h}$ and their inverses, and can be written as $\tilde{M}_{AB} = \left(SMS^T\right)_{AB}$, with the same matrix S as above. We will use the metric g_{AB} for lowering indices and g^{AB} for raising indices, with ${g_A}^B = {g_A}^B = \delta_A^B$ and $g^{AB}g_{BC} = \delta_C^A$, i.e., g^{AB} is the inverse of the matrix g_{AB}.

A. Dual κ-Minkowski spaces and κ-Poincaré algebras isomorphic to $o(1,5,g)$

Dual κ-Minkowski spaces and κ-Poincaré algebras can be defined by imposing $g_{44} = 0$ and $g_{55} = 0$, so that

$$a^2 = -u^2\frac{\kappa^2}{M^2} < 0, \quad b^2 = -v^2\frac{\tilde{\kappa}^2}{R^2} < 0. \tag{2.27}$$

Then it follows

$$\left[\tilde{X}_\mu, \tilde{X}_\nu\right] = \frac{i}{\kappa}\left(a_\mu\tilde{X}_\nu - a_\nu\tilde{X}_\mu\right) \tag{2.28}$$

and

$$\left[\tilde{P}_\mu, \tilde{P}_\nu\right] = \frac{i}{\tilde{\kappa}}\left(b_\mu\tilde{P}_\nu - b_\nu\tilde{P}_\mu\right). \tag{2.29}$$

The relation (2.28) defines κ-Minkowski space, while (2.29) describes the Born-dual $\tilde{\kappa}$-Minkowski space. Taking into account (2.27) and $g_{44} = 0, g_{55} = 0, g_{45} = uv\sin(\varphi + \psi) + abMR\frac{1}{\kappa\tilde{\kappa}}$, we get $\det g = -u^2v^2\cos^2(\varphi + \psi)$, and we demand that $\det g \neq 0$. The determinant depends on the parameters u, v and the angle $\varphi + \psi$, but does not depend on the vectors a_μ, b_μ. If $g_{44} = g_{55} = g_{45} = 0$ and $\det g \neq 0$, then it follows that $a^2 \cdot b^2 > (a \cdot b)^2$ and a_μ and b_μ cannot be collinear.

The algebra generated by $\tilde{X}_\mu, \tilde{P}_\mu, M_{\mu\nu}$ and $\tilde{H}$ containing dual κ-Minkowski spaces and κ-Poincaré algebras (2.11)–(2.18) is constructed from the Yang algebra (2.1)–(2.5) and is isomorphic to the $o(1,5,g)$ algebra. It depends on the parameters M, R, on the time-like four-vectors $\frac{a_\mu}{\kappa}, \frac{b_\mu}{\tilde{\kappa}}\left(a^2 < 0, b^2 < 0\right)$ and on g_{45}, with the condition $\det g < 0$. In the limit when $M \to \infty$, the vector a_μ becomes light-like, $a^2 = 0$. In the limit when $R \to \infty$, the vector b_μ becomes light-like, $b^2 = 0$ and in the limit when $M \to \infty$ and $R \to \infty$, both vectors a_μ, b_μ become light-like. In particular, in the limit when $R \to \infty$ and $\frac{b_\mu}{\tilde{\kappa}} = 0$ we get the κ-Minkowski space with the κ-Poincaré algebra, with $a^2 \leq 0$ and

$$\left[\tilde{X}_\mu, \tilde{X}_\nu\right] = \frac{i}{\kappa}\left(a_\mu\tilde{X}_\nu - a_\nu\tilde{X}_\mu\right), \tag{2.30}$$

$$\left[\tilde{P}_\mu, \tilde{P}_\nu\right] = 0, \tag{2.31}$$

$$\left[\tilde{X}_\mu, \tilde{P}_\nu\right] = i\left(\eta_{\mu\nu}\tilde{H} - \frac{1}{\kappa}a_\nu\tilde{P}_\mu\right), \tag{2.32}$$

$$\left[M_{\mu\nu}, \tilde{X}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{X}_\nu - \eta_{\nu\lambda}\tilde{X}_\mu + \frac{1}{\kappa}\left(a_\mu M_{\lambda\nu} - a_\nu M_{\lambda\mu}\right)\right), \tag{2.33}$$

$$\left[M_{\mu\nu}, \tilde{P}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{P}_\nu - \eta_{\nu\lambda}\tilde{P}_\mu\right), \tag{2.34}$$

$$\left[M_{\mu\nu}, \tilde{H}\right] = \frac{-i}{\kappa}\left(a_\nu\tilde{P}_\mu - a_\mu\tilde{P}_\nu\right), \tag{2.35}$$

$$\left[\tilde{H}, \tilde{X}_\mu\right] = \frac{-i}{\kappa}a_\mu\tilde{H}, \tag{2.36}$$

$$\left[\tilde{H}, \tilde{P}_\mu\right] = 0. \tag{2.37}$$

Alternatively when $M \to \infty$ and $\frac{a_\mu}{\kappa} = 0$, we get the dual κ-Minkowski momentum space with $b^2 \leq 0$. Finally, when $M \to \infty, R \to \infty$ and $\frac{a_\mu}{\kappa} = \frac{b_\mu}{\tilde{\kappa}} = 0$, we get the ordinary Heisenberg algebra, where $\tilde{H}$ is a central element commuting with $\tilde{X}_\mu, \tilde{P}_\mu$ and $M_{\mu\nu}$, implying that $\tilde{H}$ is proportional to the identity operator I. The special case $M = \kappa$ and $R = \tilde{\kappa}$ is described in Ref. 30.

III. GENERALIZED YANG MODELS ISOMORPHIC TO $o(2,4,g)$

In this section we consider the case $\epsilon_1 = 1, \epsilon_2 = -1$, i.e., $\eta_{AB} = \mathrm{diag}(-1,1,1,1,1,-1)$, while the case $\epsilon_1 = -1, \epsilon_2 = 1$ is mentioned at the end of Subsection III A. The Yang algebra isomorphic to $o(2,4)$ is defined as

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{i}{M^2}M_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = \frac{-i}{R^2}M_{\mu\nu}, \tag{3.1}$$

$$[M_{\mu\nu}, \hat{x}_\lambda] = i\big(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu\big), \quad [M_{\mu\nu}, \hat{p}_\lambda] = i\big(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\nu\lambda}\hat{p}_\mu\big), \tag{3.2}$$

$$[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}\hat{h}, \quad \Big[\hat{h}, \hat{x}_\mu\Big] = \frac{i}{M^2}\hat{p}_\mu, \quad \Big[\hat{h}, \hat{p}_\mu\Big] = \frac{i}{R^2}\hat{x}_\mu, \tag{3.3}$$

$$\Big[M_{\mu\nu}, \hat{h}\Big] = 0, \tag{3.4}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}). \tag{3.5}$$

Let us construct new generators $\tilde{X}_\mu$, $\tilde{P}_\mu$

$$\tilde{X}_\mu = u\Big(\cosh\,\varphi\hat{x}_\mu + \frac{R}{M}\sinh\,\varphi\hat{p}_\mu\Big) + \frac{1}{\kappa}a_\rho M_{\mu\rho}, \tag{3.6}$$

$$\tilde{P}_\mu = v\Big(\cosh\,\psi\hat{p}_\mu + \frac{M}{R}\sinh\,\psi\hat{x}_\mu\Big) + \frac{1}{\tilde{\kappa}}b_\rho M_{\mu\rho} \tag{3.7}$$

and

$$\tilde{M}_{\mu\nu} = M_{\mu\nu}. \tag{3.8}$$

The generators $\tilde{X}_\mu, \tilde{P}_\mu, M_{\mu\nu}$, and $\tilde{H}$ generate a new class of Lie algebras isomorphic to the $o(2,4)$ Yang algebra

$$\Big[\tilde{X}_\mu, \tilde{X}_\nu\Big] = i\left(\frac{1}{M^2}\left(u^2 + a^2\frac{M^2}{\kappa^2}\right)M_{\mu\nu} + \frac{1}{\kappa}\big(a_\mu\tilde{X}_\nu - a_\nu\tilde{X}_\mu\big)\right), \tag{3.9}$$

$$\Big[\tilde{P}_\mu, \tilde{P}_\nu\Big] = i\left(\frac{1}{R^2}\left(-v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}\right)M_{\mu\nu} + \frac{1}{\tilde{\kappa}}\big(b_\mu\tilde{P}_\nu - b_\nu\tilde{P}_\mu\big)\right), \tag{3.10}$$

$$\Big[\tilde{X}_\mu, \tilde{P}_\nu\Big] = i\left(\eta_{\mu\nu}\tilde{H} + \frac{1}{\tilde{\kappa}}b_\mu\tilde{X}_\nu - \frac{1}{\kappa}a_\nu\tilde{P}_\mu + \frac{1}{MR}\left(uv\,\sinh\,(\psi-\varphi) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)M_{\mu\nu}\right), \tag{3.11}$$

$$\Big[M_{\mu\nu}, \tilde{X}_\lambda\Big] = i\left(\eta_{\mu\lambda}\tilde{X}_\nu - \eta_{\nu\lambda}\tilde{X}_\mu + \frac{1}{\kappa}\big(a_\mu M_{\lambda\nu} - a_\nu M_{\lambda\mu}\big)\right), \tag{3.12}$$

$$\Big[M_{\mu\nu}, \tilde{P}_\lambda\Big] = i\left(\eta_{\mu\lambda}\tilde{P}_\nu - \eta_{\nu\lambda}\tilde{P}_\mu + \frac{1}{\tilde{\kappa}}\big(b_\mu M_{\lambda\nu} - b_\nu M_{\lambda\mu}\big)\right), \tag{3.13}$$

where

$$\tilde{H} = \hat{h}uv\,\cosh\,(\psi-\varphi) + \frac{1}{\kappa}a\tilde{P} - \frac{1}{\tilde{\kappa}}b\tilde{X} - \frac{1}{\kappa\tilde{\kappa}}a_\rho b_\sigma M_{\rho\sigma} \tag{3.14}$$

and

$$\Big[M_{\mu\nu}, \tilde{H}\Big] = i\left(\frac{1}{\tilde{\kappa}}\big(b_\nu\tilde{X}_\mu - b_\mu\tilde{X}_\nu\big) - \frac{1}{\kappa}\big(a_\nu\tilde{P}_\mu - a_\mu\tilde{P}_\nu\big)\right), \tag{3.15}$$

$$\Big[\tilde{H}, \tilde{X}_\mu\Big] = i\left(\frac{1}{M^2}\left(u^2 + a^2\frac{M^2}{\kappa^2}\right)\tilde{P}_\mu - \frac{1}{MR}\left(uv\,\sinh\,(\psi-\varphi) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)\tilde{X}_\mu - \frac{1}{\kappa}a_\mu\tilde{H}\right), \tag{3.16}$$

$$\Big[\tilde{H}, \tilde{P}_\mu\Big] = -i\left(\frac{1}{R^2}\left(-v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}\right)\tilde{X}_\mu - \frac{1}{MR}\left(uv\,\sinh\,(\psi-\varphi) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)\tilde{P}_\mu + \frac{1}{\tilde{\kappa}}b_\mu\tilde{H}\right). \tag{3.17}$$

The Born duality holds for $M \leftrightarrow R, \kappa \leftrightarrow \tilde{\kappa}, a_\mu \to -b_\mu, b_\mu \to a_\mu, u \leftrightarrow v, \varphi \to -\psi, \psi \to -\varphi, \tilde{X}_\mu \to -\tilde{P}_\mu, \tilde{P}_\mu \to \tilde{X}_\mu, M_{\mu\nu} \leftrightarrow M_{\mu\nu}, \tilde{H} \leftrightarrow \tilde{H}$ and $\epsilon_1 \leftrightarrow \epsilon_2$. If $\frac{a_\mu}{\kappa} = 0$ and $\frac{b_\mu}{\tilde{\kappa}} = 0$ then the above algebra reduces to a Khruschev–Leznov model[7] isomorphic to the $o(2,4)$ algebra, with $\rho = uv\sinh(\psi - \varphi)$. If we define

$$\tilde{M}_{\mu 4} = M\tilde{X}_\mu, \tilde{M}_{\mu 5} = R\tilde{P}_\mu, \tilde{M}_{45} = MR\tilde{H} \tag{3.18}$$

as in (2.21) then the algebra (3.9)–(3.17) becomes an $o(2,4,g)$ algebra and $\Big[\tilde{M}_{AB}, \tilde{M}_{CD}\Big]$ is given in (2.22), with the metric g_{AB} given by a symmetric matrix defined as

$$\begin{gathered} g_{\mu\nu} = \eta_{\mu\nu}, g_{\mu 4} = \frac{M}{\kappa}a_\mu, g_{\mu 5} = \frac{R}{\tilde{\kappa}}b_\mu, g_{44} = u^2 + a^2\frac{M^2}{\kappa^2}, g_{55} = -v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}, \\ \text{and } g_{45} = g_{54} = uv\,\sinh\,(\psi-\varphi) + abMR\frac{1}{\kappa\tilde{\kappa}}, A,B,C,D = 0,1,2,3,4,5. \end{gathered} \tag{3.19}$$

The inverse relations are given by

$$u^2 = g_{44} - g_{\mu 4}g_{\mu 4}, v^2 = g_{\mu 5}g_{\mu 5} - g_{55} \quad \text{and} \quad \sinh\,(\psi-\varphi) = \frac{1}{uv}(g_{45} - g_{\mu 4}g_{\mu 5}). \tag{3.20}$$

Note that

$$\det g = (g_{45} - g_{\mu 4} g_{\mu 5})^2 - (g_{44} - g_{\mu 4} g_{\mu 4})(g_{55} - g_{\nu 5} g_{\nu 5}) \geq 1, \tag{3.21}$$

and $g_{AB} = \left(S\eta S^T\right)_{AB}$, $\det S^2 = \det g$ with

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\frac{M}{\kappa} a_0 & \frac{M}{\kappa} a_1 & \frac{M}{\kappa} a_2 & \frac{M}{\kappa} a_3 & \sigma & 0 \\ -\frac{R}{\tilde{\kappa}} b_0 & \frac{R}{\tilde{\kappa}} b_1 & \frac{R}{\tilde{\kappa}} b_2 & \frac{R}{\tilde{\kappa}} b_3 & \upsilon & \tau \end{bmatrix}, \tag{3.22}$$

where $\sigma = u, \upsilon = v \sinh(\psi - \varphi)$ and $\tau = v\cosh(\psi - \varphi)$.

The relations between $\tilde{M}_{AB}$ and M_{AB} can be written as $\tilde{M}_{AB} = \left(SMS^T\right)_{AB}$ with the matrix S as above.

A. Dual κ-Minkowski spaces and κ-Poincaré algebras isomorphic to $o(2,4,g)$

If we impose $g_{44} = 0$ and $g_{55} = 0$, we obtain

$$a^2 = -u^2 \frac{\kappa^2}{M^2} < 0, \quad b^2 = v^2 \frac{\tilde{\kappa}^2}{R^2} > 0 \tag{3.23}$$

and $\left[\tilde{X}_\mu, \tilde{X}_\nu\right], \left[\tilde{P}_\mu, \tilde{P}_\nu\right]$ are as in Subsection II A. Taking into account (3.23) and $g_{44} = 0, g_{55} = 0, g_{45} = uv \sinh(\psi - \varphi) + abMR\frac{1}{\kappa\tilde{\kappa}}$, we get $\det g = u^2 v^2 \cosh^2(\psi - \varphi) \geq 1$, which depends on the parameters u, v and $\psi - \varphi$, but does not depend on the vectors a_μ, b_μ.

Alternatively if $\epsilon_1 = -1, \epsilon_2 = 1$ we get a new class of algebras isomorphic to $o(2,4,g)$ with a metric

$$g_{\mu\nu} = \eta_{\mu\nu}, g_{\mu 4} = \frac{M}{\kappa} a_\mu, g_{\mu 5} = \frac{R}{\tilde{\kappa}} b_\mu, g_{44} = -u^2 + a^2 \frac{M^2}{\kappa^2}, g_{55} = v^2 + b^2 \frac{R^2}{\tilde{\kappa}^2}, \text{ and}$$
$$g_{45} = g_{54} = -uv \sinh(\psi - \varphi) + ab\frac{MR}{\kappa\tilde{\kappa}}. \tag{3.24}$$

Now if we impose $g_{44} = 0$ and $g_{55} = 0$, we obtain

$$a^2 = u^2 \frac{\kappa^2}{M^2} > 0, \quad b^2 = -v^2 \frac{\tilde{\kappa}^2}{R^2} < 0. \tag{3.25}$$

Taking into account (3.25) and $g_{44} = 0, g_{55} = 0, g_{45} = -uv \sinh(\psi - \varphi) + abMR\frac{1}{\kappa\tilde{\kappa}}$ we get $\det g = u^2 v^2 \cosh^2(\psi - \varphi) \geq 1$.

IV. GENERALIZED YANG MODELS ISOMORPHIC TO $o(3,3,g)$

In this section we consider the case $\epsilon_1 = \epsilon_2 = -1$, i.e., $\eta_{AB} = (-1,1,1,1,-1,-1)$. The Yang algebra isomorphic to $o(3,3)$ is defined as

$$[\hat{x}_\mu, \hat{x}_\nu] = \frac{-i}{M^2} M_{\mu\nu}, \quad [\hat{p}_\mu, \hat{p}_\nu] = \frac{-i}{R^2} M_{\mu\nu}, \tag{4.1}$$

$$[M_{\mu\nu}, \hat{x}_\lambda] = i\left(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu\right), \quad [M_{\mu\nu}, \hat{p}_\lambda] = i\left(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\nu\lambda}\hat{p}_\mu\right), \tag{4.2}$$

$$[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu}\hat{h}, \quad \left[\hat{h}, \hat{x}_\mu\right] = \frac{-i}{M^2}\hat{p}_\mu, \quad \left[\hat{h}, \hat{p}_\mu\right] = \frac{i}{R^2}\hat{x}_\mu, \tag{4.3}$$

$$\left[M_{\mu\nu}, \hat{h}\right] = 0, \tag{4.4}$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}), \tag{4.5}$$

Let us construct the new generators

$$\tilde{X}_\mu = u\left(\cos\,\varphi\hat{x}_\mu + \frac{R}{M}\sin\,\varphi\hat{p}_\mu\right) + \frac{1}{\kappa}a_\rho M_{\mu\rho}, \tag{4.6}$$

$$\tilde{P}_\mu = v\left(\cos\,\psi\hat{p}_\mu + \frac{M}{R}\sin\,\psi\hat{x}_\mu\right) + \frac{1}{\tilde{\kappa}}b_\rho M_{\mu\rho}. \tag{4.7}$$

The generators $\tilde{X}_\mu, \tilde{P}_\mu, M_{\mu\nu}$ and $\tilde{H}$ generate a new class of Lie algebras isomorphic to the $o(3,3)$ algebra

$$\left[\tilde{X}_\mu, \tilde{X}_\nu\right] = i\left(\frac{1}{M^2}\left(-u^2 + a^2\frac{M^2}{\kappa^2}\right)M_{\mu\nu} + \frac{1}{\kappa}\left(a_\mu\tilde{X}_\nu - a_\nu\tilde{X}_\mu\right)\right), \tag{4.8}$$

$$\left[\tilde{P}_\mu, \tilde{P}_\nu\right] = i\left(\frac{1}{R^2}\left(-v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}\right)M_{\mu\nu} + \frac{1}{\tilde{\kappa}}\left(b_\mu\tilde{P}_\nu - b_\nu\tilde{P}_\mu\right)\right), \tag{4.9}$$

$$\left[\tilde{X}_\mu, \tilde{P}_\nu\right] = i\left(\eta_{\mu\nu}\tilde{H} + \frac{1}{\tilde{\kappa}}b_\mu\tilde{X}_\nu - \frac{1}{\kappa}a_\nu\tilde{P}_\mu + \frac{1}{MR}\left(-uv\,\sin(\varphi+\psi) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)M_{\mu\nu}\right), \tag{4.10}$$

$$\left[M_{\mu\nu}, \tilde{X}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{X}_\nu - \eta_{\nu\lambda}\tilde{X}_\mu + \frac{1}{\kappa}\left(a_\mu M_{\lambda\nu} - a_\nu M_{\lambda\mu}\right)\right), \tag{4.11}$$

$$\left[M_{\mu\nu}, \tilde{P}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{P}_\nu - \eta_{\nu\lambda}\tilde{P}_\mu + \frac{1}{\tilde{\kappa}}\left(b_\mu M_{\lambda\nu} - b_\nu M_{\lambda\mu}\right)\right), \tag{4.12}$$

where

$$\tilde{H} = \hat{h}uv\,\cos(\varphi+\psi) + \frac{1}{\kappa}a\tilde{P} - \frac{1}{\tilde{\kappa}}b\tilde{X} - \frac{1}{\kappa\tilde{\kappa}}a_\rho b_\sigma M_{\rho\sigma} \tag{4.13}$$

and

$$\left[M_{\mu\nu}, \tilde{H}\right] = i\left(\frac{1}{\tilde{\kappa}}\left(b_\nu\tilde{X}_\mu - b_\mu\tilde{X}_\nu\right) - \frac{1}{\kappa}\left(a_\nu\tilde{P}_\mu - a_\mu\tilde{P}_\nu\right)\right), \tag{4.14}$$

$$\left[\tilde{H}, \tilde{X}_\mu\right] = i\left(\frac{1}{M^2}\left(-u^2 + a^2\frac{M^2}{\kappa^2}\right)\tilde{P}_\mu - \frac{1}{MR}\left(-uv\,\sin(\varphi+\psi) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)\tilde{X}_\mu - \frac{1}{\kappa}a_\mu\tilde{H}\right), \tag{4.15}$$

$$\left[\tilde{H}, \tilde{P}_\mu\right] = -i\left(\frac{1}{R^2}\left(-v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}\right)\tilde{X}_\mu - \frac{1}{MR}\left(-uv\,\sin(\varphi+\psi) + abMR\frac{1}{\kappa\tilde{\kappa}}\right)\tilde{P}_\mu + \frac{1}{\tilde{\kappa}}b_\mu\tilde{H}\right). \tag{4.16}$$

The Born duality holds as in Secs. II and III. If $\frac{a_\mu}{\kappa} = 0$ and $\frac{b_\mu}{\tilde{\kappa}} = 0$ then the algebra (4.8)–(4.16) reduces to a model of Khruschev-Leznov type[7] isomorphic to an $o(3,3)$ algebra, with $\rho = -uv\sin(\varphi+\psi)$.

If we define

$$\tilde{M}_{\mu 4} = M\tilde{X}_\mu, \tilde{M}_{\mu 5} = R\tilde{P}_\mu, \tilde{M}_{45} = MR\tilde{H} \tag{4.17}$$

then the algebra (4.8)–(4.16) becomes the $o(3,3,g)$ algebra and $\left[\tilde{M}_{AB}, \tilde{M}_{CD}\right]$ is as in Secs. II and III, with the metric g_{AB} given by

$$\begin{gathered} g_{\mu\nu} = \eta_{\mu\nu}, g_{\mu 4} = \frac{M}{\kappa}a_\mu, g_{\mu 5} = \frac{R}{\tilde{\kappa}}b_\mu, g_{44} = -u^2 + a^2\frac{M^2}{\kappa^2}, g_{55} = -v^2 + b^2\frac{R^2}{\tilde{\kappa}^2}, \\ g_{45} = g_{54} = -uv\,\sin(\varphi+\psi) + abMR\frac{1}{\kappa\tilde{\kappa}}, A, B, C, D = 0,1,2,3,4,5. \end{gathered} \tag{4.18}$$

Note that $\det g < 0$ and $\det S^2 = -\det g$ with S given in (2.26) where $\sigma = u, v = v\sin(\varphi+\psi)$ and $\tau = v\cos(\varphi+\psi)$.

A. Dual κ-Minkowski spaces and κ-Poincaré algebras isomorphic to $o(3,3,g)$

When we impose $g_{44} = 0$ and $g_{55} = 0$, we obtain

$$a^2 = u^2\frac{\kappa^2}{M^2} > 0, \quad b^2 = v^2\frac{\tilde{\kappa}^2}{R^2} > 0 \tag{4.19}$$

and we get $\left[\tilde{X}_\mu, \tilde{X}_\nu\right], \left[\tilde{P}_\mu, \tilde{P}_\nu\right]$ as in Subsections II A and III A: then $\left[\tilde{X}_\mu, \tilde{X}_\nu\right]$ defines the κ-Minkowski space and $\left[\tilde{P}_\mu, \tilde{P}_\nu\right]$ the Born dual $\tilde{\kappa}$-Minkowski space. In particular, when $g_{44} = 0, g_{55} = 0, g_{45} = -uv\,\sin(\varphi+\psi) + abMR\frac{1}{\kappa\tilde{\kappa}}$, we get $\det g = -u^2v^2\cos^2(\varphi+\psi)$ and $\det g \neq 0$. The determinant depends on the parameters u, v and $\varphi+\psi$, but not on the vectors a_μ and b_μ. If $g_{44} = 0, g_{55} = 0, g_{45} = 0, \det g < 0$ we get $(ab)^2 - a^2b^2 < 0$, i.e., $a^2b^2 > 0$ and the vectors a_μ and b_μ cannot be collinear.

The algebra generated by $\tilde{X}_\mu, \tilde{P}_\mu, \tilde{H}$ and $M_{\mu\nu}$ containing dual κ-Minkowski spaces is constructed from the Yang algebra and is isomorphic to the $o(3,3,g)$ algebra. It depends on the parameters M, R and space-like vectors $\frac{a_\mu}{\kappa}, \frac{b_\mu}{\tilde{\kappa}}, a^2 > 0, b^2 > 0$, and g_{45}, with $\det g < 0$.

B. Unification of the four cases $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$

All four cases $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ considered in Secs. II–IV can be unified in the following way:

$$\begin{gathered} g_{\mu\nu} = \eta_{\mu\nu}, g_{\mu 4} = \frac{M}{\kappa} a_\mu, g_{\mu 5} = \frac{R}{\tilde{\kappa}} b_\mu, \\ g_{44} = \epsilon_1 u^2 + a^2 \frac{M^2}{\kappa^2}, g_{55} = \epsilon_2 v^2 + b^2 \frac{R^2}{\tilde{\kappa}^2}, \\ g_{45} = g_{54} = \sqrt{\epsilon_1\epsilon_2} uv \sin\left(\sqrt{\epsilon_1\epsilon_2}(\epsilon_1\varphi + \epsilon_2\psi)\right) + abMR\frac{1}{\kappa\tilde{\kappa}}, \\ \det g = -\epsilon_1\epsilon_2 \cos^2\left(\sqrt{\epsilon_1\epsilon_2}(\epsilon_1\varphi + \epsilon_2\psi)\right), \end{gathered} \tag{4.20}$$

where the square roots are taken with the positive sign.

The cases $g_{\mu 4} = g_{\mu 5} = 0, g_{45} \neq 0, \epsilon_1 = \pm 1, \epsilon_2 = \pm 1$ correspond to the Khruschev–Leznov type of models given in Ref. 7. The corresponding algebras containing dual κ-Minkowski spaces and κ-Poincaré algebras are defined as

$$\left[\tilde{X}_\mu, \tilde{X}_\nu\right] = \frac{i}{\kappa}\left(a_\mu \tilde{X}_\nu - a_\nu \tilde{X}_\mu\right), \tag{4.21}$$

$$\left[\tilde{P}_\mu, \tilde{P}_\nu\right] = \frac{i}{\tilde{\kappa}}\left(b_\mu \tilde{P}_\nu - b_\nu \tilde{P}_\mu\right), \tag{4.22}$$

$$\left[\tilde{X}_\mu, \tilde{P}_\nu\right] = i\left(\eta_{\mu\nu}\tilde{H} + \frac{1}{\tilde{\kappa}} b_\mu \tilde{X}_\nu - \frac{1}{\kappa} a_\nu \tilde{P}_\mu + \left(\frac{\sqrt{\epsilon_1\epsilon_2} uv}{MR}\sin\left(\sqrt{\epsilon_1\epsilon_2}(\epsilon_1\varphi + \epsilon_2\psi)\right) + \frac{ab}{\kappa\tilde{\kappa}}\right) M_{\mu\nu}\right), \tag{4.23}$$

$$\left[M_{\mu\nu}, \tilde{X}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{X}_\nu - \eta_{\nu\lambda}\tilde{X}_\mu + \frac{1}{\kappa}\left(a_\mu M_{\lambda\nu} - a_\nu M_{\lambda\mu}\right)\right), \tag{4.24}$$

$$\left[M_{\mu\nu}, \tilde{P}_\lambda\right] = i\left(\eta_{\mu\lambda}\tilde{P}_\nu - \eta_{\nu\lambda}\tilde{P}_\mu + \frac{1}{\tilde{\kappa}}\left(b_\mu M_{\lambda\nu} - b_\nu M_{\lambda\mu}\right)\right), \tag{4.25}$$

$$\left[M_{\mu\nu}, \tilde{H}\right] = i\left(\frac{1}{\tilde{\kappa}}\left(b_\nu \tilde{X}_\mu - b_\mu \tilde{X}_\nu\right) - \frac{1}{\kappa}\left(a_\nu \tilde{P}_\mu - a_\mu \tilde{P}_\nu\right)\right), \tag{4.26}$$

$$\left[\tilde{H}, \tilde{X}_\mu\right] = -i\left(\left(\frac{\sqrt{\epsilon_1\epsilon_2}}{MR} uv \sin\left(\sqrt{\epsilon_1\epsilon_2}(\epsilon_1\varphi + \epsilon_2\psi)\right) + \frac{ab}{\kappa\tilde{\kappa}}\right)\tilde{X}_\mu + \frac{1}{\kappa} a_\mu \tilde{H}\right), \tag{4.27}$$

$$\left[\tilde{H}, \tilde{P}_\mu\right] = i\left(\left(\frac{\sqrt{\epsilon_1\epsilon_2}}{MR} uv \sin\left(\sqrt{\epsilon_1\epsilon_2}(\epsilon_1\varphi + \epsilon_2\psi)\right) + \frac{ab}{\kappa\tilde{\kappa}}\right)\tilde{P}_\mu - \frac{1}{\tilde{\kappa}} b_\mu \tilde{H}\right), \tag{4.28}$$

where

$$\tilde{H} = \hat{h}\det S + \frac{1}{\kappa} a\tilde{P} - \frac{1}{\tilde{\kappa}} b\tilde{X} - \frac{1}{\kappa\tilde{\kappa}} a_\rho b_\sigma M_{\rho\sigma} \tag{4.29}$$

and

$$a^2 = -\epsilon_1 u^2 \frac{\kappa^2}{M^2}, \quad b^2 = -\epsilon_2 v^2 \frac{\tilde{\kappa}^2}{R^2}. \tag{4.30}$$

Generally if $\varphi = \psi = 0$ then the algebra (4.8)–(4.16) depends only on the parameters $\frac{a_\mu}{\kappa}$ and $\frac{b_\mu}{\tilde{\kappa}}$. The case $a^2 < 0$, $b^2 < 0$ corresponds to $\epsilon_1 = \epsilon_2 = 1$ and is related to the $o(1,5)$ algebra. The cases $a^2 < 0, b^2 > 0$ and $a^2 > 0, b^2 < 0$ correspond to $\epsilon_1 = 1, \epsilon_2 = -1$ and $\epsilon_1 = -1, \epsilon_2 = 1$, respectively, and are related to the $o(2,4)$ algebra. The case $a^2 > 0, b^2 > 0$, that corresponds to $\epsilon_1 = \epsilon_2 = -1$, is related to the $o(3,3)$ algebra. The case $a^2 = 0$ implies $M \to \infty$ i.e. $[\hat{x}_\mu, \hat{x}_\nu] = 0$ and the case $b^2 = 0$ implies $R \to \infty$ i.e., $[\hat{p}_\mu, \hat{p}_\nu] = 0$.

In particular, if $\tilde{X}_\mu = \hat{x}_\mu + \frac{1}{\kappa} a_\rho M_{\mu\rho}, \tilde{P}_\mu = \hat{p}_\mu, \tilde{H} = \hat{h} + \frac{1}{\kappa} a\hat{p}$ i.e., $u = v = 1, \varphi = \psi = 0$ and $\frac{b_\mu}{\tilde{\kappa}} = 0$, the above algebra becomes the algebra (2.30)–(2.37). This algebra coincides with the construction of the natural realizations for the κ-Poincaré algebra presented in Ref. 35. In addition, if $\frac{a_\mu}{\kappa} = 0$ we get the ordinary Heisenberg algebra with the Lorentz algebra, where $\tilde{H}$ is a central element commuting with $\tilde{X}_\mu, \tilde{P}_\mu$ and $M_{\mu\nu}$ and is proportional to the identity operator I.

V. WEYL REALIZATION FOR GENERALIZED YANG MODEL AND DUAL κ-MINKOWSKI SPACES

In this section we consider Weyl realization for orthogonal algebras $o(1,5), o(2,4)$, $o(3,3)$ and $o(1,5,g), o(2,4,g), o(3,3,g)$ and for dual κ-Minkowski spaces, using a formalism analogue to that of Refs. 27 and 28.

A. Weyl realizations for $o(1,5), o(2,4), o(3,3)$ algebras

Let us consider the algebra defined as

$$[M_{AB}, M_{CD}] = iC_{AB,CD}{}^{EF} M_{EF} = i(\eta_{AC}M_{BD} - \eta_{AD}M_{BC} + \eta_{BD}M_{AC} - \eta_{BC}M_{AD}), \tag{5.1}$$

where $\eta_{AB} = \mathrm{diag}(-1,1,1,1,\epsilon_1,\epsilon_2)$, corresponding to the algebra $o(1,5)$ for $\epsilon_1 = \epsilon_2 = 1$, to $o(2,4)$ for $\epsilon_1 = 1, \epsilon_2 = -1$ or $\epsilon_1 = -1, \epsilon_2 = 1$, and to $o(3,3)$ for $\epsilon_1 = \epsilon_2 = -1$. The definition (5.1) implies

$$C_{AB,CD}{}^{EF} = \frac{1}{2}\Big[-\eta_{BC}\big(\delta_A{}^E\delta_D{}^F - \delta_A{}^F\delta_D{}^E\big) + \eta_{AD}\big(\delta_C{}^E\delta_B{}^F - \delta_C{}^F\delta_B{}^E\big) - (A \leftrightarrow B)\Big]. \tag{5.2}$$

Starting with the generalized Heisenberg algebra defined with the commutative coordinates x_{AB} and their corresponding momenta k^{AB}

$$[x_{AB}, x_{CD}] = \big[k^{AB}, k^{CD}\big] = 0, \quad \big[x_{AB}, k^{CD}\big] = i\big(\delta_A{}^C\delta_B{}^D - \delta_A{}^D\delta_B{}^C\big), \tag{5.3}$$

where $k^{CD} = \eta^{CM}\eta^{DN}k_{MN}$ and using the general Weyl realization of a Lie algebras corresponding to the Weyl symmetric ordering[29,36] we express the generators M_{AB} as

$$M_{AB} = x_{CD}\left(\frac{\mathbf{C}}{1 - e^{-\mathbf{C}}}\right)_{AB}{}^{CD}, \tag{5.4}$$

where $\mathbf{C}_{AB}{}^{CD} = -\frac{1}{2}C_{AB,EF}{}^{CD}k^{EF}$.

The structure constants $C_{AB,EF}{}^{CD}$ are multiplied with $\hbar$, in our convention $\hbar = 1$ and in the classical limit when $\hbar = 0$ all the generators commute. The Weyl realization of M_{AB} in terms of the generalized Heisenberg algebra generated with x_{AB} and k^{AB} reads up to second order,

$$M_{AB} = x_{AB} + \frac{1}{2}x_{CD}\mathbf{C}_{AB}{}^{CD} + \frac{1}{12}x_{CD}\big(\mathbf{C}^2\big)_{AB}{}^{CD} \tag{5.5}$$

where

$$\mathbf{C}_{AB}{}^{EF} = \frac{1}{2}\big(\delta_A{}^E k_B{}^F + \delta_B{}^F k_A{}^E - (E \leftrightarrow F)\big), \tag{5.6}$$

$$\big(\mathbf{C}^2\big)_{AB}{}^{EF} = \frac{1}{2}\big(2k_A{}^E k_B{}^F + \delta_B{}^F k_{AC}k^{CE} + \delta_A{}^E k_{BC}k^{CF} - (E \leftrightarrow F)\big), \tag{5.7}$$

and the indices are lowered and raised by the means of the metric η_{AB} and η^{AB}, respectively. Inserting $\mathbf{C}$ in (5.5) we find up to first order

$$M_{AB} = x_{AB} + \frac{1}{2}\big(x_{AE}k_B{}^E - x_{BE}k_A{}^E\big) \tag{5.8}$$

and

$$\big[M_{AB}, k^{CD}\big] = i\big(\delta_A{}^C\delta_B{}^D - \delta_A{}^D\delta_B{}^C\big) + \frac{i}{2}\big(\delta_A{}^C k_B{}^D - \delta_B{}^C k_A{}^D + \delta_B{}^D k_A{}^C - \delta_A{}^D k_B{}^C\big). \tag{5.9}$$

We can write x_{AB} and k^{AB} in the terms of the four-dimensional variables

$$k^{\mu 4} = \frac{q^\mu}{M}, \quad k^{\mu 5} = \frac{y^\mu}{R}, \quad k^{45} = \frac{w}{MR}, \tag{5.10}$$

$$x_{\mu 4} = Mx_\mu, \quad x_{\mu 5} = Rp_\mu, \quad x_{45} = MRh, \tag{5.11}$$

so that

$$[x_\mu, q_\nu] = i\delta_\mu^\nu, \quad [p_\mu, y_\nu] = i\delta_\mu^\nu, \quad [h, w] = i. \tag{5.12}$$

B. Weyl realizations for $o(1,5,g), o(2,4,g)$ and $o(3,3,g)$ algebras

The algebras $o(1,5,g), o(2,4,g)$ and $o(3,3,g)$ generated by the $\tilde{M}_{AB}$ are defined as

$$\big[\tilde{M}_{AB}, \tilde{M}_{CD}\big] = i\tilde{C}_{AB,CD}{}^{EF}\tilde{M}_{EF} = i\big(g_{AC}\tilde{M}_{BD} - g_{AD}\tilde{M}_{BC} + g_{BD}\tilde{M}_{AC} - g_{BC}\tilde{M}_{AD}\big), \tag{5.13}$$

with $g_{AB} = \big(S\eta S^T\big)_{AB}$, corresponding to $o(1,5,g), o(2,4,g), o(3,3,g)$. As in Subsection V A, starting with the generalized Heisenberg algebra defined with the commutative coordinates X_{AB} and their corresponding momenta K^{AB}

$$[X_{AB}, X_{CD}] = \big[K^{AB}, K^{CD}\big] = 0, \quad \big[X_{AB}, K^{CD}\big] = i\big(\delta_A{}^C\delta_B{}^D - \delta_A{}^D\delta_B{}^C\big), \tag{5.14}$$

where $K^{CD} = g^{CM} g^{DN} K_{MN}$, and using the general Weyl realization of the Lie algebras corresponding to the Weyl symmetric ordering[29,36] we express the generators $\tilde{M}_{AB}$ as

$$\tilde{M}_{AB} = X_{CD} \left(\frac{\tilde{\mathbf{C}}}{1 - e^{-\tilde{\mathbf{C}}}} \right)_{AB}^{\ \ CD}, \tag{5.15}$$

where

$$\tilde{\mathbf{C}}_{AB}^{\ \ CD} = -\frac{1}{2} \tilde{C}_{AB,EF}^{\ \ CD} K^{EF} \tag{5.16}$$

and

$$\tilde{C}_{AB,EF}^{\ \ CD} = \frac{1}{2} \left[-g_{BE} \left(\delta_A^{\ C} \delta_F^{\ D} - \delta_A^{\ D} \delta_F^{\ C} \right) + g_{AF} \left(\delta_E^{\ C} \delta_B^{\ D} - \delta_E^{\ D} \delta_B^{\ C} \right) - (A \leftrightarrow B) \right]. \tag{5.17}$$

Note that $\tilde{M}_{AB} = \left(SMS^T \right)_{AB}, X_{AB} = \left(SxS^T \right)_{AB}, K^{AB} = \left(S^{\ddagger} kS^{-1} \right)^{AB}$, where $S^{\ddagger} = \left(S^{-1} \right)^T$. The Weyl realization of $\tilde{M}_{AB}$ in the terms of the generalized Heisenberg algebra in terms of X_{AB} and K^{AB} reads up to second order,

$$\tilde{M}_{AB} = X_{AB} + \frac{1}{2} X_{CD} \tilde{\mathbf{C}}_{AB}^{\ \ CD} + \frac{1}{12} X_{CD} \left(\tilde{\mathbf{C}}^2 \right)_{AB}^{\ \ CD}, \tag{5.18}$$

where $\tilde{\mathbf{C}}_{AB}^{\ \ CD}$ and $\left(\tilde{\mathbf{C}}^2 \right)_{AB}^{\ \ CD}$ are as in (5.6) and (5.7), respectively, with $k \to K$. The indices are lowered and raised with the metric g_{AB} and g^{AB}, respectively. Inserting $\mathbf{C}$ in (5.18) we find up to first order

$$\tilde{M}_{AB} = X_{AB} + \frac{1}{2} \left(X_{AE} K_B^{\ E} - X_{BE} K_A^{\ E} \right) \tag{5.19}$$

and

$$\left[\tilde{M}_{AB}, K^{CD} \right] = i \left(\delta_A^{\ C} \delta_B^{\ D} - \delta_A^{\ D} \delta_B^{\ C} \right) + \frac{i}{2} \left(\delta_A^{\ C} K_B^{\ D} - \delta_B^{\ C} K_A^{\ D} + \delta_B^{\ D} K_A^{\ C} - \delta_A^{\ D} K_B^{\ C} \right). \tag{5.20}$$

The Weyl realization $\tilde{M}^{AB}$ given in (5.15)–(5.17) enjoys the property

$$e^{\frac{1}{2} t^{AB} \tilde{M}^{AB}} \triangleright 1 = e^{\frac{1}{2} t^{AB} X^{AB}}, \tag{5.21}$$

where t_{AB} are real numbers transforming as tensors, and the action $\triangleright$ of phase space operators on functions f of the coordinates X_{AB} is defined as

$$X_{AB} \triangleright f(X_{EF}) = X_{AB} f(X_{EF}), \quad K^{AB} \triangleright f(X_{EF}) = -i \frac{\partial f(X_{EF})}{\partial X_{AB}} = \left[K^{AB}, f(X_{EF}) \right]. \tag{5.22}$$

In particular

$$X_{AB} \triangleright 1 = X_{AB}, K_{AB} \triangleright 1 = 0, \tag{5.23}$$

$$K^{AB} \triangleright e^{\frac{1}{2} t^{EF} X^{EF}} = t^{AB} e^{\frac{1}{2} t^{EF} X^{EF}}. \tag{5.24}$$

Also, as in Subsection V A, we can write the generators X_{AB} and K^{AB} in terms of the four-dimensional variables

$$K^{\mu 4} = \frac{Q^\mu}{M}, \quad K^{\mu 5} = \frac{Y^\mu}{R}, \quad K^{45} = \frac{W}{MR}, \tag{5.25}$$

$$X_{\mu 4} = M X_\mu, \quad X_{\mu 5} = R P_\mu, \quad X_{45} = MRH, \tag{5.26}$$

so that

$$[X_\mu, Q_\nu] = i\delta_\mu^\nu, \quad [P_\mu, Y_\nu] = i\delta_\mu^\nu, \quad [H, W] = i. \tag{5.27}$$

C. Weyl realization for dual κ Minkowski spaces

The generators $\tilde{M}_{AB}$ can be written in terms of the four-dimensional variables with Greek indices ($\tilde{M}_{\mu\nu}, \tilde{X}_\mu, \tilde{P}_\mu, \tilde{H}$, expressed in terms of $M_{\mu\nu}, X_\mu, P_\mu, H$, and $K_{\mu\nu}, Q_\mu, Y_\mu, W$). They are

$$\tilde{M}_{\mu\nu} = X_{\mu\nu} + \frac{1}{2} \Bigg(X_{\mu\alpha} \left(K_\nu^{\ \alpha} - \frac{1}{M} g_{\nu 4} Q^\alpha - \frac{1}{R} g_{\nu 5} Y^\alpha \right) + X_\mu \left(Q_\nu - \frac{1}{R} g_{\nu 5} W \right) + P_\mu \left(Y_\nu + \frac{1}{M} g_{\nu 4} W \right) - (u \leftrightarrow \nu) \Bigg), \tag{5.28}$$

$$\tilde{X}_\mu = X_\mu + \frac{1}{2}\Big(-\frac{1}{M}X_{\mu\nu}\Big(g_{\alpha 4}K^{\nu\alpha} + \frac{1}{M}g_{44}Q^\nu + \frac{1}{R}g_{45}Y^\nu\Big) + \frac{1}{M}X_\mu\Big(g_{\nu 4}Q^\nu - \frac{1}{R}g_{45}W\Big)$$
$$+ X_\nu\Big(K_\mu{}^\nu - \frac{1}{M}g_{\mu 4}Q^\nu - \frac{1}{R}g_{\mu 5}Y^\nu\Big) + \frac{1}{M}P_\mu\Big(g_{\nu 4}Y^\nu + \frac{1}{M}g_{44}W\Big) - H\Big(Y_\mu + \frac{1}{M}g_{\mu 4}W\Big)\Big), \tag{5.29}$$

$$\tilde{P}_\mu = P_\mu + \frac{1}{2}\Big(-\frac{1}{R}X_{\mu\nu}\Big(g_{\alpha 5}K^{\nu\alpha} + \frac{1}{M}g_{45}Q^\nu + \frac{1}{R}g_{55}Y^\nu\Big) + \frac{1}{R}P_\mu\Big(g_{\nu 5}Y^\nu + \frac{1}{M}g_{45}W\Big)$$
$$+ P_\nu\Big(K_\mu{}^\nu - \frac{1}{R}g_{\mu 5}Y^\nu - \frac{1}{M}g_{\mu 4}Q^\nu\Big) + \frac{1}{R}X_\mu\Big(g_{\nu 5}Q^\nu - \frac{1}{R}g_{55}W\Big) + H\Big(Q_\mu - \frac{1}{R}g_{\mu 5}W\Big)\Big), \tag{5.30}$$

$$\tilde{H} = H + \frac{1}{2}\Big(\frac{1}{R}X_\mu\Big(g_{\nu 5}K^{\mu\nu} + \frac{1}{M}g_{45}Q^\mu + \frac{1}{R}g_{55}Y^\mu\Big)$$
$$- \frac{1}{M}P_\mu\Big(\frac{1}{M}g_{44}Q^\mu + \frac{1}{R}g_{45}Y^\mu + g_{\nu 4}K^{\mu\nu}\Big) + H\Big(\frac{1}{R}g_{\mu 5}Y^\mu + \frac{1}{M}g_{\mu 4}Q^\mu\Big)\Big). \tag{5.31}$$

For the dual κ-Minkowski spaces $g_{44} = 0, g_{55} = 0$, see (4.21)–(4.30). The other realizations can be obtained from the Weyl realizations of $\tilde{M}^W_{AB}$ by using similarity transformation

$$\tilde{M}_{AB} = \mathcal{S}\tilde{M}^W_{AB}\mathcal{S}^{-1}, \tag{5.32}$$

where $\mathcal{S} = \exp(G)$, with $G = XF(K)$, $F(K)$ being a function of K. The corresponding realizations will be linear in X and can be written as a series in K. The corresponding coproducts, star products and twists can be obtained by using this similarity transformation.[8]

VI. COPRODUCT, STAR PRODUCT, AND TWIST IN WEYL REALIZATION

Formulae for coproduct and deformed addition of momenta for the generalized Yang model and dual κ-Minkowski spaces are written in terms of the curved metric g_{AB} and are deduced using results of Refs. 27, 28, and 36. The corresponding coproducts for the momenta, $\Delta K^{\mu\nu}, \Delta Q^\mu, \Delta Y^\mu, \Delta W$ are coassociative, see Ref. 14. Defining

$$\exp\Big(\frac{i}{2}s^{AB}\tilde{M}_{AB}\Big)\exp\Big(\frac{i}{2}t^{CD}\tilde{M}_{CD}\Big) = \exp\Big(\frac{i}{2}\big(s^{AB} \oplus t^{AB}\big)\tilde{M}_{AB}\Big) = \exp\Big(\frac{i}{2}\mathcal{D}^{AB}(s,t)\tilde{M}_{AB}\Big), \tag{6.1}$$

where s^{AB} and t^{AB} transform as tensors of corresponding orthogonal algebra. One has at first order

$$\mathcal{D}^{AB}\big(s^{CD}, t^{CD}\big) = s^{AB} + t^{AB} - \frac{1}{2}\big(s^{AC}t^B{}_C - s^{BC}t^A{}_C\big). \tag{6.2}$$

In the following all formulae are written in the first order. The coproduct ΔK^{AB} is

$$\Delta K^{AB} = \mathcal{D}^{AB}\big(K^{AB}\otimes 1, 1\otimes K^{AB}\big) = \Delta_0 K^{AB} - \frac{1}{2}\big(K^{AC}\otimes K^B{}_C - K^{BC}\otimes K^A{}_C\big), \tag{6.3}$$

where $\Delta_0 K^{AB} = K^{AB}\otimes 1 + 1\otimes K^{AB}$. In components, it reads

$$\Delta K^{\mu\nu} = \Delta_0 K^{\mu\nu} - \frac{1}{2}\Big(K^{\mu\alpha}\otimes K_\alpha{}^\nu + \frac{1}{M^2}g_{44}Q^\mu\otimes Q^\nu + \frac{1}{R^2}g_{55}Y^\mu\otimes Y^\nu$$
$$+ \frac{1}{MR}g_{45}\big(Q^\mu\otimes Y^\nu + Y^\mu\otimes Q^\nu\big) + \frac{1}{M}g_{\alpha 4}\big(K^{\mu\alpha}\otimes Q^\nu + Q^\mu\otimes K^{\nu\alpha}\big)$$
$$+ \frac{1}{R}g_{\alpha 5}\big(K^{\mu\alpha}\otimes Y^\nu + Y^\mu\otimes K^{\nu\alpha}\big) - (u\leftrightarrow\nu)\Big), \tag{6.4}$$

$$\Delta Q^\mu = \Delta_0 Q^\mu - \frac{1}{2}\Big(-K^{\mu\alpha}\otimes Q_\alpha + Q_\alpha\otimes K^{\mu\alpha} + \frac{1}{MR}g_{45}\big(Q^\mu\otimes W - W\otimes Q^\mu\big)$$
$$+ \frac{1}{R^2}g_{55}\big(Y^\mu\otimes W - W\otimes Y^\mu\big) + \frac{1}{R}g_{\alpha 5}\big(K^{\mu\alpha}\otimes W - W\otimes K^{\mu\alpha}\big)$$
$$- \frac{1}{M}g_{\alpha 4}\big(Q^\mu\otimes Q^\alpha - Q^\alpha\otimes Q^\mu\big) - \frac{1}{R}g_{\alpha 5}\big(Y^\mu\otimes Q^\alpha - Q^\alpha\otimes Y^\mu\big)\Big), \tag{6.5}$$

$$\Delta Y^\mu = \Delta_0 Y^\mu - \frac{1}{2}\Big(-K^{\mu\alpha}\otimes Y_\alpha + Y_\alpha\otimes K^{\mu\alpha} - \frac{1}{MR}g_{45}\big(Y^\mu\otimes W - W\otimes Y^\mu\big)$$

$$+\frac{1}{M^2}g_{44}\big(W\otimes Q^\mu-Q^\mu\otimes W\big)-\frac{1}{M}g_{\alpha4}\big(K^{\mu\alpha}\otimes W-W\otimes K^{\mu\alpha}\big)$$
$$-\frac{1}{M}g_{\alpha4}\big(Q^\mu\otimes Y^\alpha-Y^\alpha\otimes Q^\mu\big)-\frac{1}{R}g_{\alpha5}\big(Y^\mu\otimes Y^\alpha-Y^\alpha\otimes Y^\mu\big)\bigg), \tag{6.6}$$

$$\Delta W=\Delta_0 W-\frac{1}{2}\bigg(Q^\alpha\otimes Y_\alpha-Y^\alpha\otimes Q_\alpha+\frac{1}{R}g_{\alpha5}\big(Y^\alpha\otimes W-W\otimes Y^\alpha\big)$$
$$+\frac{1}{M}g_{\alpha4}\big(Q^\alpha\otimes W-W\otimes Q^\alpha\big)\bigg). \tag{6.7}$$

The antipodes of $K^{\mu\nu}, Q^\mu, Y^\mu$ and W are trivial. The star product is defined as

$$e^{\frac{1}{2}s^{AB}X_{AB}}\star e^{\frac{1}{2}t^{CD}X_{CD}}=e^{\frac{1}{2}\mathcal{D}^{AB}(s,t)X_{AB}}, \tag{6.8}$$

where $\mathcal{D}^{AB}(s,t)$ is given in (6.2). This star product is associative.

Defining $\mathcal{D}^\mu=\mathcal{D}^{\mu4}$, $\bar{\mathcal{D}}^\mu=\mathcal{D}^{\mu5}$, $\mathcal{D}=\mathcal{D}^{45}$, the four-dimensional expression of $\mathcal{D}^{AB}(s,t)$ is given by

$$\mathcal{D}^{\mu\nu}(s,t)=s^{\mu\nu}+t^{\mu\nu}-\frac{1}{2}\bigg(s^{\mu\rho}t^\nu{}_\rho+\frac{1}{M^2}g_{44}s^\mu t^\nu+\frac{1}{R^2}g_{55}\bar{s}^\mu\bar{t}^\nu+\frac{1}{MR}g_{45}\big(s^\mu\bar{t}^\nu+\bar{s}^\mu t^\nu\big)$$
$$+\frac{1}{M}g_{\mu4}\big(s^{\mu\rho}t^\nu+s^\mu t^{\nu\rho}\big)+\frac{1}{R}g_{\mu5}\big(s^{\mu\rho}\bar{t}^\nu+\bar{s}^\mu t^{\nu\rho}\big)-(u\leftrightarrow v)\bigg), \tag{6.9}$$

$$\mathcal{D}^\mu(s,t)=s^\mu+t^\mu-\frac{1}{2}\bigg(s^{\mu\rho}t_\rho-t^{\mu\rho}s_\rho+\frac{1}{MR}g_{45}\big(s^\mu t-st^\mu\big)+\frac{1}{R^2}g_{55}\big(\bar{s}^\mu t-s\bar{t}^\mu\big)+\frac{1}{R}g_{\mu5}\big(s^{\mu\rho}t-st^{\mu\rho}\big)\bigg), \tag{6.10}$$

$$\bar{\mathcal{D}}^\mu(s,t)=\bar{s}^\mu+\bar{t}^\mu-\frac{1}{2}\bigg(s^{\mu\rho}\bar{t}_\rho-\bar{s}_\rho t^{\mu\rho}-\frac{1}{MR}g_{45}\big(\bar{s}^\mu t-s\bar{t}^\mu\big)+\frac{1}{M^2}g_{44}\big(s^\mu t-st^\mu\big)-\frac{1}{M}g_{\mu4}\big(s^{\mu\rho}t-st^{\mu\rho}\big)\bigg), \tag{6.11}$$

$$\mathcal{D}(s,t)=s+t-\frac{1}{2}\big(s^\rho\bar{t}_\rho-\bar{s}^\rho t_\rho\big), \tag{6.12}$$

where we have defined the components of tensors t^{AB} corresponding to the orthogonal algebra as $t^\mu=t^{\mu4}, \bar{t}^\mu=t^{\mu5}, t=t^{45}$ and analogously for s^{AB}.

Now, we construct the twist operator at first order as in Refs. 27 and 36. The twist is defined as a bilinear operator such that

$$\Delta K^{AB}=\mathscr{F}\big(\Delta_0 K^{AB}\big)\mathscr{F}^{-1} \tag{6.13}$$

for momenta K^{AB} belonging to the corresponding orthogonal algebra. The twist was introduced in the context of the NC geometry in Refs. 37 and 38 as a useful tool in construction of quantum field theories. In a Hopf algebroid approach[39–41] the twist can be written as

$$\mathscr{F}^{-1}\equiv e^F=e^{-\frac{i}{2}K^{AB}\otimes X_{AB}}e^{\frac{i}{2}K^{CD}\otimes\tilde{M}_{CD}}. \tag{6.14}$$

Using the BCH formula one gets

$$F=\frac{i}{2}K^{AB}\otimes\big(\tilde{M}_{AB}-X_{AB}\big) \tag{6.15}$$

and substituing (5.19) into (6.15) one has

$$F=\frac{i}{2}K^{AC}\otimes X_{AB}K_C{}^B. \tag{6.16}$$

In terms of components, one can write

$$F=\frac{i}{2}\bigg(K^{\mu\nu}\otimes\bigg[X_{\mu\alpha}\bigg(K_\nu{}^\alpha-\frac{1}{M}g_{\nu4}Q^\alpha-\frac{1}{R}g_{\nu5}Y^\alpha\bigg)+X_\mu\bigg(Q_\nu-\frac{1}{R}g_{\nu5}W\bigg)+P_\mu\bigg(Y_\nu+\frac{1}{M}g_{\nu4}W\bigg)\bigg]$$
$$+Q^\mu\otimes\bigg[-\frac{1}{M}X_{\mu\nu}\bigg(g_{\alpha4}K^{\nu\alpha}+\frac{1}{M}g_{44}Q^\nu+\frac{1}{R}g_{45}Y^\nu\bigg)+\frac{1}{M}X_\mu\bigg(g_{\alpha4}Q^\alpha-\frac{1}{R}g_{45}W\bigg)+\frac{1}{M}P_\mu\bigg(g_{\alpha4}Y^\alpha+\frac{1}{M}g_{44}W\bigg)\bigg]$$
$$+X_\nu\bigg(K_\mu{}^\nu-\frac{1}{M}g_{\mu4}Q^\nu-\frac{1}{R}g_{\nu5}Y^\nu\bigg)+H\bigg(Y_\mu+\frac{1}{M}g_{\mu4}W\bigg)\bigg]$$
$$+P^\mu\otimes\bigg[-\frac{1}{R}X_{\mu\nu}\bigg(g_{\alpha5}K^{\nu\alpha}+\frac{1}{M}g_{45}Q^\nu+\frac{1}{R}g_{55}Y^\nu\bigg)+\frac{1}{R}X_\mu\bigg(g_{\alpha5}Q^\alpha-\frac{1}{R}g_{55}W\bigg)+\frac{1}{R}P_\mu\bigg(g_{\alpha5}Y^\alpha+\frac{1}{M}g_{45}W\bigg)\bigg]$$
$$+P_\nu\bigg(K_\mu{}^\nu-\frac{1}{M}g_{\mu4}Q^\nu-\frac{1}{R}g_{\nu5}Y^\nu\bigg)+H\bigg(Q_\mu-\frac{1}{R}g_{\mu5}W\bigg)\bigg]$$

$$+ W \otimes \left[\frac{1}{R} X_\mu \left(g_{v5} K^{\mu\nu} + \frac{1}{M} g_{45} Q^\mu + \frac{1}{R} g_{55} Y^\mu \right) - \frac{1}{M} P_\mu \left(g_{v4} K^{\mu\nu} + \frac{1}{M} g_{44} Q^\mu + \frac{1}{R} g_{45} Y^\mu \right) + H \left(\frac{1}{R} g_{\mu 5} Y^\mu + \frac{1}{M} g_{\mu 4} Q^\mu \right) \right] \Big). \tag{6.17}$$

The coproducts, star product and twist for the Yang model for different orthogonal algebras can be derived from the above results setting $u = v = 1, \varphi = \psi = 0$ and $a_\mu = b_\mu = 0$.

VII. REDUCED DUAL κ-MINKOWSKI SPACES AND κ-POINCARÉ ALGEBRAS

Let us now consider the reduced Yang models, introduced in Ref. 34. If the quadratic Casimir operator of the orthogonal algebra (2.20), $\frac{1}{2} M_{AB} M^{AB}$ with metric η_{AB}, is restricted to a constant value $\epsilon_1 \epsilon_2 M^2 R^2$, then the generator M_{45} can be expressed in terms of the other 14 generators. Consequently, for a Yang algebra (2.1)–(2.5), one can define a nonlinear algebra with 14 generators, called reduced Yang algebra, with

$$\hat{h} = \sqrt{1 - \frac{\epsilon_2}{R^2} \hat{x}_\mu \hat{x}^\mu - \frac{\epsilon_1}{M^2} \hat{p}_\mu \hat{p}^\mu - \frac{\epsilon_1 \epsilon_2}{2 M^2 R^2} M_{\alpha\beta} M^{\alpha\beta}}. \tag{7.1}$$

It has been shown in Ref. 34, that this expression for $\hat{h}$ satisfies the relations (2.3) and (2.4) and hence all the Yang algebra commutation relations (2.1)–(2.5) hold.

Following the construction in Secs. II–IV, we can define $\tilde{H}$, $\tilde{X}_\mu$, $\tilde{P}_\mu$, and $M_{\mu\nu}$, using the above expression of $\hat{h}$ (7.1). For the dual κ-Minkowski space and κ-Poincaré algebra, using the results of Subsection IV B we obtain the relations for $\tilde{H}$, $\tilde{X}_\mu$, $\tilde{P}_\mu$, $M_{\mu\nu}$ in the terms of $\hat{x}_\mu$, $\hat{p}_\mu$, $\hat{h}$, $M_{\mu\nu}$. The relation (4.29) for $\tilde{H}$ is the same but with $\hat{h}$ (7.1) expressed in the terms of $\tilde{X}_\mu$, $\tilde{P}_\mu$, $M_{\mu\nu}$.

In the special case $u = v = 1$, $\phi = \psi = 0$,

$$\tilde{X}_\mu = \hat{x}_\mu + \frac{1}{\kappa} a_\rho M_{\mu\rho}, \qquad \tilde{P}_\mu = \hat{p}_\mu + \frac{1}{\tilde{\kappa}} b_\rho M_{\mu\rho}, \tag{7.2}$$

$$\tilde{H} = \hat{h} + \frac{1}{\kappa} a \cdot \tilde{P} - \frac{1}{\tilde{\kappa}} b \cdot \tilde{X} - \frac{1}{\kappa \tilde{\kappa}} a_\rho b_\sigma M_{\rho\sigma}, \tag{7.3}$$

taking into account that in (7.1) the following changes should be made

$$-\frac{\epsilon_1}{M^2} = \frac{a^2}{\kappa^2}, -\frac{\epsilon_2}{R^2} = \frac{b^2}{\tilde{\kappa}^2}. \tag{7.4}$$

Relations (4.26)–(4.28) are still valid. Moreover, in the limit $\tilde{\kappa} \to \infty$ we obtain

$$\tilde{X}_\mu = \hat{x}_\mu + \frac{1}{\kappa} a_\rho M_{\mu\rho}, \qquad \tilde{P}_\mu = \hat{p}_\mu, \tag{7.5}$$

$$\tilde{H} = \sqrt{1 + \frac{a^2}{\kappa^2} \tilde{P}_\mu \tilde{P}^\mu} + \frac{1}{\kappa} a \cdot \tilde{P} \tag{7.6}$$

with the properties $[\tilde{H}, \tilde{X}_\mu] = -\frac{1}{\kappa} a_\mu \tilde{H}$ and $[\tilde{H}, \tilde{P}_\mu] = 0$.

The generator $\tilde{H}$ is the shift operator and the case $\tilde{\kappa} \to \infty$ corresponds to the natural realization of κ-Poincaré algebra.[35]

VIII. CONCLUDING REMARKS

The Yang model describes a noncommutative geometry defined on a curved background. Its interest for physics resides in possible applications to quantum cosmology. From a mathematical point of view, its most remarkable property is the existence of a Born duality between positions and momenta.

The model is based on the Yang algebra (2.1)–(2.5), which depends on a mass M, a length R and the discrete parameters $\epsilon_1 = \pm 1, \epsilon_2 = \pm 1$, and is isomorphic either to $o(1,5)$, $o(2,4)$, or $o(3,3)$ algebras with flat metrics η_{AB}, depending on the values of ϵ_1 and ϵ_2.

Performing general linear transformations that keep Lorentz generators unchanged, we have constructed a class of algebras depending on parameters M, R, κ, $\tilde{\kappa}$, vectors a_μ, b_μ, and scalars $u, v, \epsilon_1 \varphi + \epsilon_2 \psi$, which are isomorphic either to $o(1,5,g)$, $o(2,4,g)$, or $o(3,3,g)$ algebras with metric g_{AB} (4.20) satisfying $g_{\mu\nu} = \eta_{\mu\nu}$. This construction unifies all Lorentz-invariant models isomorphic to orthogonal algebras and generalizes the results of Ref. 30.

In particular, the dual κ-Minkowski and κ-Poincaré algebras introduced in Ref. 33 are obtained demanding $g_{44} = 0, g_{55} = 0$, with $\det g \neq 0$, leading to the relations (4.30), $\frac{a^2}{\kappa^2} = -\epsilon_1 \frac{u^2}{M^2}$ and $\frac{b^2}{\tilde{\kappa}^2} = -\epsilon_2 \frac{v^2}{R^2}$. The interest of such models is that they extend the standard κ-Minkowski space to a dual formulation independent from the Snyder framework. In the limit $M \to \infty$, the vector a_μ becomes light-like, $a^2 = 0$, while for $R \to \infty$, the vector b_μ becomes light-like, $b^2 = 0$. If in addition, $\kappa \to \infty, \tilde{\kappa} \to \infty$, the algebra reduces to the Heisenberg algebra.

We have also obtained the Weyl realization of dual κ-Minkowski and κ-Poincaré algebras in terms of the metric g and thence the corresponding coproduct, star product, and twist. Finally, following the suggestion of Ref. 34, we have constructed reduced κ-Minkowski spaces and κ-Poincaré algebras, where the generator $\tilde{H}$ is no longer taken as independent, but as a function of the other 14 generators of the algebra.

In this paper, we have considered only formal aspects of the theory. Possible physical effects can arise in the nonrelativistic limit, when one may investigate the generalization of uncertainty relations and of the dynamics coming from the deformation of the Heisenberg commutation relations, as in Refs. 34 and 42. One may also apply our results to models of doubly special relativity with a nontrivial spacetime background.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Tea Martinić Bilać: Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **Stjepan Meljanac**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Writing – original draft (equal). **Salvatore Mignemi**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Supervision (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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