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ABSTRACT

Long time ago, Yang [Phys. Rev. **72**, 874 (1947)] proposed a model of noncommutative spacetime that generalized the Snyder model to a curved background. In this paper, we review his proposal and the generalizations that have been suggested during the years. In particular, we discuss the most general algebras that contain as subalgebras both de Sitter and Snyder algebras, preserving Lorentz invariance, and are generated by a two-parameter deformation of the canonical Heisenberg algebra. We also define their realizations on quantum phase space, giving explicit examples, both exact and in terms of a perturbative expansion in deformation parameters.

I. INTRODUCTION

At the Planck scale, gravitational and quantum effects have comparable strength and can affect the structure of spacetime. In this regime, noncommutative geometries may play a relevant role.[1] The first example of noncommutative spacetime was proposed by Snyder[2] in 1947, but at that time did not receive much attention. Despite the assumed granular structure of spacetime, this model is characterized by the preservation of the Lorentz invariance.

A first generalization of Snyder's idea was advanced by Yang, who combined noncommutativity with spacetime curvature,[3] in terms of a fifteen-dimensional SO(1,5) algebra of symmetries of phase space. The generators of this algebra were identified with the generators of the de Sitter algebra and with the coordinates of the de Sitter spacetime. The remaining generator h rotates positions into momenta, but its physical interpretation was not specified. More recently, this model was slightly generalized by Khruschev and Leznov (KL).[4]

Later, in Ref. 5 was proposed a model inspired by that of Yang, which realizes the same symmetries in a nonlinear way, reducing to 14 the number of independent generators of the algebra, thus eliminating the unphysical generator h. This model was dubbed triply special relativity (TSR) because it contains three fundamental constants, identified with the speed of light, the Planck mass, and the cosmological constant, generalizing in this way the idea advanced in doubly special relativity theories[6] of deforming the Poincaré symmetry by the introduction of a new fundamental constant proportional to the Planck mass. In this model, however, the Lorentz symmetry is preserved and only translation symmetries are deformed.

It was then shown in Ref. 7 that the TSR model can be realized exactly in terms of coordinates and momenta only. This particular realization in phase space was called Snyder–de Sitter (SdS) spacetime. In Refs. 7 and 8, it was also shown that the SdS algebra can be obtained from the Snyder algebra by a nonunitary transformation.

The previous models of quantum phase space have the common property of realizing explicitly the duality introduced by Born[9] for the exchange of position and momentum operators. Recently, the possibility of studying general quantum phase spaces displaying the structure described above has been advanced in Ref. 10 and studied in more detail in Ref. 11. These papers employed a widespread approach to noncommutative geometry based on the formalism of Hopf algebras,[12] which aims to describe the symmetries of the quantum spacetime. A powerful tool in this formalism is the realization of Hopf algebras in terms of the Heisenberg algebra, which were introduced in Refs. 13–15.

The Snyder model has been extensively discussed in the literature, see, e.g., Refs. 13, 14, 16, and 17, in several theoretical and phenomenological aspects, such as algebraic representations, relativistic field equations, quantum field theory, and relations with Hopf algebras, but also its nonrelativistic limit has been widely investigated, including generalized uncertainty principles and simple models, such as the harmonic oscillator. To our knowledge, instead, in addition to Ref. 4, the Yang model has been considered only in Ref. 18, where its supersymmetric extensions were analyzed. Many authors discuss aspects of TSR; see, e.g., Refs. 7, 8, 11, and 19. Most of them treat its classical limit, either in a nonrelativistic or relativistic setting; the quantum field theory of a self-interacting scalar field in SdS spacetime has also been investigated in Ref. 20.

In the present paper, we discuss general perturbative realizations of the unified model proposed in Ref. 10 in terms of the canonical Heisenberg algebra, extending the results of Ref. 11. In Sec. II, we review the generalized Snyder spaces and their realizations. In Sec. III, we discuss the de Sitter algebras obtained by duality from the Snyder ones. In Sec. IV, a formalism unifying the two algebras is introduced, which describes a curved noncommutative spacetime and includes Yang, KL, and TSR models as special cases. Section V analyzes the perturbative realizations of these models, while in Sec. VI, we exploit the relation of TSR with the Snyder model to write down some exact realizations.

II. GENERALIZED SNYDER SPACE

In this section, we review some generalizations of Snyder's original proposal for a deformation of the momentum space, which leads to noncommutativity of spacetime. Most of the results of this section were obtained in Refs. 13, 14, and 17.

We define a generalized Snyder space introducing a Lorentz-invariant deformation parameter $\beta \sim L_P$ and postulate the commutation relations as follows:

$$[\hat{x}_\mu, \hat{x}_\nu] = i\epsilon\beta^2 M_{\mu\nu}\, \psi^p(\epsilon\beta^2 p^2), \qquad [p_\mu, p_\nu] = 0, \qquad [\hat{x}_\mu, p_\nu] = i\phi^p_{\mu\nu}(\epsilon\beta^2 p^2), \tag{1}$$

where $\epsilon = \pm 1$ and $M_{\mu\nu} = M_{\mu\nu}^\dagger$ are the generators of the Lorentz algebra, which we assume to satisfy the standard relations,

$$\begin{aligned}&[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}),\\ &[M_{\mu\nu}, \hat{x}_\lambda] = i(\eta_{\mu\lambda}\hat{x}_\nu - \eta_{\nu\lambda}\hat{x}_\mu), \qquad [M_{\mu\nu}, p_\lambda] = i(\eta_{\mu\lambda}p_\nu - \eta_{\nu\lambda}p_\mu),\end{aligned} \tag{2}$$

with $\eta_{\mu\nu}$ being the flat metric, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$, and the functions $\psi^p(\epsilon\beta^2 p^2)$ and $\phi^p_{\mu\nu}(\epsilon\beta^2 p^2)$ being constrained so that the Jacobi identities hold. Commutation relations (1) and (2) generalize those originally investigated in Ref. 2, which are recovered for $\psi^p = \text{const}$ and $\phi^p_{\mu\nu} = \eta_{\mu\nu} + \beta^2 p_\mu p_\nu$.

In the following, we shall look for realizations of the algebra defined by Eqs. (1) and (2) in terms of the Heisenberg algebra $\mathcal{H}$: in its undeformed version, the Heisenberg algebra is generated by commutative coordinates x_μ and momenta p_μ that satisfy

$$[x_\mu, x_\nu] = [p_\mu, p_\nu] = 0, \qquad [x_\mu, p_\nu] = i\eta_{\mu\nu}, \tag{3}$$

with $x_\mu^\dagger = x_\mu$ and $p_\mu^\dagger = p_\mu$.

The action of x_μ and p_μ on functions $f(x)$ belonging to the enveloping algebra $\mathcal{A}$ generated by x_μ is defined as

$$x_\mu \triangleright f(x) = x_\mu f(x), \qquad p_\mu \triangleright f(x) = -i\frac{\partial f(x)}{\partial x^\mu}. \tag{4}$$

Noncommutative coordinates $\hat{x}_\mu$ and Lorentz generators $M_{\mu\nu}$ can be expressed in terms of commutative coordinates x_μ and momenta p_μ as

$$\begin{aligned}&\hat{x}_\mu = x_\mu\varphi^p_1(\epsilon\beta^2 p^2) + \epsilon\beta^2 x\cdot p\; p_\mu\varphi^p_2(\epsilon\beta^2 p^2) + \epsilon\beta^2 p_\mu\chi^p(\epsilon\beta^2 p^2),\\ &M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \qquad M_{\mu\nu} = M^\dagger_{\mu\nu}.\end{aligned} \tag{5}$$

In terms of realizations (5), the functions $\phi^p_{\mu\nu}$ in (1) read

$$\phi^p_{\mu\nu} = \eta_{\mu\nu}\varphi^p_1 + \epsilon\beta^2 p_\mu p_\nu\varphi^p_2. \tag{6}$$

The Jacobi identities are satisfied if

$$\psi^p = -2\varphi^p_1(\varphi^p_1)' + \varphi^p_1\varphi^p_2 - 2\epsilon\beta^2 p^2(\varphi^p_1)'\varphi^p_2, \tag{7}$$

where a prime denotes a derivative with respect to $\epsilon\beta^2 p^2$. The function ψ^p is independent of χ^p.

From (7), it follows that the $\hat{x}_\mu$ are commutative for

$$\varphi^p_2 = \frac{2\varphi^p_1(\varphi^p_1)'}{\varphi^p_1 - 2\epsilon\beta^2 p^2(\varphi^p_1)'}$$

and correspond to Snyder space if

$$\varphi_2^p = \frac{1 + 2\varphi_1^p(\varphi_1^p)'}{\varphi_1^p - 2\epsilon\beta^2 p^2(\varphi_1^p)'}.$$

Some interesting special cases arise for particular values of the functions φ_i^p. The Snyder realization[2,14] is recovered for $\varphi_1^p = \varphi_2^p = 1$, and the Maggiore realization[14,21] is recovered for $\varphi_1^p = \sqrt{1 - \epsilon\beta^2 p^2}$ and $\varphi_2^p = 0$. Another interesting case of the realization of Snyder space[17] is for $\varphi_1^p = s = \text{const}$, $\chi^p = 0$,

$$\hat{x}_\mu = x_\mu + \frac{\epsilon\beta^2 s}{4} K_\mu, \tag{8}$$

where $K_\mu = x_\mu p^2 - 2x \cdot p\, p_\mu$ are the generators of conformal transformations in momentum space, with $[K_\mu, K_\nu] = 0$.

Algebras (1) and (2) unify commutative space, $\psi^p = 0$, and Snyder space, $\psi^p = 1$. Since the Lorentz transformations are not deformed, its Casimir operator is $\mathcal{C}^p = p^2$. Algebras (1) and (2) are invariant under Hermitian conjugation if $p_\mu^\dagger = p_\mu$, $\hat{x}_\mu^\dagger = \hat{x}_\mu$, and $M_{\mu\nu}^\dagger = M_{\mu\nu}$.

Hermitian realizations of $\hat{x}_\mu(\beta)$ are obtained from (5) as

$$\hat{x}_\mu(\beta) = \frac{1}{2}\left[x_\mu\varphi_1^p + \varphi_1^p x_\mu + \epsilon\, \beta^2(x\cdot p\; p_\mu\varphi_2^p + \varphi_2^p p_\mu p\cdot x)\right] + \epsilon\, \beta^2 p_\mu \chi^p, \tag{9}$$

where φ_1^p, φ_2^p, and χ^p are real functions.

III. NONCOMMUTATIVE MOMENTA OF GENERALIZED SNYDER TYPE

By duality, one can obtain different Snyder-like algebras with noncommutative momenta, which coincide with generalized de Sitter algebras and can be associated with the symmetries of spacetimes of constant curvature, namely, de Sitter or anti-de Sitter. The relevance of de Sitter spacetime in general relativity and quantum gravity is well-known,[22] especially in the cosmological context. The algebra of symmetries of de Sitter spacetime is also well-known; see, e.g., Ref. 23. In addition, deformations of special relativity based on the de Sitter algebra have been investigated.[24] The models presented here correspond to unusual parameterizations of the de Sitter manifold, which are isotropic in the four-dimensional spacetime.[25]

To obtain these algebras, we introduce a new Lorentz-invariant deformation parameter α of dimension inverse length so that α^2 may be identified with the cosmological constant and assume

$$[\hat{p}_\mu, \hat{p}_\nu] = i\epsilon'\alpha^2 M_{\mu\nu}\psi^x(\epsilon'\alpha^2 x^2), \qquad [x_\mu, x_\nu] = 0, \qquad [x_\mu, \hat{p}_\nu] = i\phi_{\mu\nu}^x(\epsilon'\alpha^2 x^2), \tag{10}$$

with $\epsilon' = \pm 1$. Moreover, the Lorentz generators satisfy

$$[M_{\mu\nu}, \hat{p}_\lambda] = i(\eta_{\mu\lambda}\hat{p}_\nu - \eta_{\nu\lambda}\hat{p}_\mu) \tag{11}$$

and the other standard relations in (2). The functions $\psi^x(\epsilon'\alpha^2 x^2)$ and $\phi_{\mu\nu}^x(\epsilon'\alpha^2 x^2)$ are chosen so that the Jacobi identities are satisfied.

The noncommutative momenta $\hat{p}_\mu$ and the Lorentz generators $M_{\mu\nu}$ can then be expressed in terms of commutative coordinates and momenta as

$$\begin{aligned}\hat{p}_\mu &= p_\mu\varphi_1^x(\epsilon'\alpha^2 x^2) + \epsilon'\alpha^2 p\cdot x\; x_\mu\varphi_2^x(\epsilon'\alpha^2 x^2) + \epsilon'\alpha^2 x_\mu\chi^x(\epsilon'\alpha^2 x^2),\\ M_{\mu\nu} &= x_\mu p_\nu - x_\nu p_\mu, \qquad M_{\mu\nu} = M_{\mu\nu}^\dagger.\end{aligned} \tag{12}$$

In terms of this realization, the functions $\phi_{\mu\nu}^x$ read

$$\phi_{\mu\nu}^x = \eta_{\mu\nu}\varphi_1^x + \epsilon'\alpha^2 x_\mu x_\nu\varphi_2^x, \tag{13}$$

and the Jacobi identities are satisfied if

$$\psi^x = -2\varphi_1^x(\varphi_1^x)' + \varphi_1^x\varphi_2^x - 2\epsilon'\alpha^2 x^2(\varphi_1^x)'\varphi_2^x, \tag{14}$$

where now a prime denotes a derivative with respect to $\epsilon'\alpha^2 x^2$. The function ψ^x does not depend on χ^x.

Since the Lorentz transformations are not deformed, the Casimir operator of this algebra is $\mathcal{C} = x^2$. Algebras (10) and (11) are invariant under Hermitian conjugation if $\hat{p}_\mu^\dagger = \hat{p}_\mu$, $\hat{x}_\mu^\dagger = \hat{x}_\mu$, and $M_{\mu\nu}^\dagger = M_{\mu\nu}$.

Hermitian realizations of $\hat{p}_\mu(\alpha)$ are obtained from (12) as

$$\hat{p}_\mu(\alpha) = \frac{1}{2}\left[p_\mu\varphi_1^x + \varphi_1^x p_\mu + \epsilon'\alpha^2(p\cdot x\; x_\mu\varphi_2^x + \varphi_2^x x_\mu x\cdot p)\right] + \epsilon'\alpha^2 x_\mu\chi^x. \tag{15}$$

IV. QUANTUM DEFORMED PHASE SPACES DEPENDING ON TWO PARAMETERS

The deformed phase spaces of Secs. II and III can be unified by introducing both fundamental Lorentz-invariant parameters α and β so that[10,11]

$$[\hat{x}_\mu, \hat{x}_\nu] = i\epsilon\, \beta^2 M_{\mu\nu}\, \psi^p(\epsilon\, \beta^2 p^2), \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\epsilon' \alpha^2 M_{\mu\nu}\, \psi^x(\epsilon' \alpha^2 x^2), \qquad [\hat{x}_\mu, \hat{p}_\nu] = i g_{\mu\nu}, \tag{16}$$

where

$$g_{\mu\nu} = h_0 \eta_{\mu\nu} + \alpha^2 X_{\mu\nu} + \beta^2 P_{\mu\nu} + \alpha\beta H_{\mu\nu}, \tag{17}$$

with $X_{\mu\nu} = h_1 \hat{x}_\mu \hat{x}_\nu + \hat{x}_\nu \hat{x}_\mu h_1^\dagger$, $P_{\mu\nu} = h_2 \hat{p}_\mu \hat{p}_\nu + \hat{p}_\nu \hat{p}_\mu h_2^\dagger$, and $H_{\mu\nu} = h_3(\hat{x}_\mu \hat{p}_\nu + \hat{p}_\nu \hat{x}_\mu) + h_4(\hat{x}_\nu \hat{p}_\mu + \hat{p}_\mu \hat{x}_\nu) + \text{h.c.}$, and h_i are Lorentz-invariant functions of x_μ and p_μ depending on $\hat{x}^2$, $\hat{p}^2$, and $D = \frac{1}{2}(\hat{x} \cdot \hat{p} + \hat{p} \cdot \hat{x})$.

If $\hat{x}_\mu$, $\hat{p}_\mu$, and $M_{\mu\nu}$ are Hermitian operators, then $g_{\mu\nu}^\dagger = g_{\mu\nu}$, $h_0 = h_0$, $X_{\mu\nu}^\dagger = X_{\mu\nu}$, $P_{\mu\nu}^\dagger = P_{\mu\nu}$, and $H_{\mu\nu}^\dagger = H_{\mu\nu}$. The operators $g_{\mu\nu}$, $X_{\mu\nu}$, $P_{\mu\nu}$, and $H_{\mu\nu}$ transform as second rank tensors under Lorentz transformations, while h_0 is invariant, $[M_{\mu\nu}, h_0] = 0$. We also require that all Jacobi identities hold. For simplicity, from now on, we set $\epsilon = \epsilon' = 1$.

Under suitable conditions on h_i, algebras (16) and (17) satisfy the Born duality, defined as invariance for $\beta \leftrightarrow \alpha$, $\hat{x}_\mu \to -\hat{p}_\mu$, $\hat{p}_\mu \to \hat{x}_\mu$, $M_{\mu\nu} \leftrightarrow M_{\mu\nu}$, and $g_{\mu\nu} \leftrightarrow g_{\nu\mu}$.[11] Moreover, it is easy to see that

$$g_{\mu\nu} - g_{\nu\mu} = \alpha\beta(H_{\mu\nu} - H_{\nu\mu}) = \alpha\beta F M_{\mu\nu} \tag{18}$$

with $[M_{\mu\nu}, F] = 0$.

The above quantum deformed phase spaces include as special cases Yang[3] and TSR[5,7] models and their generalizations.[4,11] The algebras with $X_{\mu\nu} = P_{\mu\nu} = 0$ and $H_{\mu\nu}$ proportional to $M_{\mu\nu}$ reduce to Lie algebras, generated by $\hat{x}_\mu$, $\hat{p}_\mu$ $M_{\mu\nu}$, and h_0. These algebras were introduced in Ref. 4 and are defined by

$$\begin{aligned} &[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 M_{\mu\nu}, \qquad [\hat{p}_\mu, \hat{p}_\nu] = i\alpha^2 M_{\mu\nu}, \qquad [\hat{x}_\mu, \hat{p}_\nu] = i(h_0 \eta_{\mu\nu} - 2\alpha\beta\rho M_{\mu\nu}), \\ &[h_0, \hat{x}_\mu] = i(\beta^2 \hat{p}_\mu + 2\alpha\beta\rho\ \hat{x}_\mu), \qquad [h_0, \hat{p}_\mu] = -i(\alpha^2 \hat{x}_\mu + 2\alpha\beta\rho\ \hat{p}_\mu), \end{aligned} \tag{19}$$

where α, β, and ρ are real parameters. For $\rho = 0$, one gets the Yang model, and for $\alpha = 0$ or $\beta = 0$, one gets the Snyder or de Sitter algebra, respectively.

If instead $X_{\mu\nu}$ or $P_{\mu\nu}$ do not vanish, as in the case of TSR or SdS, algebras (16) and (17) are not Lie algebras. In the KL model, $X_{\mu\nu} = P_{\mu\nu} = 0$, but $H_{\mu\nu}$ is proportional to $M_{\mu\nu}$, and this gives rise to a Lie algebra.

A. General Hermitian realizations

The most general Hermitian realizations of deformed phase spaces (16) and (17) are given by

$$\hat{x}_\mu(\alpha,\beta) = e^{iG_1} \hat{x}_\mu(\beta)\ e^{-iG_1}, \qquad \hat{p}_\mu(\alpha,\beta) = e^{iG_2} \hat{p}_\mu(\alpha)\ e^{-iG_2}, \tag{20}$$

where $\hat{x}_\mu(\beta)$ and $\hat{p}_\mu(\alpha)$ are given in (9) and (15), respectively, and

$$G_1 = G_1{}^\dagger = G_1(\alpha^2 x^2, \alpha\beta D, \beta^2 p^2), \qquad G_2 = G_2{}^\dagger = G_2(\alpha^2 x^2, \alpha\beta D, \beta^2 p^2), \tag{21}$$

with $D = \frac{1}{2}(x \cdot p + p \cdot x)$.

In order for the Jacobi identities to hold, the functions φ_1^p, φ_2^p and φ_1^x, φ_2^x have to satisfy relations (7) and (14), respectively.

In these realizations, the functions $X_{\mu\nu}$, $P_{\mu\nu}$, $H_{\mu\nu}$, and h depend on φ_1^p, φ_2^p, φ_1^x, φ_2^x, and G_1, G_2. The Born duality applied to the above realization generates new ones.

B. Hermitian realization of the Yang model

An Hermitian realization for the Yang model is given by

$$\begin{aligned} \hat{x}_\mu(\beta) &= \frac{1}{2}\left(x_\mu \sqrt{1 - \beta^2 p^2} + \sqrt{1 - \beta^2 p^2}\ x_\mu\right) + \beta^2 p_\mu \chi^p(\beta^2 p^2), \\ \hat{p}_\mu(\alpha) &= \frac{1}{2}\left(p_\mu \sqrt{1 - \alpha^2 x^2} + \sqrt{1 - \alpha^2 x^2}\ p_\mu\right) + \alpha^2 x_\mu \chi^x(\alpha^2 x^2). \end{aligned} \tag{22}$$

The Yang model is obtained for G_1, G_2 such that

$$[\hat{x}_\mu, \hat{p}_\nu] = i\eta_{\mu\nu} h_0, \quad [h_0, \hat{x}_\mu] = i\beta^2 \hat{p}_\mu, \quad [h_0, \hat{p}_\mu] = -i\alpha^2 \hat{x}_\mu \tag{23}$$

hold. The explicit form at fourth order for $\chi^p = \chi^x = 0$ is reported in Ref. 11. In the limit $\alpha = \beta = 0$, $\hat{x}_\mu(\alpha,\beta) = x_\mu$ and $\hat{p}_\mu(\alpha,\beta) = p_\mu$.

Similar constructions can be applied to TSR and SdS models, where

$$\hat{x}_\mu(\beta) = x_\mu + \frac{\beta^2}{2}(x\cdot p_\mu + p_\mu p\cdot x) + \beta^2 p_\mu \chi^p(\beta^2 p^2),$$
$$\hat{p}_\mu(\alpha) = p_\mu + \frac{\alpha^2}{2}(p\cdot x x_\mu + x_\mu x\cdot p) + \alpha^2 x_\mu \chi^x(\alpha^2 x^2). \tag{24}$$

V. PERTURBATIVE EXPANSION OF HERMITIAN REALIZATIONS

We shall now consider Hermitian realizations in a perturbative expansion in α and β of the two-parameter model introduced in Sec. IV B, extending the results of Ref. 11.

A. Second-order expansion

At second order in α and β, we use the ansatz

$$\hat{x}_\mu = x_\mu + \frac{a_1}{2}\alpha\beta(x_\mu x\cdot p + p\cdot x x_\mu) + \frac{a_2}{2}\beta^2(x_\mu p^2 + p^2 x_\mu) + \frac{a_3}{2}\beta^2(p_\mu p\cdot x + x\cdot p p_\mu) + \frac{a_4}{2}\alpha\beta(p_\mu x^2 + x^2 p_\mu), \tag{25}$$
$$\hat{p}_\mu = p_\mu + \frac{b_1}{2}\alpha\beta(p_\mu p\cdot x + x\cdot p p_\mu) + \frac{b_2}{2}\alpha^2(p_\mu x^2 + x^2 p_\mu) + \frac{b_3}{2}\alpha^2(x_\mu x\cdot p + p\cdot x x_\mu) + \frac{b_4}{2}\alpha\beta(x_\mu p^2 + p^2 x_\mu), \tag{26}$$

where a_i, b_i are real constants. (Notice that in $\hat{x}_\mu$, we could also add $\alpha\beta x_\mu$, $\alpha^2 x_\mu x^2$, and $\beta^2 p_\mu$, and in $\hat{p}_\mu$, we could also add $\alpha\beta p_\mu$, $\beta^2 p_\mu p^2$, and $\alpha^2 x_\mu$.)

Substituting (25) and (26) in the relations $[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 M_{\mu\nu}$ and $[\hat{p}_\mu, \hat{p}_\nu] = i\alpha^2 M_{\mu\nu}$, we find $a_3 - 2a_2 = 1$, $b_3 - 2b_2 = 1$, while calculating $[\hat{x}_\mu, \hat{p}_\nu]$, we get

$$g_{\mu\nu} = \eta_{\mu\nu}\big(1 + 2\tau\alpha\beta D + a_2\beta^2 p^2 + b_2\alpha^2 x^2\big) + b_3\alpha^2 x_\mu x_\nu + a_3\beta^2 p_\mu p_\nu$$
$$+ \alpha\beta[\tau(x_\mu p_\nu + p_\nu x_\mu) + \rho(x_\nu p_\mu + p_\mu x_\nu)], \tag{27}$$

where $\rho = a_4 + b_4$, $\tau = \frac{1}{2}(a_1 + b_1)$, and $D = \frac{1}{2}(x\cdot p + p\cdot x)$.

Hence,

$$h_0 = 1 + 2\tau\alpha\beta D + a_2\beta^2 p^2 + b_2\alpha^2 x^2, \qquad X_{\mu\nu} = b_3 x_\mu x_\nu, \qquad P_{\mu\nu} = a_3 p_\mu p_\nu,$$
$$H_{\mu\nu} = \tau(x_\mu p_\nu + p_\nu x_\mu) + \rho(x_\nu p_\mu + p_\mu x_\nu). \tag{28}$$

In the following, for simplicity, we shall consider the symmetric solutions $a_i = b_i$.

For KL models, $a_2 = b_2 = -\frac{1}{2}$, $a_3 = b_3 = 0$, $a_1 = b_1 = -\rho$, $a_4 = b_4 = \frac{\rho}{2}$, $\tau = -\rho$, and

$$g_{\mu\nu} = \eta_{\mu\nu}\left(1 - 2\rho\alpha\beta D - \frac{1}{2}(\alpha^2 x^2 + \beta^2 p^2)\right) - 2\rho\alpha\beta M_{\mu\nu} = \eta_{\mu\nu} h_0 - 2\rho\alpha\beta M_{\mu\nu}, \tag{29}$$

with $h_0 = 1 - 2\rho\alpha\beta D - \frac{1}{2}(\alpha^2 x^2 + \beta^2 p^2)$ and

$$[h_0, \hat{x}_\mu] = i(2\rho\alpha\beta x_\mu + \beta^2 p_\mu), \qquad [h_0, \hat{p}_\mu] = -i(2\rho\alpha\beta p_\mu + \alpha^2 x_\mu). \tag{30}$$

For Yang models, $\rho = 0$ and $h_0 = 1 - \frac{1}{2}(\alpha^2 x^2 + \beta^2 p^2)$. For TSR models, $a_2 = b_2 = 0$, $a_3 = b_3 = 1$, $a_1 = b_1 = 0$, $a_4 = b_4 = \frac{1}{2}$, $\tau = 0$, and $\rho = 1$.

B. Fourth-order expansion

At fourth order, we use the ansatz

$$\hat{x}_\mu^{(4)} = \frac{c_1}{2}\alpha^3\beta(x_\mu x^2 x\cdot p + p\cdot x x^2 x_\mu) + \frac{c_2}{2}\alpha^2\beta^2(x_\mu x^2 p^2 + p^2 x^2 x_\mu) + \frac{c_3}{2}\alpha^2\beta^2(x_\mu x\cdot p p\cdot x + p\cdot x x\cdot p x_\mu)$$
$$+ \frac{c_4}{2}\alpha\beta^3(x_\mu x\cdot p p^2 + p^2 p\cdot x x_\mu) + \frac{c_5}{2}\beta^4(x_\mu p^4 + p^4 x_\mu) + \frac{c_6}{2}\beta^4(p_\mu p^2 p\cdot x + x\cdot p p^2 p_\mu)$$
$$+ \frac{c_7}{2}\alpha\beta^3(p_\mu p^2 x^2 + x^2 p^2 p_\mu) + \frac{c_8}{2}\alpha\beta^3(p_\mu p\cdot x x\cdot p + p\cdot x x\cdot p p_\mu)$$
$$+ \frac{c_9}{2}\alpha^2\beta^2(p_\mu p\cdot x x^2 + x^2 x\cdot p p_\mu), \tag{31}$$

$$\hat{p}_\mu^{(4)} = \frac{d_1}{2}\alpha\beta^3(p_\mu p^2 p\cdot x + x\cdot p p^2 p_\mu) + \frac{d_2}{2}\alpha^2\beta^2(p_\mu p^2 x^2 + x^2 p^2 p_\mu) + \frac{d_3}{2}\alpha^2\beta^2(p_\mu p\cdot x x\cdot p + x\cdot p p\cdot x p_\mu)$$
$$+ \frac{d_4}{2}\alpha^3\beta(p_\mu p\cdot x x^2 + x^2 x\cdot p p_\mu) + \frac{d_5}{2}\alpha^4(p_\mu x^4 + x^4 p_\mu) + \frac{d_6}{2}\alpha^4(x_\mu x^2 x\cdot p + p\cdot x x^2 x_\mu)$$
$$+ \frac{d_7}{2}\alpha^3\beta(x_\mu x^2 p^2 + p^2 x^2 x_\mu) + \frac{d_8}{2}\alpha^3\beta(x_\mu x\cdot p p\cdot x + x\cdot p p\cdot x x_\mu)$$
$$+ \frac{d_9}{2}\alpha^2\beta^2(x_\mu x\cdot p p^2 + p^2 p\cdot x x_\mu), \tag{32}$$

where c_i, d_i are real constants. (In this case also we have omitted some terms that do not contribute significantly, such as $\alpha^3\beta\, p_\mu x^4$, $\alpha^4 x_\mu x^4$, and $\beta^4 p_\mu p^2$ in $\hat{x}_\mu$ and analogous terms in $\hat{p}_\mu$.)

Inserting (31) and (32) into the relations $[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 M_{\mu\nu}$ and $[\hat{p}_\mu, \hat{p}_\nu] = i\alpha^2 M_{\mu\nu}$, we find $c_6 = 4a_2^2 + 4c_5 + a_2$, $d_6 = 4b_2^2 + 4d_5 + b_2$.

For KL models, assuming $d_i = c_i$, one has $c_1 = c_4 = c_6 = c_8 = 0$, $c_2 = \frac{\rho^2}{8}$, $c_3 = \frac{1}{4} - \frac{\rho^2}{2}$, $c_5 = -\frac{1}{8}$, $c_7 = \frac{\rho}{4}$, and $c_9 = \frac{\rho^2}{4}$, and calculating $[\hat{x}_\mu, \hat{p}_\nu]$, one obtains

$$h_0 = 1 - 2\rho\alpha\beta D - \frac{1}{2}(\alpha^2 x^2 + \beta^2 p^2) - \frac{1}{8}(\alpha^4 x^4 + \beta^4 p^4) + \frac{\rho}{4}\alpha^3\beta(x^2 x\cdot p + p\cdot x\, x^2)$$
$$+ \frac{\rho}{4}\alpha\beta^3(p^2 p\cdot x + x\cdot p\, p^2) + \frac{1}{8}\alpha^2\beta^2(x^2 p^2 + p^2 x^2) + \frac{1}{2}\alpha^2\beta^2 D^2 \tag{33}$$

and

$$[h_0, \hat{x}_\mu] = i\Bigg[2\rho\alpha\beta x_\mu + \beta^2 p_\mu + \left(\frac{\rho^2}{2} - \frac{1}{4}\right)\alpha^2\beta^2(x^2 p_\mu + p_\mu x^2) - \frac{\rho}{2}\alpha\beta^3(x\cdot p p_\mu + p_\mu p\cdot x)$$
$$- \rho^2\alpha^2\beta^2(p\cdot x x_\mu + x_\mu x\cdot p)\Bigg] = i(\beta^2\hat{p}_\mu + 2\alpha\beta\rho\ \hat{x}_\mu), \tag{34}$$

$$[h_0, \hat{p}_\mu] = -i\Bigg[2\rho\alpha\beta p_\mu + \alpha^2 x_\mu + \left(\frac{\rho^2}{2} - \frac{1}{4}\right)\alpha^2\beta^2(p^2 x_\mu + x_\mu p^2) - \frac{\rho}{2}\alpha^3\beta(p\cdot x x_\mu + x_\mu x\cdot p)$$
$$- \rho^2\alpha^2\beta^2(p\cdot x p_\mu + p_\mu p\cdot x)\Bigg] = -i(\alpha^2\hat{x}_\mu + 2\alpha\beta\rho\ \hat{p}_\mu) \tag{35}$$

in accordance with (19). The Yang model is obtained for $\rho = 0$.

For SdS, the results are reported in Ref. 11. In particular, for $c_i = d_i$, the coefficients depend on the free parameter c_1, with $c_5 = c_6 = c_7 = 0$, $c_2 = \frac{1}{8}$, $c_3 = \frac{1}{2}$, $c_4 = \frac{1}{2} - c_1$, $c_8 = 1 - c_1$, and $c_9 = \frac{3}{4}$.

VI. EXACT RESULTS ON GENERALIZED TSR

In this section, we present some exact realizations of generalized TSR obtained by exploiting a method proposed in Refs. 7 and 8.

Let us start with the Snyder algebra

$$[\hat{x}_\mu, \hat{x}_\nu] = i\beta^2 M_{\mu\nu}, \qquad \beta \neq 0. \tag{36}$$

A class of realizations of $\hat{x}_\mu$ is given by

$$\hat{x}_\mu = X_\mu\varphi_1(\beta^2 P^2) + \beta^2 X\cdot P\ P_\mu\varphi_2(\beta^2 P^2), \qquad M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu = M_{\mu\nu}^\dagger, \tag{37}$$

with $\hat{x}^\dagger \neq \hat{x}$, where X_μ and P_μ satisfy

$$[X_\mu, X_\nu] = [P_\mu, P_\nu] = 0, \qquad [X_\mu, P_\nu] = i\eta_{\mu\nu} \tag{38}$$

and

$$\varphi_2 = \frac{1 + 2\varphi_1\varphi_1'}{\varphi_1 - 2\beta^2 P^2\varphi_1'}, \qquad \text{with } \varphi_1' = \frac{d\varphi_1}{d(\beta^2 P^2)}, \quad \varphi(0) = 1. \tag{39}$$

Let us define

$$\hat{p}_\mu = P_\mu - \epsilon\frac{\alpha}{\beta}\hat{x}_\mu, \qquad \epsilon^2 = 1,\ \alpha \neq 0, \tag{40}$$

and using $[\hat{x}_\mu, P_\nu] = [\hat{x}_\nu, P_\mu]$, we obtain

$$[\hat{p}_\mu, \hat{p}_\nu] = i\alpha^2 M_{\mu\nu}, \qquad [\hat{x}_\mu, \hat{p}_\nu] = ig_{\mu\nu} = i\eta_{\mu\nu}\varphi_1(\beta^2 P^2) + \beta^2 P_\mu P_\nu \varphi_2(\beta^2 P^2) - \epsilon\ \alpha\beta M_{\mu\nu}. \tag{41}$$

Hence, $g_{\mu\nu} - g_{\nu\mu} = -2\epsilon\ \alpha\beta M_{\mu\nu}$.

Note that

$$P_\mu = \hat{p}_\mu + \epsilon\frac{\alpha}{\beta}\hat{x}_\mu \tag{42}$$

and

$$M_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu = (\hat{x}_\mu P_\nu - \hat{x}_\nu P_\mu)\frac{1}{\varphi_1(\beta^2 P^2)} = (\hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu - 2i\epsilon\ \alpha\beta M_{\mu\nu})\frac{1}{\varphi_1(\beta^2 P^2)}. \tag{43}$$

Hence,

$$M_{\mu\nu} = \frac{1}{2}(\hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu + \hat{p}_\nu \hat{x}_\mu - \hat{p}_\mu \hat{x}_\nu)\frac{1}{\varphi_1(\beta^2 P^2)}. \tag{44}$$

The algebra generated by $\hat{x}_\mu$, $\hat{p}_\mu$, and $M_{\mu\nu}$ and all its realizations are invariant under the Born duality $\alpha \leftrightarrow \beta$, $\hat{x}_\mu \to -\hat{p}_\mu$, $\hat{p}_\mu \to \hat{x}_\mu$, $M_{\mu\nu} \leftrightarrow M_{\mu\nu}$, and $g_{\mu\nu} \leftrightarrow g_{\nu\mu}$. The relation between $\hat{x}_\mu$ and $\hat{p}_\mu$ can be written as

$$\hat{p}_\mu = -\epsilon\frac{\alpha}{\beta}\ S\hat{x}_\mu S^{-1}, \tag{45}$$

where

$$S = \exp\left(-\frac{iZ}{2\epsilon\ \alpha\beta}\right), \qquad \frac{dZ}{d(\beta^2 P^2)} = \frac{1}{\varphi_1(\beta^2 P^2) + \beta^2 P^2 \varphi_2(\beta^2 P^2)}. \tag{46}$$

Note that $Z^\dagger = Z$, $S^{-1} = S^\dagger$.

Clearly, Hermitian realizations of $\hat{x}_\mu$ and $\hat{p}_\mu$ are given by

$$\hat{x}_\mu^H = \frac{1}{2}(\hat{x}_\mu + \hat{x}_\mu^\dagger), \qquad \hat{p}_\mu^H = \frac{1}{2}(\hat{p}_\mu + \hat{p}_\mu^\dagger). \tag{47}$$

A. Special cases

For $\varphi_1 = \varphi_2 = 1$, we obtain SdS,[7,11]

$$\begin{aligned}
[\hat{x}_\mu, \hat{p}_\nu] &= i(\eta_{\mu\nu} + \beta^2 P_\mu P_\nu - \epsilon\alpha\beta M_{\mu\nu}) = i(\eta_{\mu\nu} + (\alpha\hat{x}_\mu + \beta\hat{p}_\mu)(\alpha\hat{x}_\nu + \beta\hat{p}_\nu) - \epsilon\alpha\beta M_{\mu\nu}),\\
M_{\mu\nu} &= \frac{1}{2}(\hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu - \hat{p}_\mu \hat{x}_\nu + \hat{p}_\nu \hat{x}_\mu).
\end{aligned} \tag{48}$$

In this case,

$$Z = \ln(1 + \beta^2 P^2), \qquad S = \exp\left(-\frac{i}{2\epsilon\alpha\beta}\ln(1 + \beta^2 P^2)\right). \tag{49}$$

For $\varphi_1 = \sqrt{1 - \beta^2 P^2}$, $\varphi_2 = 0$, we obtain

$$\begin{aligned}
[\hat{x}_\mu, \hat{p}_\nu] &= i\left(\eta_{\mu\nu}\sqrt{1 - \beta^2 P^2} - \epsilon\alpha\beta M_{\mu\nu}\right),\\
M_{\mu\nu} &= \frac{1}{2}(\hat{x}_\mu \hat{p}_\nu - \hat{x}_\nu \hat{p}_\mu - \hat{p}_\mu \hat{x}_\nu + \hat{p}_\nu \hat{x}_\mu)\frac{1}{\sqrt{1 - \beta^2 P^2}}.
\end{aligned} \tag{50}$$

In this case,

$$Z = -2\sqrt{1 - \beta^2 P^2}, \qquad S = \exp\left(\frac{i}{\epsilon\alpha\beta}\sqrt{1 - \beta^2 P^2}\right). \tag{51}$$

This example corresponds to a special case of KL model[4] with $h_0 = \sqrt{1 - \beta^2 P^2}$, $\epsilon = 2\rho = \pm 1$, and

$$[h_0, \hat{x}_\mu] = i(\beta^2 \hat{p}_\mu + \epsilon\ \alpha\beta\hat{x}_\mu), \qquad [h_0, \hat{p}_\mu] = -i(\alpha^2 \hat{x}_\mu + \epsilon\ \alpha\beta\hat{p}_\mu). \tag{52}$$

VII. CONCLUSIONS

Models of noncommutative geometry in curved spacetime have recently attracted much interest because of possible applications to astrophysical observations[26] and to the measurement of the time delay in the propagation of photons by cosmic sources.

In this paper, we have examined a class of these models, characterized by a high degree of symmetry, which generalize an early proposal by Yang[3] and include TSR[5,7] among others. The main feature of these models is that the defining algebra contains both the Snyder[2] and the de Sitter[23] algebra, and in particular, the Lorentz invariance is preserved. They may, therefore, be relevant in a low-energy limit of quantum gravity, for which theoretical arguments suggest that both a noncommutative parameter and a cosmological constant should be relevant.[5]

We have presented quantum realizations of these algebras in canonical phase space, starting from the simpler cases of Snyder or its de Sitter dual. However, their structure, which involves the full phase space, renders problematic the definition of a star product or of a Hopf algebroid structure, such as those introduced in Ref. 27. More general mathematical constructions should be introduced if one wishes to include analogous notions in this formalism.

A possible area of application of our results is quantum field theory. A field theory based on the SdS algebra was discussed in Ref. 20 using some rough approximations. In that paper, its similitude with the Grosse–Wulkenhaar model,[28] a renormalizable and exactly solvable theory, was noted, which, in analogy with SdS field theory, can be interpreted as a field theory in curved noncommutative space.[29] An investigation of the field theory on spacetimes of Yang type that exploit the general results obtained in this paper would, therefore, be a promising development of the present research.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

S. Meljanac: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal). **S. Mignemi**: Conceptualization (equal); Formal analysis (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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