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Moduli of polarised Enriques surfaces — Computational aspects

Mathieu Dutour Sikirić[1] | **Klaus Hulek**[2]

[1]Rudjer Bosković Institute, Zagreb, Croatia

[2]Institut für Algebraische Geometrie, Leibniz Universität Hannover, Hannover, Germany

Correspondence
Klaus Hulek, Institut für Algebraische Geometrie, Leibniz Universität Hannover, D-30060 Hannover, Germany.
Email: hulek@math.uni-hannover.de

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Abstract

Moduli spaces of (polarised) Enriques surfaces can be described as open subsets of modular varieties of orthogonal type. It was shown by Gritsenko and Hulek that there are, up to isomorphism, only finitely many different moduli spaces of polarised Enriques surfaces. Here, we investigate the possible arithmetic groups and show that there are exactly 87 such groups up to conjugacy. We also show that all moduli spaces are dominated by a moduli space of polarised Enriques surfaces of degree 1240. Ciliberto, Dedieu, Galati and Knutsen have also investigated moduli spaces of polarised Enriques surfaces in detail. We discuss how our enumeration relates to theirs. We further compute the Tits building of the groups in question. Our computation is based on groups and indefinite quadratic forms and the algorithms used are explained.

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1 | INTRODUCTION

The moduli space $\mathcal{M}^0_{\mathrm{En}}$ of Enriques surfaces is an open subset of a 10-dimensional orthogonal modular variety, which was shown by Kondō [22] to be rational. This description is obtained by considering the universal cover of Enriques surfaces, which leads to the moduli space of $K3$

surfaces with a fixed-point free involution. Indeed, $\mathcal{M}^0_{\mathrm{En}}$ can be viewed as the moduli space of N-polarised $K3$ surfaces where

$$N = U + U(2) + E_8(-2). \tag{1}$$

Here, U denotes a hyperbolic plane, $E_8(-1)$ is the negative-definite E_8-lattice and $U(2)$ and $E_8(-2)$ means that the bilinear forms are multiplied by 2. These $K3$ surfaces carry a non-symplectic free involution giving rise to a quotient which is an Enriques surface.

Taking a slightly different viewpoint, one can also consider moduli spaces of *polarised* Enriques surfaces, that is, Enriques surfaces with an ample line bundle. These moduli spaces come in two flavours, namely as moduli spaces of *polarised* or *numerically polarised* Enriques surfaces, depending on whether one considers the polarisation as an element in the Néron–Severi group $\mathrm{NS}(S)$ or the group $\mathrm{Num}(S)$ of divisors modulo numerical equivalence. We recall that $\mathrm{Num}(S)$ is the quotient of $\mathrm{NS}(S)$ by the 2-torsion element given by the canonical class K_S. It was shown in [17] that the moduli spaces $\mathcal{M}^a_{\mathrm{En},h}$ of numerically polarised Enriques surfaces are open subsets of 10-dimensional orthogonal modular varieties $\mathcal{M}_{\mathrm{En},h}$ (see (3)) and the moduli spaces $\widehat{\mathcal{M}}^a_{\mathrm{En},h}$ of polarised Enriques surfaces are étale $2:1$ covers $\widehat{\mathcal{M}}^a_{\mathrm{En},h} \to \mathcal{M}^a_{\mathrm{En},h}$. In [17], we also asked the question when this covering is connected. A complete answer was given in [21, Theorem 1.1]: the space $\mathcal{M}^a_{\mathrm{En},h}$ is connected if and only if the class h is not 2-divisible in $\mathrm{Num}(S)$.

Moduli spaces of polarised Enriques surfaces behave in some ways very differently from moduli spaces of polarised $K3$ surfaces. Indeed, it was shown in [17, Theorem 1.1] that there are only finitely many moduli spaces, up to isomorphism, of (numerically) polarised Enriques surfaces. The starting point of this paper is the question: how many different moduli spaces of Enriques surfaces exist? Here, we shall treat this question from the point of view of orthogonal modular varieties.

To describe the results of this paper, we need some more details concerning moduli spaces of numerically polarised Enriques surfaces, which are all open subsets of orthogonal modular varieties. As usual (see also Section 2 for more details), we denote by $\mathcal{D}_N$ a connected component of the 10-dimensional type IV domain Ω_N associated to N and by $\mathrm{O}(N)$ and $\mathrm{O}^+(N)$ the orthogonal group and the orthogonal group of transformations with real spinor norm 1. These act on Ω_N and $\mathcal{D}_N$, respectively, and we set

$$\mathcal{M}_{\mathrm{En}} := \mathrm{O}^+(N)\backslash\mathcal{D}_N.$$

The moduli space $\mathcal{M}^0_{\mathrm{En}}$ of Enriques surfaces is the open subset of $\mathcal{M}_{\mathrm{En}}$

$$\mathcal{M}^0_{\mathrm{En}} := \mathcal{M}_{\mathrm{En}} \setminus \Delta_{-2},$$

where Δ_{-2} is the image of all hyperplanes orthogonal to roots r in N. This is necessary to ensure that we really have period points on Enriques surfaces. By [28, Theorem 2.13], the hypersurface Δ_{-2} is irreducible.

There is also the notion of moduli spaces of Enriques surfaces with a level-2 structure. For this, we consider the dual lattice of N, which we denote by $N^\vee$, and the stable orthogonal group $\widetilde{\mathrm{O}}(N)$, which is defined as the group of all elements in $\mathrm{O}(N)$ acting trivially on the discriminant $D(N) = N^\vee/N$. We set

$$\widetilde{\mathrm{O}}^+(N) := \mathrm{O}^+(N) \cap \widetilde{\mathrm{O}}(N),$$

and note that this is an index 2 subgroup since the reflection with respect to a vector of length 2 in the summand U of N gives an element in $\widetilde{\mathrm{O}}(N)$ with real spinor norm -1. Let

$$\widetilde{\mathcal{M}}_{\mathrm{En}} := \widetilde{\mathrm{O}}^+(N)\backslash\mathcal{D}_N.$$

The open subset

$$\widetilde{\mathcal{M}}^0_{\mathrm{En}} := \widetilde{\mathcal{M}}_{\mathrm{En}} \setminus \widetilde{\Delta}_{-2}$$

defined as the complement of the hypersurfaces orthogonal to the roots can be interpreted as the *moduli space of Enriques surfaces with a level 2 structure*.

We recall that $D(N) = N^\vee/N \cong (\mathbb{F}_2)^{10}$ and

$$\mathrm{O}(D(N)) \cong \mathrm{O}^+(\mathbb{F}_2^{10})$$

is the orthogonal group of *even* type whose order is $|\,\mathrm{O}^+(\mathbb{F}_2^{10})| = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$. For details, see [23, §1] and [13, Chap. I, §16, Chap. II. §10]. We also recall that $\mathrm{O}(N) \to \mathrm{O}(D(N))$ is surjective (see [29, Theorem 3.6.3] and

$$\mathrm{O}(D(N)) \cong \mathrm{O}(N)/\widetilde{\mathrm{O}}(N) \cong \mathrm{O}^+(N)/\widetilde{\mathrm{O}}^+(N). \tag{2}$$

We will describe the construction of moduli spaces of polarised Enriques surface in more detail in Section 2. Here, we only want to state that all moduli spaces $\mathcal{M}^a_{\mathrm{En},h}$ are open subsets of orthogonal modular varieties

$$\mathcal{M}_{\mathrm{En},h} := \Gamma_h^+\backslash\mathcal{D}_N, \tag{3}$$

where

$$\widetilde{\mathrm{O}}^+(N) \subset \Gamma_h^+ \subset \mathrm{O}^+(N). \tag{4}$$

From this, one has to remove the hyperplanes orthogonal to the roots and some hyperplanes which are orthogonal to certain (-4)-vectors. The latter is necessary to ensure that h represents an ample class, not removing these hyperplanes means that we are also considering quasi-polarisations, that is, nef and big line bundles. Here, we are exclusively concerned with the orthogonal varieties $\mathcal{M}_{\mathrm{En},h}$. Obviously, there are only finitely many possible choices of subgroups Γ_h. Each such choice defines an orthogonal modular variety $\mathcal{M}_{\mathrm{En},h}$ which is covered by $\widetilde{\mathcal{M}}_{\mathrm{En}}$ and covers $\mathcal{M}_{\mathrm{En}}$, in turn,

$$\widetilde{\mathcal{M}}_{\mathrm{En}} \to \mathcal{M}_{\mathrm{En},h} \to \mathcal{M}_{\mathrm{En}}.$$

Note that the maps involved here are not necessarily Galois coverings.

In the situation described here, a number of natural questions arise which we want to address in this paper. The first question is to ask for the number of possible modular varieties which arise in connection with moduli spaces of polarised Enriques surfaces. We rephrase this question in terms of arithmetic groups.

Question 1.1. How many subgroups Γ_h^+, arising from moduli spaces $\mathcal{M}_{\mathrm{En},h}$ of polarised Enriques surfaces (cf. (3)), exist (up to conjugacy)?

We will see in Theorem 3.5 that there are 87 such conjugacy classes. In Tables 1 and 2, we will provide further information about these groups, in particular, their index in $\mathrm{O}^+(N)$ (which is equivalent to knowing the index of $\tilde{\mathrm{O}}^+(N)$ in Γ_h^+). This can be rephrased in terms of subgroups of the finite group orthogonal group $\mathrm{O}^+(\mathbb{F}_2^{10})$, see Question 2.1.

The next question concerns the relation between the degree $h^2 = 2d$ of a polarisation and the possible groups Γ_h^+. In the case of $K3$ surface, given the degree $h^2 = 2d$ of a primitive polarisation, we obtain an irreducible moduli space of $2d$-polarised $K3$ surfaces. The reason is that the $K3$-lattice $L_{K3} = 3U + 2E_8(-1)$ is unimodular and the group $\mathrm{O}(L_{K3})$ acts transitively on all primitive vectors of fixed norm. This is no longer true in the case of Enriques surfaces. Indeed, for given degree $h^2 = 2d > 2$, one has to expect many primitive vectors h which are not equivalent modulo the action of the isometry group of the Néron–Severi lattice $M(1/2) := U + E_8(-1)$.

This leads us to our next as follows.

Question 1.2. Enumerate all inequivalent primitive vectors $h \in U + E_8(-1)$ of given (small) degree $h^2 = 2d$ under the action of the group $\mathrm{O}(U + E_8(-1))$.

We will give an answer to this in Theorem 3.3. In Table 3, we gather the information as to which polarisations define conjugate groups Γ_h^+.

In [17, Proposition 5.7], the existence of a polarisation h_0 was shown such that $\Gamma_{h_0}^+$ is minimal, that is, $\Gamma_{h_0}^+ = \widetilde{\mathrm{O}}^+(N)$. This is of interest as the corresponding modular variety $\mathcal{M}_{\mathrm{En},h_0} = \widetilde{\mathcal{M}}_{\mathrm{En}}$ covers all varieties $\mathcal{M}_{\mathrm{En},h}$. Hence, it is natural to ask the following.

Question 1.3. What is the minimal degree $d_{\min} = h_0^2$ such that there exists a vector h_0 with $\Gamma_{h_0}^+ = \widetilde{\mathrm{O}}^+(N)$, that is, $\mathcal{M}_{\mathrm{En},h_0} = \widetilde{\mathcal{M}}_{\mathrm{En}}$? Is such a vector of minimal degree unique?

We shall provide an answer to this question in Theorem 3.1 where we will see that there is a unique such vector h_0 of minimal degree $h_0^2 = 1240$.

Naturally, moduli spaces of polarised Enriques surfaces of small degree have been studied classically. For a discussion of classical constructions for $d \leqslant 10$, we refer to Dolgachev's paper [14]. In the case of degree 4, Casnati [4] studied polarisations which are base-point free and lead to a 4:1 cover of the projective plane (also called Cossec–Verra polarisations). He showed that this defines an irreducible moduli space which is also rational. There are also degree 4 polarisations (ample line bundles) which are not base point free. These are sometimes not considered to be polarisations in the literature (see [4, Section 1]). The case of (base point free) polarisations of degree 6 is the classical case representing Enriques surfaces as singular sextic surfaces in $\mathbb{P}^3$. For degree 10, there exists one polarisation with generically very ample line bundle. This leads to Reye congruences, respectively, degree 10 models in $\mathbb{P}^5$. We shall discuss these cases and the relation with our calculations more systematically in Section 3.7.

Ciliberto, Dedieu, Galati and Knutsen undertook a very systematic enumeration of moduli spaces of polarised Enriques surfaces in [5], based on the ϕ-invariant of a polarisation. This is the minimal degree of a polarisation on an effective elliptic curve. This enumeration was taken further in [21] where it was shown that the moduli spaces depend on a finer invariant, called the

ϕ-vector, which is the minimal (defined in a proper way) degree of the polarisation with respect to a whole isotropic 10-sequence (see [21, Theorem 1.4]). This leads us to the following.

Question 1.4. How can the enumerations given by our methods and that of Ciliberto et al. be matched?

A complete matching will be provided in Tables 5 and 6.

When one wants to study the geometry of moduli spaces, one typically has to work with projective compactifications of the modular varieties $\Gamma_h^+\backslash\mathcal{D}_N$. The natural choices here are the Baily–Borel and toroidal compactifications. The first is canonically defined for all orthogonal modular varieties, and the second involves a choice of fans. In either case, it is important to know the cusps as these are in $1:1$ correspondence with the boundary components of the Baily–Borel compactification. In the orthogonal case, one has zero-dimensional cusps (points) and one-dimensional cusps (modular curves) which have to be added to the orthogonal modular variety to obtain the Baily–Borel compactification. We recall that for all arithmetic orthogonal groups Γ of lattices of signature $(2, n)$, the zero and one-dimensional cusps are in $1:1$ correspondence with the Γ-orbits of rational isotropic lines and rational isotropic planes, respectively. More generally, a zero-dimensional cusp is contained in the closure of a one-dimensional cusp if and only if $l \subset e$ for some representatives of the corresponding isotropic line and plane, respectively. The Tits building is the 1-complex whose vertices are the orbits of isotropic lines and planes and whose edges are given by the inclusion relation. The Tits building $\mathcal{T}(\Gamma_h^+)$ encodes the combinatorial structure of the boundary of the Baily–Borel compactification of $\Gamma_h^+\backslash\mathcal{D}_N$. This leads to the following.

Question 1.5. How many zero- and one-dimensional cusps do the varieties $\Gamma_h^+\backslash\mathcal{D}_N$ have? More generally, what can we say about the Tits building $\mathcal{T}(\Gamma_h^+)$?

This question will be addressed in Section 3.6.

2 | CONSTRUCTION OF THE MODULI SPACES

In this section, we want to give more details on the construction of the moduli spaces and the groups involved. The starting point is the fact that for an Enriques surface S, the group of divisors modulo numerical equivalence is

$$H^2(S, \mathbb{Z})_f = \mathrm{Num}(S) \cong U + E_8(-1).$$

The fact that the canonical class K_S is 2-torsion implies the existence of an étale $2:1$ cover $p : X \to S$ where X is a K3 surface. We denote the corresponding involution on X by $\sigma : X \to X$. It is well known that the intersection form equips $H^2(X, \mathbb{Z})$ with the structure of a lattice, namely

$$H^2(X, \mathbb{Z}) \cong 3U + 2E_8(-1) =: L_{\mathrm{K3}},$$

where we refer to L_{K3} as the K3 lattice. Under the $2:1$ cover $p : X \to S$, the intersection form is multiplied by a factor 2, and thus,

$$p^*(H^2(S, \mathbb{Z})) \cong U(2) + E_8(-2) =: M.$$

By [29, Theorem 1.14.4], the primitive embedding of the lattice $U(2) + E_8(-2)$ into the K3 lattice L_{K3} is unique (up to the action of $\mathrm{O}(L_{K3})$). Hence, we may assume that M is embedded into L_{K3} by the embedding $(x, u) \mapsto (x, 0, x, u, u)$ where $x \in U(2), u \in E_8(-2)$. When we refer to the sublattice M of L_{K3} we will always assume this embedding. The sublattice M also has an interpretation in terms of the involution

$$\rho : L_{\mathrm{K3}} = 3U + 2E_8(-1) \to L_{\mathrm{K3}} = 3U + 2E_8(-1),$$

$$\rho(x, y, z, u, v) = (z, -y, x, v, u).$$

Clearly, M can be identified with the $(+1)$-eigenspace $\mathrm{Eig}(\rho)^+$ of ρ. The (-1)-eigenspace $\mathrm{Eig}(\rho)^-$ can be identified with the lattice N as defined in (1). Indeed, we can embed the lattice N primitively into L_{K3} by $(y, z, v) \mapsto (z, y, -z, v, -v)$ and this gives

$$\mathrm{Eig}(\rho)^- = M^{\perp}_{L_{\mathrm{K3}}} \cong N.$$

We shall now explain how the groups Γ_h^+ arise. When one wants to construct moduli spaces of numerically polarised Enriques surfaces, one considers pairs (S, h) where h is the class of a numerical polarisation. This defines an element $h \in U + E_8(-1) = M(1/2)$ of positive degree $h^2 = 2d > 0$. For what follows we can and will assume that this vector is primitive. Given h, one has to consider the stabiliser

$$\mathrm{O}(M(1/2), h) = \mathrm{O}(M, h) = \{g \in \mathrm{O}(M(1/2)) = \mathrm{O}(M) \mid g(h) = h\}.$$

The natural maps $\pi_M : \mathrm{O}(M) \to \mathrm{O}(D(M))$ and $\pi_N : \mathrm{O}(N) \to \mathrm{O}(D(N))$ are surjective. Since M and N are orthogonal to each other in the $K3$ lattice L_{K3}, the discriminant groups $D(M)$ and $D(N)$ are naturally isomorphic:

$$(D(M), q_M) \cong (D(N), -q_N).$$

Here, q_M and q_N are the induced quadratic forms. We shall forthwith identify these discriminant groups and hence also $\mathrm{O}(D(M))$ and $\mathrm{O}(D(N))$.

The crucial definition is

$$\Gamma_h := \pi_N^{-1}(\pi_M(\mathrm{O}(M, h))) \subset \mathrm{O}(N). \tag{5}$$

Since $\widetilde{\mathrm{O}}(N) \subset \Gamma_h$ is a normal subgroup of $\mathrm{O}(N)$ of finite index, the group Γ_h is an arithmetic subgroup of $\mathrm{O}(N)$. We again note that the subgroup

$$\Gamma_h^+ = \Gamma_h \cap \mathrm{O}^+(N)$$

has index 2.

In fact, we can rephrase our Question 1.1 on the groups Γ_h^+ entirely in terms of subgroups of $\mathrm{O}^+(\mathbb{F}_2^{10})$. For this, let

$$\bar{\Gamma}_h := \pi_M(\mathrm{O}(M, h) \subset \mathrm{O}^+(\mathbb{F}_2^{10}).$$

Since the natural map $\mathrm{O}^+(N) \to \mathrm{O}(D(N)) \cong \mathrm{O}^+(\mathbb{F}_2^{10})$ is surjective, Question 1.1 can be solved by giving an answer to the following.

Question 2.1. How many subgroups (up to conjugacy) of the form $\bar{\Gamma}_h$ are there in $O^+(\mathbb{F}_2^{10})$?

3 | THE COMPUTATIONS

3.1 | Some basic facts and roots

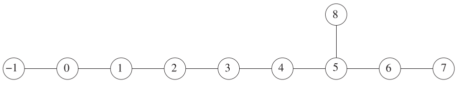
The lattice $M(-1/2) = U + E_8$ is known under different names. It is actually the root lattice of the hyperbolic Coxeter group E_{10} (with E_n, $n \leqslant 8$ being the classical ones and E_9 being the affine extension of E_8). It is also the even Lorentzian lattice $II_{9,1}$. Another common name is E_8^{++}, see [20] for more details. We will use the following Gram matrix for $M(1/2)$:

$$G = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -2 & -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -2 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & -2 & 1 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & -1 & 0 & -2 & 1 \\ 0 & 0 & 0 & -1 & 1 & 1 & 1 & 0 & 1 & -2 \end{pmatrix}. \tag{6}$$

We shall first collect some basic facts about this lattice. It is known to be 2-reflective, see [30]. Hence, Vinberg's algorithm [38] can be employed to compute a fundamental domain D of the Weyl group of the lattice. Here, we give a list of roots that define the facets of (a possible choice of) D, which is a list of simple roots. The roots r satisfy $r^2 = -2$, are numbered from -1 to 8 and have the coordinates:

$$\begin{aligned} -1 &= (-1, 1, 0, 0, 0, 0, 0, 0, 0, 0), \\ 0 &= (0, -1, -1, 1, 1, -2, -2, -2, 2, -1), \\ 1 &= (0, 0, 0, 0, -1, 1, 1, 1, -1, 0), \\ 2 &= (0, 0, 0, 0, 1, -1, 0, 0, 0, 0), \\ 3 &= (0, 0, 0, 0, 0, 0, -1, 0, 1, 0), \\ 4 &= (0, 0, 1, 0, -1, 1, 1, 0, -2, -1), \\ 5 &= (0, 0, -1, -1, 0, -1, -1, 0, 2, 1), \\ 6 &= (0, 0, 0, 0, 0, 1, 0, 0, -1, 0), \\ 7 &= (0, 0, 1, 1, 0, 0, 1, 0, -1, 0), \\ 8 &= (0, 0, 0, 1, 1, 0, 1, 0, 0, 0). \end{aligned}$$

The associated Coxeter–Dynkin diagram is



Since there are 10 simple roots, it follows that the fundamental domain is simplicial. The generators g_i of the extreme rays are the following:

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & -2 & -2 & 1 & 1 & -3 & -3 & -2 & 3 & -2 \\ -2 & -2 & -1 & 1 & 1 & -2 & -2 & -2 & 2 & -1 \\ -3 & -3 & -2 & 2 & 1 & -4 & -4 & -3 & 3 & -3 \\ -3 & -3 & -2 & 2 & 1 & -3 & -3 & -3 & 3 & -2 \\ -4 & -4 & -3 & 3 & 2 & -6 & -5 & -4 & 5 & -4 \\ -4 & -4 & -3 & 3 & 2 & -5 & -4 & -4 & 4 & -3 \\ -5 & -5 & -4 & 4 & 3 & -7 & -6 & -5 & 6 & -4 \\ -6 & -6 & -4 & 5 & 3 & -8 & -7 & -6 & 6 & -6 \end{pmatrix}.$$

A straightforward computation shows that these generators even define a $\mathbb{Z}$-basis of the lattice $U + E_8(-1)$, and hence, the fundamental domain is in fact a basic cone.

The symmetrix matrix $W = g_i \cdot g_j$ with respect to the simple roots g_i is easily computed to be

$$W = \begin{pmatrix} 0 & 1 & 2 & 2 & 3 & 3 & 4 & 4 & 5 & 6 \\ 1 & 2 & 4 & 4 & 6 & 6 & 8 & 8 & 10 & 12 \\ 2 & 4 & 4 & 6 & 7 & 8 & 9 & 10 & 12 & 14 \\ 2 & 4 & 6 & 6 & 9 & 9 & 12 & 12 & 15 & 18 \\ 3 & 6 & 7 & 9 & 10 & 12 & 14 & 15 & 18 & 21 \\ 3 & 6 & 8 & 9 & 12 & 12 & 16 & 16 & 20 & 24 \\ 4 & 8 & 9 & 12 & 14 & 16 & 18 & 20 & 24 & 28 \\ 4 & 8 & 10 & 12 & 15 & 16 & 20 & 20 & 25 & 30 \\ 5 & 10 & 12 & 15 & 18 & 20 & 24 & 25 & 30 & 36 \\ 6 & 12 & 14 & 18 & 21 & 24 & 28 & 30 & 36 & 42 \end{pmatrix}. \tag{7}$$

From the above presentation, one can also conclude that the Coxeter–Dynkin E_{10} has only trivial symmetries. It thus follows that the isometry group and the Coxeter group coincide:

$$\mathrm{O}(U + E_8(-1)) = W(U + E_8(-1)). \tag{8}$$

We also mention that the Coxeter–Dynkin is simply laced, that is, has no multiple edges (but we will not make use of this fact).

This information already allows us to give an answer to Question 1.3.

Theorem 3.1. *The minimal norm of integer vectors with trivial stabiliser in $U + E_8(-1)$ is 1240 and in this degree, there is a unique such vector.*

Proof. Since the Coxeter–Dynkin diagram has no symmetries, we have already concluded in (8) that the isometry group and the Coxeter group of the lattice $U + E_8(-1)$ coincide. Hence, a vector h has trivial stabiliser if and only if it is in the interior of the fundamental domain. Since the g_i

form a $\mathbb{Z}$-basis of the lattice, it follows that

$$v = \sum_{i=1}^{10} a_i g_i \text{ for some } a_i \in \mathbb{N}_{>0}.$$

It then follows from the form of W in (7), notably the observation that all entries in the matrix W are non-negative and only one entry is 0, that the minimum value for v^2 is obtained if and only if all $a_i = 1$. We can then conclude, again from (7), that this minimum value is $v^2 = 1240$. □

Remark 3.2. The vector h with norm $h^2 = 1240$ is characterised by the property that $(h, r) = 1$ for every root r defining a wall of the Weyl chamber. This is called the *Weyl vector* in [7, Chapter 27, §2, Theorem 1].

We note that this fits very well with the results obtained by Knutsen in [21, Proposition 1.5] where a geometric construction of a divisor class h_0 was given such that h_0^2 is minimal and the corresponding moduli space dominates all moduli spaces of numerically polarised Enriques surfaces. The divisor found by Knutsen also satisfies $h_0^2 = 1240$, and we checked by computer that his polarisation and the polarisation h from Theorem 3.1 are equivalent, confirming that the corresponding modular varieties are the same.

3.2 | Enumerating polarisations of small degree

Our next aim is to enumerate the number of inequivalent polarisations in a given degree (for small values of d).

Theorem 3.3. *The list of non-isotropic vectors of norm at most 30 in the fundamental domain is given in Table 3.*

Proof. The matrix of scalar products $(g_i \cdot g_j)$ is positive except for the isotropic vector. We can enumerate the vectors w of the form

$$\sum_{k=2}^{10} \alpha_k g_k \ \text{ for } \alpha_k \in \mathbb{N}$$

with $w \cdot w \leqslant 30$. Since $g_k \cdot g_k > 0$ for $k \geqslant 2$, we have a finite set of possible solutions. For such a w, we consider the vectors $t = \beta g_1 + w$ for $\beta \in \mathbb{N}$. Since we want to find non-isotropic vectors, we have $w \neq 0$. We have $t \cdot t = 2\beta g_1 \cdot w + w \cdot w$. Since $w \neq 0$, we also have $g_1 \cdot w > 0$ and thus a finite number of possibilities to consider. □

Remark 3.4. We postpone the table to Subsection 3.3 because we will then also add the information about which polarisations will lead to the same modular varieties.

We note that there are two different polarisations in degree 4. The first is is given by $h = g_1 + g_2$, and the second by $h = g_3$. Another representation of the first polarisation is $h = e + 2f$ where e, f are a standard basis of the hyperbolic plane U, that is, $e^2 = f^2 = 0$ and $e \cdot f = 1$. Indeed, if one sets $e = g_2 - g_1$ and $f = g_1$, then one gets that e, f define a hyperbolic plane. This leads to a polarisation with base points (since it has degree 1 on an elliptic curve). The second polarisation is the one treated by Casnati. Similarly, there are two polarisations in degree 6, one corresponding to $h = 2g_1 + g_2$ or, alternatively, $h = e + 3f$. This is again not base point free, the other polarisation is $h = g_4$ and leads to sextic surfaces in $\mathbb{P}^3$. In general, there are the non-base-point-free polarisations $h = kg_1 + g_2$, or equivalently, $h = e + (k+1)f$ of degree $2k + 2$. We shall see later that all polarisations $h = kg_1 + g_2$ lead to equivalent subgroups Γ_h^+ and thus to isomorphic modular varieties. We note that in the (classical) literature, non-base-point-free polarisations are sometimes excluded. We will return to the connection with the classical cases in more detail in Subsection 3.7.

3.3 | Enumerating moduli spaces

We will now start enumerating the conjugacy classes of the groups Γ_h^+. By Section 2, this is equivalent to enumerating all conjugacy classes of the groups $\bar{\Gamma}_h \subset \mathrm{O}^+(\mathbb{F}_2^{10})$. We shall give detailed information on the groups in Tables 1 and 2.

Theorem 3.5. *There are 87 conjugacy classes of subgroups of the form* Γ_h^+.

Proof. Let $H = \{\sum_{i=1}^{10} \alpha_i g_i \mid \alpha_i \in \mathbb{R}_{\geqslant 0}\}$ be our chosen fundamental domain of the group $O(U + E_8(-1))$. The crucial fact which we use is the following: the stabiliser of a point x in H is generated by the reflections corresponding to the facets of H in which x is contained. For a proof, see [18, Theorem 4.8], which, in turn, refers to [18, Theorem 1.12.c]. Hence, a group Γ_h^+ is determined by the set of roots to which h is orthogonal to. There are exactly 10 roots for the fundamental domain. We note that the isotropic vector g_1 cannot represent a polarisation. Further, h cannot be orthogonal to all roots (as these span the lattice). This leaves us with $2^{10} - 1 - 1$ possibilities.

All remaining sets give us potential subgroups Γ_h^+. We shall now work with the groups $\bar{\Gamma}_h$, which makes this a finite problem. These groups can be represented as a permutation group acting on 2^{10} elements. By using [16], we can check when two subgroups are conjugate and thus reduce from 1022 to 87 subgroups. In order to do this practically, one needs to compute suitable invariants. The level 1 invariants are the order and the size of the orbits of these groups. For groups with less than 1000 elements, we compute all their subgroups and their associated level 1 invariants. This gets us a more powerful invariant that would not be possible to compute for the larger groups of this enumeration. □

In Tables 1 and 2, we provide detailed information on the groups $\bar{\Gamma}_h$ (and thus equivalently for Γ_h^+). For this, we use our description that the group Γ_h^+ is completely determined by the facets of the fundamental domain containing h. This allows us to describe these groups in terms of admissible subsets of the Dynkin diagram, that is, subsets which are neither the set of all roots nor consist of only the isotropic vector. If the number of generating elements is greater than 5, then we take the complement of the subset of the Dynkin diagram and indicate this by a line over the set given in the table. We then give the number of subsets defining the same conjugacy class of subgroups. The next columns give the order of $\bar{\Gamma}_h$ and we then provide the number of orbits of isotropic vectors and planes in the lattice N with respect to the group Γ_h^+ (we will return to the latter in more detail in Subsection 3.6). Our computations also show that for each group Γ_h, there is a unique orbit of a vector $h_{\min}$ with $h_{\min}^2$ minimal and $\Gamma_h = \Gamma_{h_{\min}}$ (up to conjugation). In the last column, we provide the ϕ-invariant of the vector $h_{\min}$ representing the group Γ_h^+.

TABLE 1 The group $\bar{\Gamma}_h$. For each group, we give one representative as a subset S of the diagram. If the number of generating elements is greater than 5, then we take the complement and denote this by $\overline{S}$. We then give the number $\#S$ of subsets leading to the same group. The next column is the order of $\bar{\Gamma}_h$. The next three columns give the numbers $\#I_1$, $\#I_2$ and $\#I_{12}$ of orbits of isotropic lines, planes and flags in N. The penultimate column gives the degree of the (unique) smallest realisation $h_{\min}$ having this group and the last column gives $\phi(h_{\min})$ (Part 1).

Nr	S	$\#S$	$\lvert\bar{\Gamma}_h\rvert$	$\#I_1$	$\#I_2$	$\#I_{12}$	min deg	$\phi(h_{\min})$
1	$\overline{\{0\}}$	1	$2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$	5	9	18	2	1
2	$\overline{\{-1,0\}}$	1	$2^{13} \cdot 3^5 \cdot 5^2 \cdot 7$	7	13	28	4	1
3	$\overline{\{7\}}$	1	$2^{15} \cdot 3^4 \cdot 5 \cdot 7$	5	10	19	4	2
4	$\overline{\{1\}}$	1	$2^{11} \cdot 3^5 \cdot 5 \cdot 7$	6	14	28	6	2
5	$\overline{\{-1,7\}}$	1	$2^{14} \cdot 3^2 \cdot 5 \cdot 7$	9	23	49	8	2
6	$\overline{\{0,1\}}$	2	$2^{11} \cdot 3^4 \cdot 5 \cdot 7$	9	23	51	10	2
7	$\overline{\{8\}}$	1	$2^8 \cdot 3^4 \cdot 5^2 \cdot 7$	6	13	28	10	3
8	$\overline{\{2\}}$	1	$2^{10} \cdot 3^5 \cdot 5$	8	22	47	12	3
9	$\overline{\{0,7\}}$	1	$2^{11} \cdot 3^2 \cdot 5 \cdot 7$	11	34	77	14	3
10	$\overline{\{7,8\}}$	3	$2^7 \cdot 3^4 \cdot 5 \cdot 7$	11	31	74	16	3
11	$\overline{\{6\}}$	1	$2^8 \cdot 3^4 \cdot 5 \cdot 7$	8	22	49	18	4
12	$\overline{\{1,2\}}$	2	$2^8 \cdot 3^5 \cdot 5$	12	38	89	18	3
13	$\overline{\{-1,0,7\}}$	1	$2^{10} \cdot 3^2 \cdot 5 \cdot 7$	15	48	116	20	3
14	$\overline{\{3\}}$	1	$2^{10} \cdot 3^2 \cdot 5^2$	10	34	76	20	4
15	$\overline{\{-1,0,1\}}$	1	$2^{10} \cdot 3^4 \cdot 5 \cdot 7$	12	32	76	22	3
16	$\overline{\{1,7\}}$	1	$2^{10} \cdot 3^3 \cdot 5$	13	49	113	22	4
17	$\overline{\{0,8\}}$	3	$2^8 \cdot 3^2 \cdot 5 \cdot 7$	14	51	125	24	4
18	$\overline{\{0,2\}}$	1	$2^9 \cdot 3^4 \cdot 5$	13	46	108	26	4
19	$\overline{\{2,7\}}$	3	$2^{10} \cdot 3^2 \cdot 5$	16	67	162	28	4
20	$\overline{\{4\}}$	1	$2^7 \cdot 3^3 \cdot 5^2$	11	41	96	30	5
21	$\overline{\{0,1,7\}}$	2	$2^{10} \cdot 3^2 \cdot 5$	19	83	204	30	4
22	$\overline{\{-1,0,8\}}$	4	$2^7 \cdot 3^2 \cdot 5 \cdot 7$	19	74	191	32	4
23	$\overline{\{0,1,2\}}$	3	$2^8 \cdot 3^4 \cdot 5$	17	64	159	34	4
24	$\overline{\{5,8\}}$	2	$2^5 \cdot 3^3 \cdot 5 \cdot 7$	16	68	171	34	5
25	$\overline{\{5,7\}}$	3	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	18	85	213	36	5
26	$\overline{\{0,3\}}$	2	$2^9 \cdot 3^2 \cdot 5$	18	87	213	38	5
27	$\overline{\{3,8\}}$	3	$2^6 \cdot 3^2 \cdot 5^2$	19	91	233	40	5
28	$\overline{\{5\}}$	1	$2^6 \cdot 3^3 \cdot 5 \cdot 7$	12	49	115	42	6
29	$\overline{\{1,2,7\}}$	5	$2^8 \cdot 3^2 \cdot 5$	24	124	320	42	5
30	$\overline{\{3,7\}}$	1	$2^9 \cdot 3^2 \cdot 5$	18	85	208	44	6
31	$\overline{\{0,1,8\}}$	9	$2^5 \cdot 3^2 \cdot 5 \cdot 7$	24	122	322	44	5
32	$\overline{\{2,8\}}$	3	$2^7 \cdot 3^3 \cdot 5$	18	85	215	46	6
33	$\overline{\{-1,5\}}$	4	$2^6 \cdot 3^3 \cdot 5$	21	115	294	48	6
34	$\overline{\{-1,0,1,7\}}$	1	$2^9 \cdot 3^2 \cdot 5$	25	118	305	50	5
35	$\overline{\{2,6\}}$	4	$2^7 \cdot 3^2 \cdot 5$	23	136	351	52	6
36	$\overline{\{0,2,7\}}$	4	$2^9 \cdot 3 \cdot 5$	26	151	388	54	6

(Continues)

TABLE 1 (Continued)

Nr	S	$\#S$	$\lvert\bar{\Gamma}_h\rvert$	$\#I_1$	$\#I_2$	$\#I_{12}$	min deg	$\phi(h_{\min})$
37	$\overline{\{2,3,7\}}$	2	$2^9 \cdot 3^2$	29	179	466	56	6
38	$\overline{\{1,2,8\}}$	8	$2^5 \cdot 3^3 \cdot 5$	28	167	448	58	6
39	$\overline{\{0,2,8\}}$	9	$2^6 \cdot 3^2 \cdot 5$	31	207	553	60	6
40	$\overline{\{-1,0,1,8\}}$	5	$2^4 \cdot 3^2 \cdot 5 \cdot 7$	32	181	495	62	6
41	$\overline{\{2,7,8\}}$	13	$2^6 \cdot 3^2 \cdot 5$	31	200	540	64	6

Note that this information immediately gives the degree of the maps

$$\widetilde{\mathcal{M}}_{\mathrm{En}} \to \mathcal{M}_{\mathrm{En},h} \to \mathcal{M}_{\mathrm{En}}.$$

The first is the order of $\bar{\Gamma}_h$, the latter the index $[\mathrm{O}^+(\mathbb{F}_2^{10}) : \bar{\Gamma}_h]$, where we recall that $\lvert \mathrm{O}^+(\mathbb{F}_2^{10})\rvert = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31$.

3.4 | Degree of the polarisation and number of moduli spaces

As the degree of the polarisation increases, the number of inequivalent polarisations will also grow. At the same time, the number of conjugacy classes of groups $\bar{\Gamma}_h$, and hence of modular varieties $\mathcal{M}_{\mathrm{En},h}$ is limited by 87. This means that inequivalent polarisations must give rise to isomorphic modular varieties. We will now discuss this in more detail.

In Table 3, we list a representative for each polarisation class in given low degree. We also enumerate the different conjugacy classes of the groups $\bar{\Gamma}_h$. The entries $2 : \{g_1 + g_2\}$ and $2 : \{2g_1 + g_2\}$, for example, mean that the degree 4 polarisation $g_1 + g_2$ and the degree 6 polarisation $2g_1 + g_2$ define conjugate subgroups $\bar{\Gamma}_h$.

Table 4 gives the following information for each degree $2d = 2, \dots, 72$ and corresponding genus $g = 2, \dots, 37$: the first line shows the number $\#h$ of orbits of primitive vectors in a given degree. We note that these numbers agree exactly with the corresponding list in [5, Appendix]. The second line $\#\bar{\Gamma}_h$ gives the number of conjugacy classes of groups $\bar{\Gamma}_h$ for given degree. We note that $\#\bar{\Gamma}_h \leqslant \#h$ and that strict inequality will occur when different orbits of primitive vectors h give rise to conjugate subgroups $\bar{\Gamma}_h$. This phenomenon first appears in degree 12 where the stabiliser groups of the two polarisations $5g_1 + g_2$ and $g_1 + 2g_2$ actually agree. As the degree grows, the number of orbits $\#h$ will grow much faster than $\#\bar{\Gamma}_h$. We note that the numbers in this list agree with those given in [17, Corollary 5.6].

In the next two lines, we compare how in degree at most $2d$ the number of orbits and the number of conjugacy classes increase. We see that we have found 312 different classes of polarisations and 46 different conjugacy classes of groups $\bar{\Gamma}_h$ in degree $\leqslant 72$. The number of subgroups will finally stabilise to 87 by Theorem 3.5. This happens in degree 1240, which is a lower limit by Theorem 3.1. The above lists can easily be extended to higher degree (genus) using the programs we have. The norm $2d = 72$ is the first one for which there is no new group occurring.

3.5 | Connection with the ϕ-invariant

In [5, Appendix], Ciliberto et al. gave a systematic enumeration of polarisations for genus up to 30. We will now match their enumeration with our results.

TABLE 2 Continue (Part 2).

Nr	S	$\#S$	$\lvert\bar{\Gamma}_h\rvert$	$\#I_1$	$\#I_2$	$\#I_{12}$	min deg	$\phi(h_{\min})$
42	$\overline{\{1,4\}}$	1	$2^5\cdot 3^3\cdot 5$	24	149	389	66	7
43	$\overline{\{0,1,2,7\}}$	7	$2^8\cdot 3\cdot 5$	34	219	585	66	6
44	$\overline{\{0,5\}}$	2	$2^6\cdot 3^2\cdot 5$	27	187	487	68	7
45	$\overline{\{-1,0,1,2\}}$	1	$2^7\cdot 3^4\cdot 5$	22	92	238	70	6
46	$\overline{\{0,3,7\}}$	2	$2^8\cdot 3^2$	33	239	629	70	7
47	$\overline{\{0,5,8\}}$	19	$2^5\cdot 3^2\cdot 5$	36	274	746	76	7
48	$\overline{\{0,1,2,8\}}$	17	$2^5\cdot 3^2\cdot 5$	41	307	849	78	7
49	$\overline{\{1,5\}}$	2	$2^6\cdot 3^3$	30	231	608	84	8
50	$\overline{\{1,2,3,7\}}$	3	$2^7\cdot 3^2$	44	354	967	84	7
51	$\overline{\{2,6,8\}}$	7	$2^7\cdot 3^2$	38	306	832	88	8
52	$\overline{\{0,5,7\}}$	6	$2^6\cdot 3\cdot 5$	40	342	927	92	8
53	$\overline{\{1,2,7,8\}}$	20	$2^4\cdot 3^2\cdot 5$	48	413	1159	96	8
54	$\overline{\{0,1,5\}}$	14	$2^6\cdot 3^2$	45	429	1175	100	8
55	$\overline{\{0,1,3,7\}}$	3	$2^8\cdot 3$	48	435	1184	102	8
56	$\overline{\{2,3,7,8\}}$	11	$2^6\cdot 3^2$	51	463	1298	104	8
57	$\overline{\{1,5,8\}}$	4	$2^5\cdot 3^3$	40	341	937	106	9
58	$\overline{\{0,5,7,8\}}$	30	$2^5\cdot 3\cdot 5$	53	512	1433	108	8
59	$\overline{\{-1,0,1,2,8\}}$	6	$2^4\cdot 3^2\cdot 5$	54	465	1315	110	8
60	$\overline{\{-1,0,1,2,7\}}$	2	$2^7\cdot 3\cdot 5$	44	325	890	114	8
61	$\overline{\{1,2,5\}}$	4	$2^4\cdot 3^3$	48	484	1339	120	9
62	$\overline{\{0,1,5,8\}}$	46	$2^5\cdot 3^2$	60	649	1832	124	9
63	$\overline{\{0,1,2,7,8\}}$	32	$2^4\cdot 3\cdot 5$	70	782	2235	132	9
64	$\overline{\{0,1,2,3,7\}}$	4	$2^7\cdot 3$	63	653	1825	138	9
65	$\overline{\{0,2,5\}}$	3	$2^5\cdot 3^2$	54	611	1685	140	10
66	$\overline{\{0,1,5,7\}}$	20	$2^6\cdot 3$	67	814	2291	148	10
67	$\overline{\{1,2,3,7,8\}}$	30	$2^4\cdot 3^2$	80	1002	2884	156	10
68	$\overline{\{0,1,2,5\}}$	30	$2^4\cdot 3^2$	72	929	2637	160	10
69	$\overline{\{0,1,5,7,8\}}$	57	$2^5\cdot 3$	89	1252	3599	180	11
70	$\overline{\{1,2,5,8\}}$	5	$2^3\cdot 3^3$	64	737	2097	184	12
71	$\overline{\{0,2,5,7\}}$	11	$2^5\cdot 3$	81	1174	3323	196	12
72	$\{4,6,3,5\}$	8	$2^3\cdot 3\cdot 5$	92	1210	3503	198	11
73	$\overline{\{0,1,2,5,8\}}$	48	$2^3\cdot 3^2$	96	1440	4166	208	12
74	$\overline{\{0,1,2,5,7\}}$	64	$2^4\cdot 3$	108	1818	5251	220	12
75	$\{4,6,-1,5\}$	44	$2^4\cdot 3$	118	1953	5690	228	12
76	$\{4,6,8,5\}$	1	$2^6\cdot 3$	82	996	2829	234	12
77	$\{4,6,3,7\}$	19	$2^2\cdot 3^2$	128	2263	6625	260	13
78	$\overline{\{0,1,3,5,7\}}$	9	2^5	122	2306	6647	280	14
79	$\{4,6,-1,3\}$	99	$2^3\cdot 3$	144	2856	8357	292	14
80	$\{4,6,-1,2\}$	39	2^4	163	3626	10 599	340	15
81	$\{4,6,5\}$	9	$2^3\cdot 3$	156	3074	9031	342	15

(Continues)

TABLE 2 (Continued)

Nr	S	$\#S$	$\lvert\bar{\Gamma}_h\rvert$	$\#I_1$	$\#I_2$	$\#I_{12}$	min deg	$\phi(h_{\min})$
82	$\{4, 6, 3\}$	54	$2^2 \cdot 3$	192	4532	13 369	380	16
83	$\{4, 6, -1\}$	57	2^3	218	5766	17 003	460	18
84	$\{4, 3\}$	9	$2 \cdot 3$	256	7242	21 471	532	19
85	$\{4, 6\}$	36	2^2	292	9246	27 411	580	20
86	$\{4\}$	10	2	392	14 926	44 387	820	24
87	$\emptyset$	1	1	528	24 242	72 199	1240	30

A crucial role in [5] is played by the minimal degree of a polarisation h on an effective elliptic curve E, namely

$$\phi(h) = \min\{h \cdot E \mid E^2 = 0, E > 0\}.$$

Using this parameter and the genus, they consider the moduli spaces $\widehat{\mathcal{E}}_{g,\phi}$ of polarised Enriques surfaces with given ϕ and genus g. The crucial tool in their enumeration is the notion of *decomposition type* given in [5, Definition 4.13]. This can lead to more than one component of a moduli space $\widehat{\mathcal{E}}_{g,\phi}$, in our cases denoted by $\widehat{\mathcal{E}}^{(I)}_{g,\phi}$ and $\widehat{\mathcal{E}}^{(II)}_{g,\phi}$.

Another essential technical tool is the notion of an *isotropic 10-sequence* as defined in [5, Definition 3.2], and which goes back to Cossec and Dolgachev [9, p. 122]. This is a collection of effective isotropic classes $E_i, i = 1, \dots, 10$ which span the lattice $M(1/2) = U + E_8(-1)$ over the rationals with the additional property that $E_i \cdot E_j = 1$ for $i \neq j$. By [5, Lemma 3.4], see also [8, Lemma 1.6.2 (i)] or [9, Corollary 2.5.5], an isotropic 10-sequence has the further property that $\sum E_i$ is 3-divisible, that is, there is a divisor D with $3D \equiv \sum_i E_i$. By the defining property of an isotropic 10-sequence, it then follows that $D^2 = 10$. This observation also implies that the E_i form a $\mathbb{Q}$-basis, but not a $\mathbb{Z}$-basis of $M(1/2)$.

We further note that the list of [5, Appendix] also contains non-primitive numerical polarisations, which we disregard in our approach since they do not lead to new moduli spaces.

We now want to provide a precise matching between the (components of the) moduli spaces $\widehat{\mathcal{E}}_{g,\phi}$ and our modular varieties $\mathcal{M}_{\mathrm{En},h}$. To do this, we first introduce a new integral basis u_i, $i = 1, \dots, 10$ of the lattice $M(1/2) = U + E_8(-1)$ with Gram matrix

$$G_u = \begin{pmatrix} 0 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 3 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 3 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 3 & 3 & 10 & 3 & 3 & 3 & 3 & 3 \\ 1 & 1 & 1 & 1 & 3 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}. \tag{9}$$

The equivalence between the basis u_i and the standard basis of the lattice $U + E_8(-1)$ is provided by the matrix

TABLE 3 The primitive vectors of norms at most 30 in the fundamental domain expressed in the basis (g_i). The vectors are first grouped by norm and then all together. If two or more vectors give rise to conjugate groups, then they are grouped in a set whose index corresponds to the one of Tables 1 and 2.

deg	Polarisations
2	$1:\{g_2\}$.
4	$2:\{g_1+g_2\}, 3:\{g_3\}$.
6	$2:\{2g_1+g_2\}, 4:\{g_4\}$.
8	$2:\{3g_1+g_2\}, 5:\{g_1+g_3\}$.
10	$2:\{4g_1+g_2\}, 6:\{g_1+g_4\}, 7:\{g_5\}$.
12	$2:\{5g_1+g_2, g_1+2g_2\}, 5:\{2g_1+g_3\}, 8:\{g_6\}$.
14	$2:\{6g_1+g_2\}, 6:\{2g_1+g_4\}, 9:\{g_2+g_3\}$.
16	$2:\{7g_1+g_2\}, 5:\{3g_1+g_3\}, 6:\{g_2+g_4\}, 10:\{g_1+g_5\}$.
18	$2:\{8g_1+g_2\}, 6:\{3g_1+g_4\}, 11:\{g_7\}, 12:\{g_1+g_6\}$.
20	$2:\{9g_1+g_2, 3g_1+2g_2\}, 5:\{4g_1+g_3\}, 13:\{g_1+g_2+g_3\}, 14:\{g_8\}$.
22	$2:\{10g_1+g_2\}, 6:\{4g_1+g_4\}, 10:\{2g_1+g_5\}, 15:\{g_1+g_2+g_4\}, 16:\{g_3+g_4\}$.
24	$2:\{11g_1+g_2, g_1+3g_2\}, 5:\{5g_1+g_3, g_1+2g_3\}, 12:\{2g_1+g_6\}, 17:\{g_2+g_5\}$.
26	$2:\{12g_1+g_2\}, 6:\{5g_1+g_4\}, 13:\{2g_1+g_2+g_3\}, 17:\{g_1+g_7\}, 18:\{g_2+g_6\}$.
28	$2:\{13g_1+g_2, 5g_1+2g_2\}, 5:\{6g_1+g_3\}, 9:\{2g_2+g_3\}, 10:\{3g_1+g_5, g_3+g_5\}, 15:\{2g_1+g_2+g_4\}, 19:\{g_1+g_8\}$.
30	$2:\{14g_1+g_2, 2g_1+3g_2\}, 6:\{6g_1+g_4, 2g_2+g_4\}, 12:\{3g_1+g_6\}, 20:\{g_9\}, 21:\{g_1+g_3+g_4\}$.
all	$1:\{g_2\}, 2:\{g_1+g_2, 2g_1+g_2, 3g_1+g_2, 4g_1+g_2, 5g_1+g_2, 6g_1+g_2, 7g_1+g_2, 8g_1+g_2, 9g_1+g_2, 10g_1+g_2, 11g_1+g_2, 12g_1+g_2, 13g_1+g_2, 14g_1+g_2, g_1+2g_2, 3g_1+2g_2, 5g_1+2g_2, g_1+3g_2, 2g_1+3g_2\}, 3:\{g_3\}, 4:\{g_4\}, 5:\{g_1+g_3, 2g_1+g_3, 3g_1+g_3, 4g_1+g_3, 5g_1+g_3, 6g_1+g_3, g_1+2g_3\}, 6:\{g_1+g_4, 2g_1+g_4, 3g_1+g_4, 4g_1+g_4, 5g_1+g_4, 6g_1+g_4, g_2+g_4, 2g_2+g_4\}, 7:\{g_5\}, 8:\{g_6\}, 9:\{g_2+g_3, 2g_2+g_3\}, 10:\{g_1+g_5, 2g_1+g_5, 3g_1+g_5, g_3+g_5\}, 11:\{g_7\}, 12:\{g_1+g_6, 2g_1+g_6, 3g_1+g_6\}, 13:\{g_1+g_2+g_3, 2g_1+g_2+g_3\}, 14:\{g_8\}, 15:\{g_1+g_2+g_4, 2g_1+g_2+g_4\}, 16:\{g_3+g_4\}, 17:\{g_2+g_5, g_1+g_7\}, 18:\{g_2+g_6\}, 19:\{g_1+g_8\}, 20:\{g_9\}, 21:\{g_1+g_3+g_4\}$.

TABLE 4 The number of primitive polarisations and modular groups.

g	2	3	4	5	6	7	8	9	10	11	12	13
$2d$	2	4	6	8	10	12	14	16	18	20	22	24
$\#h$	1	2	2	2	3	4	3	4	4	5	5	6
$\#\bar{\Gamma}_h$	1	2	2	2	3	3	3	4	4	4	5	4
$\#h, h^2 \leqslant 2d$	1	3	5	7	10	14	17	21	25	30	35	41
$\#\bar{\Gamma}_h, h^2 \leqslant 2d$	1	3	4	5	7	8	9	10	12	14	16	17
g	14	15	16	17	18	19	20	21	22	23	24	25
$2d$	26	28	30	32	34	36	38	40	42	44	46	48
$\#h$	5	8	7	6	8	8	7	10	10	10	11	11
$\#\bar{\Gamma}_h$	5	6	5	6	8	6	7	7	7	7	10	7
$\#h, h^2 \leqslant 2d$	46	54	61	67	75	83	90	100	110	120	131	142
$\#\bar{\Gamma}_h, h^2 \leqslant 2d$	18	19	21	22	24	25	26	27	29	31	32	33
g	26	27	28	29	30	31	32	33	34	35	36	37
$2d$	50	52	54	56	58	60	62	64	66	68	70	72
$\#h$	9	14	11	12	14	16	13	15	16	16	18	16
$\#\bar{\Gamma}_h$	9	10	11	8	12	9	11	14	11	11	12	10
$\#h, h^2 \leqslant 2d$	151	165	176	188	202	218	231	246	262	278	296	312
$\#\bar{\Gamma}_h, h^2 \leqslant 2d$	34	35	36	37	38	39	40	41	43	44	46	46

$$T = \begin{pmatrix} 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 2 & 1 & -2 & 1 & 2 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & 3 & -1 & -1 & -1 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}, \tag{10}$$

meaning that $T^t G_u T = G$ with G as in (6). The columns of the matrix T are the vectors u_i.

The isotropic 10-sequence which we are looking for can be obtained from the basis u_i by means of the transition matrix

$$I = \begin{pmatrix} -1 & -1 & -1 & -1 & 3 & -1 & - & -1 & -1 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0, & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0, & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

One sees immediately that $\det(I) = 3$. The Gram matrix of the vectors E_i is given by

$$G_E = (1)_{1\leqslant i,j\leqslant 10} - I_n, \tag{12}$$

which immediately shows that the defining conditions of an isotropic 10-sequence are satisfied.

We can now make the correspondence between the varieties $\mathcal{M}_{\mathrm{En},h}$ and $\widehat{\mathcal{E}}_{g,\phi}$ for $g \leqslant 30$ explicit. The result is given in Tables 5 and 6. Note that we also list non-primitive polarisations here in order to have a full matching with [5, Appendix].

3.6 | The Tits building

As we already recalled, the zero- and one-dimensional cusps of the modular varieties $\mathcal{M}_{\mathrm{En},h} = \Gamma_h^+\backslash\mathcal{D}_N$ correspond to the orbits of isotropic vectors l and isotropic planes h in the lattice $N = U + U(2) + E_8(-2)$ with respect to the groups Γ_h^+. The inclusions $l \subset h$ characterise when a zero-dimensional cusp is contained in a one-dimensional cusp. Taking orbits modulo the group Γ_h^+ defines the Tits building which thus incorporates the entire combinatorial structure of the boundary. We will now investigate this systematically for small degrees.

The classical case of unpolarised Enriques surfaces is well known. We start by recalling that $\mathcal{M}_{\mathrm{En}} = \mathrm{O}^+(N)\backslash\mathcal{D}_N$ has two zero-dimensional and two 1-dimensional cusps each. Proofs of this were given by Sterk [37, Propositions 4.5 and 4.6] and Allcock [1, Corollary 4]. By applying the computational techniques of Section 4, we confirm these results. It is not difficult to give explicit representatives of these orbits. The orbits of isotropic lines are spanned by $L_1 = \mathbb{Z}e_1$ and $L_2 = \mathbb{Z}e_3$. The orbits of isotropic planes of N are $P_1 = \mathbb{Z}e_1 + \mathbb{Z}e_3$ and $P_2 = \mathbb{Z}(2e_1 + 2e_2 + w) + \mathbb{Z}e_3$ with w a vector of norm 4 in E_8, which then viewed as a vector in $E_8(-2)$ has norm $w^2 = -8$ in $E_8(-2)$. Here, (e_1, e_2) and (e_3, e_4) are standard bases of U and $U(2)$. The Tits building is displayed in Table 1. The stabiliser of L_1 has an image in the discriminant group which is equal to the full group, whereas the image of the stabiliser of L_2 has index $527 = 17 \cdot 31$. The stabiliser of the plane P_1 has an image of index 527, whereas the image of the stabiliser of P_2 has an image of index $23\,715 = 3^2 \cdot 5 \cdot 17 \cdot 31$.

Here, we would also like to mention that the space $\widetilde{\mathcal{M}}_{\mathrm{En}} = \widetilde{\mathrm{O}}^+(N)\backslash\mathcal{D}_N$ has 528 zero-dimensional cusps (corresponding to isotropic lines) and 24 242 one-dimensional cusps (corresponding isotropic planes). Under the group $\mathrm{O}^+(\mathbb{F}_2)$, the 528 zero-dimensional cusps decompose into two orbits, one of length 527 and one of length 1, see the above discussion and [10, p. 534]. The set of 24 242 isotropic planes also has two orbits, and these are of length 527 and 23 715, respectively.

When working with the group Γ_h, we need to compute the orbits for a subgroup of the full isometry group G. That is, given an orbit xG, we write G_x for the stabiliser of x by G. The decomposition $xG = \cup_{i\in I} x_i\Gamma_h$ corresponds to a double coset decomposition

$$G = \cup_{i\in I} G_x h_i \Gamma_h \text{ for } x_i = xh_i.$$

Thus orbit splitting can be done with double coset decomposition. This is in general a difficult problem, but in the case of finite groups, there are well-known algorithms [33, Sec. 8.1.1].

Lemma 3.6. *Let G be a group, U a normal subgroup and K and H two subgroups of G such that $U \subset K$. Then, the quotient map $G \to G/U$ establishes a many-to-one correspondence between double coset decompositions of G by (K, H) and of G/U by $(K/U, \overline{H})$ where $\overline{H}$ is the image of H in G/U.*

TABLE 5 Expression of the entries of [5, Appendix] in terms of the g vectors (Part 1). Here, the first column gives the genus g and the second the value of ϕ. The next two columns give the description of the polarisation according to [5] and in terms of the g_i.

g	ϕ	CDGK	DH	g	ϕ	CDGK	DH	g	ϕ	CDGK	DH
2	1	$\widehat{E}_{2,1}$	g_2	3	1	$\widehat{E}_{3,1}$	g_1+g_2	3	2	$\widehat{E}_{3,2}$	g_3
4	1	$\widehat{E}_{4,1}$	$2g_1+g_2$	4	2	$\widehat{E}_{4,2}$	g_4	5	1	$\widehat{E}_{5,1}$	$3g_1+g_2$
5	2	$\widehat{E}^{(I)}_{5,2}$	g_1+g_3	5	2	$\widehat{E}^{(II)}_{5,2}$	$2g_2$	6	1	$\widehat{E}_{6,1}$	$4g_1+g_2$
6	2	$\widehat{E}_{6,2}$	g_1+g_4	6	3	$\widehat{E}_{6,3}$	g_5	7	1	$\widehat{E}_{7,1}$	$5g_1+g_2$
7	2	$\widehat{E}^{(I)}_{7,2}$	$2g_1+g_3$	7	2	$\widehat{E}^{(II)}_{7,2}$	g_1+2g_2	7	3	$\widehat{E}_{7,3}$	g_6
8	1	$\widehat{E}_{8,1}$	$6g_1+g_2$	8	2	$\widehat{E}_{8,2}$	$2g_1+g_4$	8	3	$\widehat{E}_{8,3}$	g_2+g_3
9	1	$\widehat{E}_{9,1}$	$7g_1+g_2$	9	2	$\widehat{E}^{(I)}_{9,2}$	$3g_1+g_3$	9	2	$\widehat{E}^{(II)}_{9,2}$	$2g_1+2g_2$
9	3	$\widehat{E}^{(I)}_{9,3}$	g_1+g_5	9	3	$\widehat{E}^{(II)}_{9,3}$	g_2+g_4	9	4	$\widehat{E}_{9,4}$	$2g_3$
10	1	$\widehat{E}_{10,1}$	$8g_1+g_2$	10	2	$\widehat{E}_{10,2}$	$3g_1+g_4$	10	3	$\widehat{E}^{(I)}_{10,3}$	g_1+g_6
10	3	$\widehat{E}^{(II)}_{10,3}$	$3g_2$	10	4	$\widehat{E}_{10,4}$	g_7	11	1	$\widehat{E}_{11,1}$	$9g_1+g_2$
11	2	$\widehat{E}^{(I)}_{11,2}$	$4g_1+g_3$	11	2	$\widehat{E}^{(II)}_{11,2}$	$3g_1+2g_2$	11	3	$\widehat{E}_{11,3}$	$g_1+g_2+g_3$
11	4	$\widehat{E}_{11,4}$	g_8	12	1	$\widehat{E}_{12,1}$	$10g_1+g_2$	12	2	$\widehat{E}_{12,2}$	$4g_1+g_4$
12	3	$\widehat{E}^{(I)}_{12,3}$	$g_1+g_2+g_4$	12	3	$\widehat{E}^{(II)}_{12,3}$	$2g_1+g_5$	12	4	$\widehat{E}_{12,4}$	g_3+g_4
13	1	$\widehat{E}_{13,1}$	$11g_1+g_2$	13	2	$\widehat{E}^{(I)}_{13,2}$	$5g_1+g_3$	13	2	$\widehat{E}^{(II)}_{13,2}$	$4g_1+2g_2$
13	3	$\widehat{E}^{(I)}_{13,3}$	$2g_1+g_6$	13	3	$\widehat{E}^{(II)}_{13,3}$	g_1+3g_2	13	4	$\widehat{E}^{(I)}_{13,4}$	g_2+g_5
13	4	$\widehat{E}^{(II)}_{13,4}$	$2g_4$	13	4	$\widehat{E}^{(III)}_{13,4}$	g_1+2g_3	14	1	$\widehat{E}_{14,1}$	$12g_1+g_2$
14	2	$\widehat{E}_{14,2}$	$5g_1+g_4$	14	3	$\widehat{E}_{14,3}$	$2g_1+g_2+g_3$	14	4	$\widehat{E}^{(I)}_{14,4}$	g_2+g_6
14	4	$\widehat{E}^{(II)}_{14,4}$	g_1+g_7	15	1	$\widehat{E}_{15,1}$	$13g_1+g_2$	15	2	$\widehat{E}^{(I)}_{15,2}$	$6g_1+g_3$
15	2	$\widehat{E}^{(II)}_{15,2}$	$5g_1+2g_2$	15	3	$\widehat{E}^{(I)}_{15,3}$	$2g_1+g_2+g_4$	15	3	$\widehat{E}^{(II)}_{15,3}$	$3g_1+g_5$
15	4	$\widehat{E}^{(I)}_{15,4}$	g_1+g_8	15	4	$\widehat{E}^{(II)}_{15,4}$	$2g_2+g_3$	15	5	$\widehat{E}_{15,5}$	g_3+g_5
16	1	$\widehat{E}_{16,1}$	$14g_1+g_2$	16	2	$\widehat{E}_{16,2}$	$6g_1+g_4$	16	3	$\widehat{E}^{(I)}_{16,3}$	$3g_1+g_6$
16	3	$\widehat{E}^{(II)}_{16,3}$	$2g_1+3g_2$	16	4	$\widehat{E}^{(I)}_{16,4}$	$2g_2+g_4$	16	4	$\widehat{E}^{(II)}_{16,4}$	$g_1+g_3+g_4$
16	5	$\widehat{E}_{16,5}$	g_9	17	1	$\widehat{E}_{17,1}$	$15g_1+g_2$	17	2	$\widehat{E}^{(I)}_{17,2}$	$7g_1+g_3$
17	2	$\widehat{E}^{(II)}_{17,2}$	$6g_1+2g_2$	17	3	$\widehat{E}_{17,3}$	$3g_1+g_2+g_3$	17	4	$\widehat{E}^{(I)}_{17,4}$	g_1+2g_4
17	4	$\widehat{E}^{(II)}_{17,4}$	$g_1+g_2+g_5$	17	4	$\widehat{E}^{(III)}_{17,4}$	$2g_1+2g_3$	17	4	$\widehat{E}^{(IV)}_{17,4}$	$4g_2$
17	5	$\widehat{E}_{17,5}$	g_3+g_6	18	1	$\widehat{E}_{18,1}$	$16g_1+g_2$	18	2	$\widehat{E}_{18,2}$	$7g_1+g_4$
18	3	$\widehat{E}^{(I)}_{18,3}$	$3g_1+g_2+g_4$	18	3	$\widehat{E}^{(II)}_{18,3}$	$4g_1+g_5$	18	4	$\widehat{E}^{(I)}_{18,4}$	$g_1+g_2+g_6$
18	4	$\widehat{E}^{(II)}_{18,4}$	$2g_1+g_7$	18	5	$\widehat{E}^{(I)}_{18,5}$	g_2+2g_3	18	5	$\widehat{E}^{(II)}_{18,5}$	g_4+g_5
19	1	$\widehat{E}_{19,1}$	$17g_1+g_2$	19	2	$\widehat{E}^{(I)}_{19,2}$	$8g_1+g_3$	19	2	$\widehat{E}^{(II)}_{19,2}$	$7g_1+2g_2$
19	3	$\widehat{E}^{(I)}_{19,3}$	$4g_1+g_6$	19	3	$\widehat{E}^{(II)}_{19,3}$	$3g_1+3g_2$	19	4	$\widehat{E}^{(I)}_{19,4}$	$2g_1+g_8$
19	4	$\widehat{E}^{(II)}_{19,4}$	$g_1+2g_2+g_3$	19	5	$\widehat{E}^{(I)}_{19,5}$	g_4+g_6	19	5	$\widehat{E}^{(II)}_{19,5}$	g_2+g_7
19	6	$\widehat{E}_{19,6}$	$3g_3$	20	1	$\widehat{E}_{20,1}$	$18g_1+g_2$	20	2	$\widehat{E}_{20,2}$	$8g_1+g_4$
20	3	$\widehat{E}_{20,3}$	$4g_1+g_2+g_3$	20	4	$\widehat{E}^{(I)}_{20,4}$	$g_1+2g_2+g_4$	20	4	$\widehat{E}^{(II)}_{20,4}$	$2g_1+g_3+g_4$
20	5	$\widehat{E}^{(I)}_{20,5}$	g_2+g_8	20	5	$\widehat{E}^{(II)}_{20,5}$	$g_1+g_3+g_5$	21	1	$\widehat{E}_{21,1}$	$19g_1+g_2$

Proof. Let us take a double coset decomposition

$$G = \cup_{i\in I} K g_i H.$$

Then, mapping to the quotient, we obtain

$$G/U = \cup_{i\in I} K/U\overline{g_i}\overline{H}.$$

TABLE 6 Expression of the entries of [5, Appendix] in terms of the g vectors (Part 2).

g	ϕ	CDGK	DH	g	ϕ	CDGK	DH	g	ϕ	CDGK	DH
21	2	$\widehat{E}^{(I)}_{21,2}$	$9g_1+g_3$	21	2	$\widehat{E}^{(II)}_{21,2}$	$8g_1+2g_2$	21	3	$\widehat{E}^{(I)}_{21,3}$	$5g_1+g_5$
21	3	$\widehat{E}^{(II)}_{21,3}$	$4g_1+g_2+g_4$	21	4	$\widehat{E}^{(I)}_{21,4}$	g_1+4g_2	21	4	$\widehat{E}^{(II)}_{21,4}$	$3g_1+2g_3$
21	4	$\widehat{E}^{(III)}_{21,4}$	$2g_1+2g_4$	21	4	$\widehat{E}^{(IV)}_{21,4}$	$2g_1+g_2+g_5$	21	5	$\widehat{E}^{(I)}_{21,5}$	g_1+g_9
21	5	$\widehat{E}^{(II)}_{21,5}$	$g_2+g_3+g_4$	21	6	$\widehat{E}_{21,6}$	$2g_5$	22	1	$\widehat{E}_{22,1}$	$20g_1+g_2$
22	2	$\widehat{E}_{22,2}$	$9g_1+g_4$	22	3	$\widehat{E}^{(I)}_{22,3}$	$5g_1+g_6$	22	3	$\widehat{E}^{(II)}_{22,3}$	$4g_1+3g_2$
22	4	$\widehat{E}^{(I)}_{22,4}$	$2g_1+g_2+g_6$	22	4	$\widehat{E}^{(II)}_{22,4}$	$3g_1+g_7$	22	5	$\widehat{E}^{(I)}_{22,5}$	$g_1+g_3+g_6$
22	5	$\widehat{E}^{(II)}_{22,5}$	$2g_2+g_5$	22	5	$\widehat{E}^{(III)}_{22,5}$	g_2+2g_4	22	6	$\widehat{E}_{22,6}$	g_{10}
23	1	$\widehat{E}_{23,1}$	$21g_1+g_2$	23	2	$\widehat{E}^{(I)}_{23,2}$	$10g_1+g_3$	23	2	$\widehat{E}^{(II)}_{23,2}$	$9g_1+2g_2$
23	3	$\widehat{E}_{23,3}$	$5g_1+g_2+g_3$	23	4	$\widehat{E}^{(I)}_{23,4}$	$2g_1+2g_2+g_3$	23	4	$\widehat{E}^{(II)}_{23,4}$	$3g_1+g_8$
23	5	$\widehat{E}^{(I)}_{23,5}$	$g_1+g_2+2g_3$	23	5	$\widehat{E}^{(II)}_{23,5}$	$g_1+g_4+g_5$	23	5	$\widehat{E}^{(III)}_{23,5}$	$2g_2+g_6$
23	6	$\widehat{E}_{23,6}$	g_3+g_8	24	1	$\widehat{E}_{24,1}$	$22g_1+g_2$	24	2	$\widehat{E}_{24,2}$	$10g_1+g_4$
24	3	$\widehat{E}^{(I)}_{24,3}$	$6g_1+g_5$	24	3	$\widehat{E}^{(II)}_{24,3}$	$5g_1+g_2+g_4$	24	4	$\widehat{E}^{(I)}_{24,4}$	$2g_1+2g_2+g_4$
24	4	$\widehat{E}^{(II)}_{24,4}$	$3g_1+g_3+g_4$	24	5	$\widehat{E}^{(I)}_{24,5}$	$g_1+g_4+g_6$	24	5	$\widehat{E}^{(II)}_{24,5}$	$g_1+g_2+g_7$
24	5	$\widehat{E}^{(III)}_{24,5}$	$3g_2+g_3$	24	6	$\widehat{E}^{(I)}_{24,6}$	$2g_3+g_4$	24	6	$\widehat{E}^{(II)}_{24,6}$	g_5+g_6
25	1	$\widehat{E}_{25,1}$	$23g_1+g_2$	25	2	$\widehat{E}^{(I)}_{25,2}$	$11g_1+g_3$	25	2	$\widehat{E}^{(II)}_{25,2}$	$10g_1+2g_2$
25	3	$\widehat{E}^{(I)}_{25,3}$	$6g_1+g_6$	25	3	$\widehat{E}^{(II)}_{25,3}$	$5g_1+3g_2$	25	4	$\widehat{E}^{(I)}_{25,4}$	$2g_1+4g_2$
25	4	$\widehat{E}^{(II)}_{25,4}$	$4g_1+2g_3$	25	4	$\widehat{E}^{(III)}_{25,4}$	$3g_1+2g_4$	25	4	$\widehat{E}^{(IV)}_{25,4}$	$3g_1+g_2+g_5$
25	5	$\widehat{E}^{(I)}_{25,5}$	$2g_1+g_3+g_5$	25	5	$\widehat{E}^{(II)}_{25,5}$	$g_1+g_2+g_8$	25	5	$\widehat{E}^{(III)}_{25,5}$	$3g_2+g_4$
25	6	$\widehat{E}^{(I)}_{25,6}$	g_1+3g_3	25	6	$\widehat{E}^{(II)}_{25,6}$	$2g_6$	25	6	$\widehat{E}^{(III)}_{25,6}$	g_4+g_7
26	1	$\widehat{E}_{26,1}$	$24g_1+g_2$	26	2	$\widehat{E}_{26,2}$	$11g_1+g_4$	26	3	$\widehat{E}_{26,3}$	$6g_1+g_2+g_3$
26	4	$\widehat{E}^{(I)}_{26,4}$	$3g_1+g_2+g_6$	26	4	$\widehat{E}^{(II)}_{26,4}$	$4g_1+g_7$	26	5	$\widehat{E}^{(I)}_{26,5}$	$2g_1+g_9$
26	5	$\widehat{E}^{(II)}_{26,5}$	$g_1+g_2+g_3+g_4$	26	5	$\widehat{E}^{(III)}_{26,5}$	$5g_2$	26	6	$\widehat{E}^{(I)}_{26,6}$	$g_2+g_3+g_5$
26	6	$\widehat{E}^{(II)}_{26,6}$	g_4+g_8	27	1	$\widehat{E}_{27,1}$	$25g_1+g_2$	27	2	$\widehat{E}^{(I)}_{27,2}$	$12g_1+g_3$
27	2	$\widehat{E}^{(II)}_{27,2}$	$11g_1+2g_2$	27	3	$\widehat{E}^{(I)}_{27,3}$	$7g_1+g_5$	27	3	$\widehat{E}^{(II)}_{27,3}$	$6g_1+g_2+g_4$
27	4	$\widehat{E}^{(I)}_{27,4}$	$3g_1+2g_2+g_3$	27	4	$\widehat{E}^{(II)}_{27,4}$	$4g_1+g_8$	27	5	$\widehat{E}^{(I)}_{27,5}$	$2g_1+g_3+g_6$
27	5	$\widehat{E}^{(II)}_{27,5}$	$g_1+2g_2+g_5$	27	5	$\widehat{E}^{(III)}_{27,5}$	$g_1+g_2+2g_4$	27	6	$\widehat{E}^{(I)}_{27,6}$	g_1+2g_5
27	6	$\widehat{E}^{(II)}_{27,6}$	g_3+2g_4	27	6	$\widehat{E}^{(III)}_{27,6}$	g_2+g_9	28	1	$\widehat{E}_{28,1}$	$26g_1+g_2$
28	2	$\widehat{E}_{28,2}$	$12g_1+g_4$	28	3	$\widehat{E}^{(I)}_{28,3}$	$7g_1+g_6$	28	3	$\widehat{E}^{(II)}_{28,3}$	$6g_1+3g_2$
28	4	$\widehat{E}^{(I)}_{28,4}$	$3g_1+2g_2+g_4$	28	4	$\widehat{E}^{(II)}_{28,4}$	$4g_1+g_3+g_4$	28	5	$\widehat{E}^{(I)}_{28,5}$	$2g_1+g_2+2g_3$
28	5	$\widehat{E}^{(II)}_{28,5}$	$2g_1+g_4+g_5$	28	5	$\widehat{E}^{(III)}_{28,5}$	$g_1+2g_2+g_6$	28	6	$\widehat{E}^{(I)}_{28,6}$	g_1+g_{10}
28	6	$\widehat{E}^{(II)}_{28,6}$	$3g_4$	28	6	$\widehat{E}^{(III)}_{28,6}$	$g_2+g_3+g_6$	28	7	$\widehat{E}_{28,7}$	$2g_3+g_5$
29	1	$\widehat{E}_{29,1}$	$27g_1+g_2$	29	2	$\widehat{E}^{(I)}_{29,2}$	$13g_1+g_3$	29	2	$\widehat{E}^{(II)}_{29,2}$	$12g_1+2g_2$
29	3	$\widehat{E}_{29,3}$	$7g_1+g_2+g_3$	29	4	$\widehat{E}^{(I)}_{29,4}$	$3g_1+4g_2$	29	4	$\widehat{E}^{(II)}_{29,4}$	$5g_1+2g_3$
29	4	$\widehat{E}^{(III)}_{29,4}$	$4g_1+2g_4$	29	4	$\widehat{E}^{(IV)}_{29,4}$	$4g_1+g_2+g_5$	29	5	$\widehat{E}^{(I)}_{29,5}$	$2g_1+g_4+g_6$
29	5	$\widehat{E}^{(II)}_{29,5}$	$2g_1+g_2+g_7$	29	5	$\widehat{E}^{(III)}_{29,5}$	$g_1+3g_2+g_3$	29	6	$\widehat{E}^{(I)}_{29,6}$	$g_1+g_3+g_8$
29	6	$\widehat{E}^{(II)}_{29,6}$	$2g_2+2g_3$	29	6	$\widehat{E}^{(III)}_{29,6}$	$g_2+g_4+g_5$	30	1	$\widehat{E}_{30,1}$	$28g_1+g_2$
30	2	$\widehat{E}_{30,2}$	$13g_1+g_4$	30	3	$\widehat{E}^{(I)}_{30,3}$	$7g_1+g_2+g_4$	30	3	$\widehat{E}^{(II)}_{30,3}$	$8g_1+g_5$
30	4	$\widehat{E}^{(I)}_{30,4}$	$4g_1+g_2+g_6$	30	4	$\widehat{E}^{(II)}_{30,4}$	$5g_1+g_7$	30	5	$\widehat{E}^{(I)}_{30,5}$	$3g_1+g_3+g_5$
30	5	$\widehat{E}^{(II)}_{30,5}$	$2g_1+g_2+g_8$	30	5	$\widehat{E}^{(III)}_{30,5}$	$g_1+3g_2+g_4$	30	6	$\widehat{E}^{(I)}_{30,6}$	$g_1+2g_3+g_4$
30	6	$\widehat{E}^{(II)}_{30,6}$	$g_1+g_5+g_6$	30	6	$\widehat{E}^{(III)}_{30,6}$	$g_2+g_4+g_6$	30	6	$\widehat{E}^{(IV)}_{30,6}$	$2g_2+g_7$
30	7	$\widehat{E}^{(I)}_{30,7}$	g_3+g_9	30	7	$\widehat{E}^{(II)}_{30,7}$	$2g_1+g_2+g_8$				

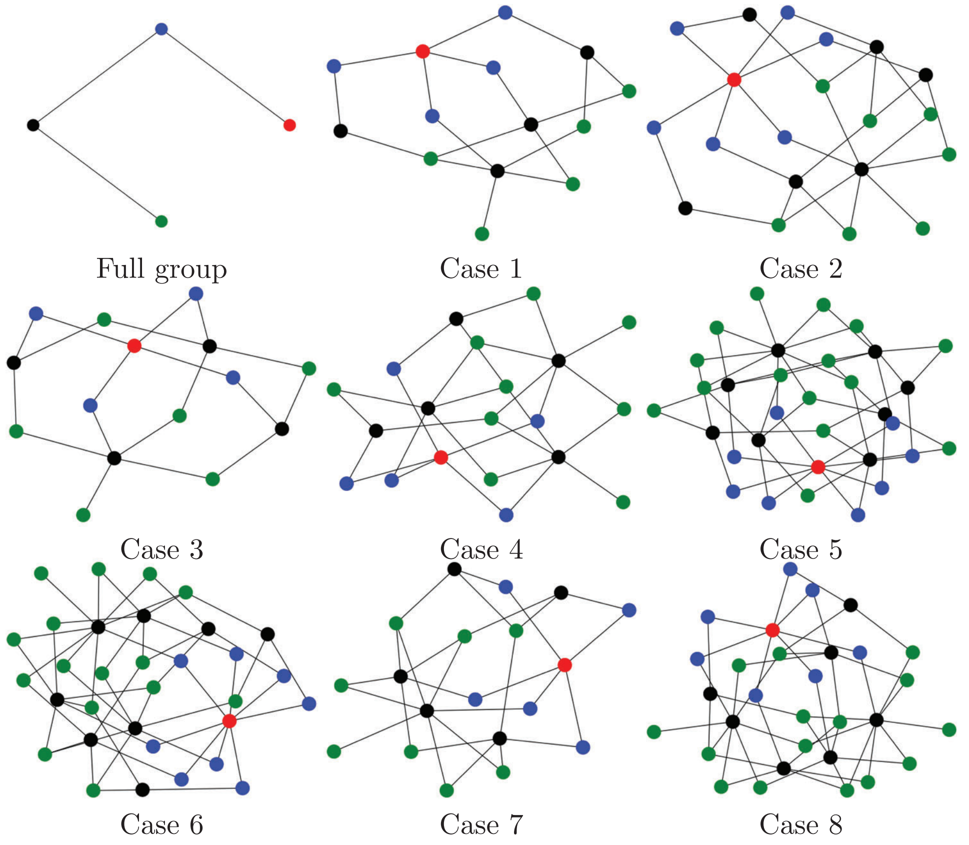


FIGURE 1 Coset graphs of the first eight coset graphs from Table 1. The red, blue, black and green dots correspond to the orbit arising from the orbit of L_1, P_1, L_2 and P_2.

Since $U \subset K$, the double coset Kg_iH is actually a union of left U cosets. Therefore, the double cosets $C_i = K/U\overline{g_i}\overline{H}$ are disjoint, that is, $C_i \cap C_j = \emptyset$ if $i \neq j$. This establishes that the mapping is well defined and it is surjective by construction. □

We can apply the above lemma to our case with $U = \widetilde{\mathrm{O}}(N)$ the kernel of the action of $\mathrm{O}(N)$ on the discriminant $K = \Gamma_h$ and $H = G_x$. The second key ingredient is that the quotient $\mathrm{O}(N)/\widetilde{\mathrm{O}}(N) \cong \mathrm{O}^+(\mathbb{F}_2)$ is finite. We can apply the existing approach for finite groups, as implemented in [16], and thus, reduce the orbit splitting from $\mathrm{O}(N)$ to the group Γ_h.

This approach allows us to compute the number of orbits of lines, planes and flags, and thus, the Tits building, for each subgroup Γ_h. The obtained data on the orbit splitting, that is, the number of isotropic lines and planes as well as inclusions are given in Tables 1 and 2 where they are labelled by $\#I_1$, $\#I_2$ and $\#I_{12}$, respectively. The pictures of the first 8 coset graphs are given in Figure 1. We note that Case 1 coincides with the Tits building found in [37, Fig 14].

3.7 | Classical cases

Here, we briefly discuss how our computations fit in with some classical results.

3.7.1 | The degree 2 case

There is only one degree 2 polarisation h, namely $h = g_2$. This linear system is not base point free, which follows, for example, because $\phi = 1$, that is, there is an elliptic curve on which h has degree 1. The linear system $|2h|$, however, is base-point-free and gives rise to what is classically known a double-plane representation. More precisely, $|2h|$ maps a general Enriques surface $2 : 1$ onto a del Pezzo surface in $\mathbb{P}^4$ which is the intersection of two rank 3 quadrics, see [2, Section 3.3] and [10, Section 3.5]. By [2, Theorem 3.9], a generic Enriques surface admits $2^7 \cdot 17 \cdot 31$ different double-plane representations. This implies that $[\mathrm{O}^+(N) : \Gamma_h] = 2^7 \cdot 17 \cdot 31$ or alternatively, that

$$[\Gamma_h : \tilde{\mathrm{O}}^+(N)] = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31/2^7 \cdot 17 \cdot 31 = 2^{14} \cdot 3^5 \cdot 5^2 \cdot 7$$

This is Case 1 in Table 1.

We note that this case was also treated by Sterk [37] who referred to degree 2 polarisations as almost polarisations. There it is also proved, see [37, Section 4.4], that the corresponding modular variety has five 0-dimensional and nine 1-dimensional cusps, in agreement with our results in Table 1. As we already mentioned above, the Tits building computed by Sterk [37, Fig 14] agrees with our graph for Case 1.

3.7.2 | Enriques realisations

In degree 6, we have two polarisations, which are distinguished by the ϕ-invariant which can be 1 or 2. These are given by $g_1 + g_2$ and g_3, respectively. In the first case, the linear system is not base-point-free, in the other case, it defines, for a generic Enriques surface, a birational map onto a non-normal sextic surface in $\mathbb{P}^3$ with double locus along the edges of a tetrahedron, see [2, Section 3.1] and [10, Section 3.5]. This is historically the first realisation of an Enriques surface. By [2, Theorem 3.10], a general Enriques surface S admits $2^{11} \cdot 5 \cdot 17 \cdot 31$ such realisations. Note, however, that h and $h + K_S$ define projectively inequivalent models. For us, this means that the morphism $\mathcal{M}_{\mathrm{En},h} \to \mathcal{M}_{\mathrm{En}}$ has degree $2^{10} \cdot 5 \cdot 17 \cdot 31$, and hence,

$$[\Gamma_h : \tilde{\mathrm{O}}^+(N)] = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31/2^{10} \cdot 5 \cdot 17 \cdot 31 = 2^{11} \cdot 3^5 \cdot 5 \cdot 7$$

agreeing with Case 4 in Table 1.

3.7.3 | Reye congruences

It follows from Table 3 that we have three different polarisations in degree 10, namely $4g_1 + g_2$, $g_1 + g_4$ and g_5. According to Table 5, these have ϕ-invariants 1, 2 and 3, respectively. In the first case, the linear system $|h|$ is not base-point free, and in the second, it cannot be ample. In the third case, the linear system $|h|$ defines an embedding for the general Enriques surface S and thus a degree 10 model in $\mathbb{P}^5$, see also [2, Section 3] and [10, Section 3.5]. This is classically known as a Reye congruence, or, according to [10] as a Fano model. By [2, Theorem 3.11], a general Enriques surface admits $2^{14} \cdot 3 \cdot 17 \cdot 31$ inequivalent representations as a degree 10 surface in $\mathbb{P}^5$. Since $|h|$ and $|h + K_S|$ define different models, we can conclude that the morphism $\mathcal{M}_{\mathrm{En},h} \to \mathcal{M}_{\mathrm{En}}$ has degree $2^{13} \cdot 3 \cdot 17 \cdot 31$, and hence,

$$[\Gamma_h : \tilde{\mathrm{O}}^+(N)] = 2^{21} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 17 \cdot 31/2^{13} \cdot 3 \cdot 17 \cdot 31 = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7,$$

which agrees exactly with Case 7 in Table 1.

4 | ALGORITHMS FOR WORKING WITH INDEFINITE FORMS

The methods used in this work are, we believe, of wider interest, and thus, we explain in this section in some detail how they work. The code is available via [35] in both a GAP and C++ version. An integrated system is available for Oscar in [39]. The emphasis is on practical techniques. In this section, a lattice is a free $\mathbb{Z}$-module of rank n, which will often be identified with $\mathbb{Z}^n$, equipped with an integer quadratic form A which can possibly be degenerate.

The first class of problems is related to groups. That is, given an integer quadratic form A, we want to compute a generating set of the integral automorphism group $\mathrm{O}(A)$. Next, we want to test the equivalence of two integral quadratic forms A_1 and A_2 by an integral transformation and, if such an isomorphism exists, produce it explicitly.

The second class of problems considered concerns vector representations. That is, given an integer quadratic form A and an integer $\beta \neq 0$, we ask to find all orbit representatives of solutions x of the equation $A[x] := xAx^T = \beta$. For $\beta = 0$, we are looking for primitive solutions. We are also interested in finding k-planes of totally isotropic vectors.

As it turns out, both classes are closely related in our algorithmic approach. In Subsection 4.1, we explain the group techniques used. We then discuss the case of positive and hyperbolic lattices, for which well-known algorithms exist, in Subsection 4.2. In Subsection 4.3, we introduce the notion of approximate model of a lattice, and finally, we show in Subsection 4.4 how all techniques together allow us to solve the above problems.

4.1 | Integral group algorithms

The matrix groups $G \subset \mathrm{GL}_n(\mathbb{Q})$ that we will consider will be in general infinite and will preserve a rank n lattice $L \subset \mathbb{Z}^n$. In particular, this implies that $G \cap \mathrm{GL}_n(\mathbb{Z})$ is a finite index subgroup in G. We will need an algorithmic solution for the following problems:

Alg 1 Compute a generating set of the intersection $G \cap \mathrm{GL}_n(\mathbb{Z})$.
Alg 2 For $x \in \mathrm{GL}_n(\mathbb{Q})$, decide whether there is some $g \in G$ such that $gx \in \mathrm{GL}_n(\mathbb{Z})$ and compute one such g.
Alg 3 Compute the right cosets of $G \cap \mathrm{GL}_n(\mathbb{Z})$ in G.

Without the condition that a lattice L is preserved by G, there is no reason to think that there is a general algorithm as the groups are just too wild. We will limit our exposition to **Alg 1**. The other algorithms use the same ideas and are suitable adaptations to the relevant context.

Let us take L an integral rank n lattice invariant under G and denote by L' the lattice $\mathbb{Z}^n$. Obviously, we have $L \subset L'$ and there exists an integer $d > 0$ such that $L \subset L' \subset L/d$. When expressed in a basis of L, the group G becomes an integral subgroup of $\mathrm{GL}_n(\mathbb{Z})$. By quotienting by dL, we obtain a map $\phi : G \mapsto \mathrm{GL}_n(\mathbb{Z}/d\mathbb{Z})$ mapping the lattice L' to a subset S of $(\mathbb{Z}/d\mathbb{Z})^n$ and the problem can be rephrased as first finding the stabiliser of S under Im ϕ and then computing its pre-image in G.

The group $\mathrm{GL}_n(\mathbb{Z}/d\mathbb{Z})$ is a finite group and finding set-stabilisers is a well-known problem with efficient algorithms [19, 24, 25]. To find the pre-image of a group, the natural way is to use Schreier's lemma [33, Lemma 4.2.1]. If the group G is finite and has a faithful permutation representation on a set W, then we can amalgamate the set-stabiliser and pre-image operations in just one set-stabiliser operation on a finite permutation group acting on $|W| + d^n$ points.

Because of its practical importance, it is essential to accelerate this algorithm as much as possible. A possible speed-up is to use the factorisation of the divisor d into prime factors as $d = p_1 \dots p_r$ and to iterate the computation prime by prime, starting by the smallest occurring prime. Another speed-up is not to consider the full set $(\mathbb{Z}/d\mathbb{Z})^n$ of vectors, but instead to select a vector x in S whose orbit O_x is not contained in S. Then, we compute the stabiliser for $S' = O_x \cap S$. In this way, d^n is reduced to $|O_x|$ which is much smaller. Of course, some additional iterations may be needed for the stabiliser to be computed since there could be other vectors in S whose orbit is contained in S. If we know that a filtration is preserved by G, then it is good to start the search of such x in the smallest subspaces. The commonality between all these approaches is that they replace a big computation with a dominating term d^n into smaller computations though at the expense of having many. This algorithm is an evolution of the last one of [3, Section 3.1] where the problem of finding the group of integral symmetries of a polytope was considered. The GAP and the independent C++ version of the code are available at [35].

4.2 | Positive definite and hyperbolic forms

For positive definite quadratic forms, there are well-known methods [32] for the equivalence problem and for computing a generating set. For the problem of finding representative solutions of $A[x] = \beta$, we can use the Fincke–Pohst algorithm (cf. [6, Algorithm 2.7.7]).

For the case of hyperbolic lattices, this becomes more involved, but is still doable using the method of perfect forms. This is an inefficient technique, but it has the advantage that there are no limitations regarding its use. In this work, we have used the Coxeter group structure for the lattice $U + E_8(-1)$. This is possible because it is a reflective lattice, but most lattices do not have this property, and so, the perfect form method has to be used.

The enumeration of perfect forms is done via a variant of Mertens' algorithm [26]. The main changes are an improvement in the way the facets of the perfect domain are enumerated up to symmetry (see [12] for a description of the algorithm and [35] for implementations) and the use of the method of 4.1 for finding automorphisms and testing isomorphisms of perfect domains.

4.3 | Approximate models and the case of signature $p, q \geqslant 2$

Having dealt with definite forms and hyperbolic lattices, we now turn to signature (p, q) with $p, q \geqslant 2$. The definition below provides the main tool for our work.

Definition 4.1. Given an integral lattice L, an *approximate model* is defined by:

- a set of generators $\{g_1, \dots, g_m\}$ of a subgroup $\mathrm{Ap}(L)$ of $\mathrm{O}(L)$ named *approximate subgroup*,

- an *oracle function* $\mathrm{Ap}(L, \beta)$ that, given a $\beta \neq 0$, returns a finite list $v_1, \ldots, v_{k(\beta)}$ such that any vector of norm β is equivalent by $\mathrm{Ap}(L)$ to one of the v_i. For $\beta = 0$, the oracle function returns a list of primitive vectors of norm 0 such that any primitive vector of norm 0 is equivalent to one such vector by an element of $\mathrm{Ap}(L)$.

It is important to note that a lattice can potentially have an infinite number of approximate models and that we do not claim that every lattice has an approximate model. The approximate subgroup is a finite index subgroup of $\mathrm{O}(L)$ in all cases considered, here but we do not know if that is always the case and we do not use this property here.

Lemma 4.2. *If L is an integral non-degenerate even lattice, then $U + U + L$ has an approximate model.*

Proof. Eichler's criterion [15, §3] applies to this class of lattices and provides an algorithm for obtaining the approximate orbit representatives. We need to prove that we have a finite set of generators of a suitable approximate subgroup. In this case, we can take a suitable subgroup of the group $\mathrm{O}(U + U)$ together with the Eichler transvections. In [34, Example 3.7.2], the lattice $U + U$ is identified with the determinant form on $M_{2,2}(\mathbb{Z})$. Thus, $\mathrm{SL}_2(\mathbb{Z})$ has an action on the left and an action on the right on $M_{2,2}(\mathbb{Z})$. In particular, $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ is a subgroup of $\mathrm{O}(U + U)$. Since $\mathrm{SL}_2(\mathbb{Z})$ is generated by two elements, this subgroup is generated by four generators.

The second step of Eichler's criterion is to apply the Eichler transvections $E_{e_i,x}$ (see [34, Section 3.7]) for e_i with $1 \leqslant i \leqslant 4$ one of the four canonical isotropic vectors coming from the two hyperbolic planes U and x a vector orthogonal to e_i. The Eichler transvections satisfy $E_{e,x}E_{e,y} = E_{e,x+y}$ for e isotropic and x, y orthogonal to e. According to [34, Proposition 3.7.3], we simply need the generators of $\mathrm{SL}_2(\mathbb{Z})^2$ and the transvections $E_{e_i,v_{i,j}}$ with $1 \leqslant i \leqslant 4$, $1 \leqslant j \leqslant n-1$ and $(v_{i,j})_{1 \leqslant j \leqslant n-1}$ forming a $\mathbb{Z}$-basis of $e_i^{\perp}$. Thus, if the dimension of $U + U + L$ is n, we need four generators from $\mathrm{SL}_2(\mathbb{Z})^2$ and $4(n-1)$ from the transvections and so $4n$ together. The proof of Proposition 3.7.3 in [34] provides an explicit way of computing a set of possible vector representatives and so the oracle function. □

It is important to note that the approximate model provided by the above lemma can be improved significantly in some cases. The group provided by the Eichler algorithm acts trivially on the discriminant. For a case such as $U + U + E_8(-2)$, this gets us 2^8 orbit representatives. By adding the isometries of the E_8 component to the approximate subgroup, we are reduced to just three representatives which is far better for computational purposes. This is because $W(E_8)$ has three orbits in its action on $E_8/2E_8$, their sizes being 1, 120 and 135.

Theorem 4.3. *Suppose that L' and L are two integral lattices of rank n with $L' \subset L$ and we have an approximate model for L. Then we have an approximate model for L'.*

Proof. We can compute the stabiliser S of L' under $\mathrm{Ap}(L)$ by **Alg 1** and this gets us an approximate subgroup $\mathrm{Ap}(L')$. By using **Alg 3**, we compute the right coset decomposition of $\mathrm{Ap}(L)$ under S with coset representatives g_1, …, g_m. For $\beta \in \mathbb{Z}$, the approximate model of L gives us representatives x_1, …, x_t of the orbits of vectors of norm β. We then consider all the elements of the form $g_j x_i$ and keep the ones that are contained in L'. This gets us our approximate orbit representatives $\mathrm{Ap}(L', \beta)$. □

In particular, the above shows that any lattice $cU + dU + W$ with $c, d \in \mathbb{N}$ and W integral and even has an approximate model via the following embedding in $U + U + W$:

$$(x_1, x_2, y_1, y_2, w) \mapsto (cx_1, x_2, dy_1, y_2, w).$$

In fact much more is true.

Theorem 4.4. *Let L be an integral lattice of signature $p, q \geqslant 2$ of dimension at least 7. Then L has an approximate model.*

Proof. Let us take the dual $L^\vee$. Since integral indefinite lattices of dimension at least 5 have isotropic vectors (see [27]), there is an isotropic vector v_1 in $L^\vee$. Let us take a vector g not orthogonal to v_1. Then the vector $v_2 = 2(g.v_1)g - (g.g)v_1$ is also isotropic and not orthogonal to v_1.

We then iterate this operation on $L^\vee \cap (\mathbb{Z}v_1 + \mathbb{Z}v_2)^\perp$, where this notation indicates that the orthogonal complement is taken in $L^\vee$, and find two isotropic vectors v_3, v_4. We define $K = L^\vee \cap (\mathbb{Z}v_1 + \mathbb{Z}v_2 + \mathbb{Z}v_3 + \mathbb{Z}v_4)^\perp$ and taking the dual, we obtain

$$L \subset U(c) + U(d) + K^\vee \text{ for some } c, d \in \mathbb{Q}_+.$$

By multiplying by a factor α, we can obtain that αc, αd are integers and that $K^\vee(\alpha)$ is an even integral lattice. Rescaling a lattice leaves its property of having an approximate model invariant.

Finally, we have the embedding

$$U(\alpha c) + U(\alpha d) + K^\vee(\alpha) \subset U + U + K^\vee(\alpha),$$

and we can conclude from Lemma 4.2 that L has an approximate model. □

Lemma 4.2 provides an approximate model for lattices of the form $U + U + L$ with L integral even. The lattices that we are going to consider are not necessarily even nor admit a decomposition $U + U + L$ but we can find an approximate model for them.

The above existence theorem is not necessarily optimal in the sense that the obtained oracle function may get us a large numbers of possible solutions. In our application, we are in the fortunate situation that the lattice $U + U(2) + E_8(-2)$ can be trivially embedded into $U + U + E_8(-2)$ by our previous remark and so no additional work is needed. For finding the isotropic vectors, we use the algorithm of [36] implemented in [31].

4.4 | Solution of the problems

We now use approximate models to solve the equivalence/automorphism and representative problems that we explained at the beginning of this section. The solutions that we provide are effective in the sense that they can be computed on computers, but we do not make any claim on complexity, though runtime is clearly one of our priorities.

For a lattice L of signature (p, q), we define $s(L) = \min(p, q)$. For an integral lattice L, a *splitting integer* is a $\beta \in \mathbb{Z} \setminus \{0\}$ such that there exists a vector v of norm β with $v^\perp$ a lattice satisfying $s(v^\perp) = s(L) - 1$. Clearly, such a number exists if $s(L) \geqslant 1$. We also define $r(L) = \max(p, q)$.

Theorem 4.5. *There exist algorithms solving the equivalence and automorphism group problems for integral non-degenerate lattices with* $r(L) \geqslant 5$.

Proof. The solution to those problems depends on each other, which is why they are stated together.

- **Orth(s)**: The problem of determining a generating set of automorphism groups for non-degenerate lattices L with $s(L) = s$.
- **Equi(s)**: Given two non-degenerate lattices L_1, L_2 with $s(L_1) = s(L_2) = s$ test whether they are isomorphic and if isomorphic find an isomorphism.

For $s(L) = 0$ or 1, Subsection 4.2 provides algorithms. Our solution is inductive in s. Since in the sequel, we will have $s \geqslant 2$, the condition of dimension at least 7 required by Theorem 4.4 is satisfied.

If we can solve **Orth(s-1)** and **Equi(s-1)**, then we can solve **Orth(s)**. To see this, let us take a lattice L with $s(L) = s$ and β a splitting integer. Let us choose an approximate model $\mathrm{Ap}(L)$ of L. The oracle function will provide a set of vectors $\mathrm{Ap}(L, \beta) = \{v_1, \dots, v_m\}$. The lattice $v_1^{\perp}$ has $s(v_1^{\perp}) = s - 1$. Therefore, by **Orth(s-1)**, we can find $\mathrm{O}(v_1^{\perp})$. For $v \in L$, define the sublattice $L_v = v^{\perp} + \mathbb{Z}v$ of L. The group $\mathrm{O}(v_1^{\perp})$ is naturally embedded as a subgroup G of $\mathrm{O}(L_{v_1})$ by sending v_1 to v_1. We want to determine the subgroup H of G that preserves L. Since L_{v_1} is a finite index sublattice of L, this can be done by applying **Alg 1**. Now we need to determine which transformations could map v_1 to one of $v_2, \dots, v_m$. If v_1 is equivalent to some v_i, then $v_1^{\perp}$ is equivalent to $v_i^{\perp}$. This can be tested using **Equi(s-1)**. We get a corresponding map ϕ from L_{v_1} to L_{v_i}. Then by applying **Alg 2** to G and ϕ, we can test whether there exists a map from L to L mapping v_1 to v_i. By taking those transformations when they exist and a generating set of H, we actually find a generating set of $\mathrm{O}(L)$.

If we can solve **Orth(s-1)** and **Equi(s-1)**, then we can solve **Equi(s)**. Let us take two lattices L and L' with $s(L) = s(L') = s$ and β a splitting integer of L. We can assume β is a splitting integer of L' since otherwise they are not equivalent. Take a vector v of norm β in L and an approximate list $\{v'_1, \dots, v'_m\}$ of representatives in L'. We compute the automorphism group $\mathrm{O}(v^{\perp})$ using **Orth(s-1)** and then the corresponding subgroup G of $\mathrm{O}(L_v)$. We simply iterate over the v'_i, form the lattices $v^{\perp}$ and $v_i'^{\perp}$ and check if there is an isomorphism using **Equi(s-1)**. If there is an isomorphism h, we extend it to an isomorphism of L_v to $L'_{v'_i}$. Then, we use **Alg 2** with h and G to check if we can obtain an isomorphism of L to L' mapping v to v'_i. If at some point, we find an equivalence, then we conclude that L and L' are equivalent. If not then, the lattices are not.

By the work done for hyperbolic lattices, we have the solution for **Orth(1)** and **Equi(1)**. Therefore, we have the solution of **Orth(s)** and **Equi(s)** for any $s \geqslant 2$. □

We next show that the assumption that L be non-degenerate is actually not necessary.

Theorem 4.6. *There exist algorithms for solving the equivalence and the automorphism problems for integral lattices with* $r(L) \geqslant 5$.

Proof. If we equip $\mathbb{Z}^n$ with a degenerate quadratic form A, then we can still compute the automorphism group of this lattice. To see this, we first notice that the integral kernel $\ker(A)$ has to be preserved. The group $\mathrm{GL}(\ker(A))$ is isomorphic to $\mathrm{GL}_k(\mathbb{Z})$ with $k = \dim \ker(A)$. We can always find a submodule L' of $\mathbb{Z}^n$ such that A restricted to L' is non-degenerate and $\mathbb{Z}^n = \ker(A) + L'$.

We compute the automorphism group of A restricted to L' by using Theorem 4.5. Then the group $\mathrm{O}(L)$ is isomorphic to

$$\mathrm{GL}_k(\mathbb{Z}) \rtimes \{(\mathbb{Z}^{n-k})^k \rtimes \mathrm{O}(L')\},$$

and so, we can easily get a generating set of that group. This method also works for isomorphism checks. □

Lemma 4.7. *If L is a lattice and v a non-zero isotropic vector in L, then any automorphism of $v^\perp$ extends uniquely to an automorphism of $L \otimes \mathbb{Q}$.*

Proof. If L is of dimension n, then $H = v^\perp$ is $(n-1)$-dimensional. Let g be an isometry of H. We want to extend this to an isometry of $L \otimes \mathbb{Q}$. We select a vector u not in H which gives the condition

$$x \cdot g(w) = u \cdot w \text{ for } w \in H \text{ and } x = g(u).$$

This is an affine system for the unknown x. The kernel corresponds to the vectors orthogonal to $g(w)$ for $w \in H$. Since g is an automorphism of H, this means that the kernel is $H^\perp = \mathbb{Q}v$. Let us take a basis h_1, …, h_{n-1} of H. The system becomes equivalent to

$$x \cdot g(h_i) = u \cdot h_i \text{ for } 1 \leqslant i \leqslant n-1.$$

Since the linear system has n unknowns and $n-1$ equations, a solution $x = u'$ exists by the rank theorem. Since $(u, h_1, \dots, h_{n-1})$ is of full rank, $(u', g(h_1), \dots, g(h_{n-1}))$ is also of full rank and thus $u' \notin H$.

Thus, we can write $g(u) = u' + Cv$ for some $C \in \mathbb{Q}$. The equation $g(u) \cdot g(u) = u \cdot u$ is expressed as $u \cdot u = u' \cdot u' + 2Cv \cdot u'$. We have $u' \cdot v \neq 0$ because $u' \notin H$. Thus, a unique solution C exists. □

Theorem 4.8. *There exists an algorithm for computing orbit representatives of vectors of given norm $\beta \in \mathbb{R}^*$ for integral non-degenerate lattices with $r(L) \geqslant 6$ and $s(L) \geqslant 2$. For $\beta = 0$, the algorithm gives the orbit representatives of primitive vectors.*

Proof. Let us take a lattice L of dimension n with $s(L) = s \geqslant 2$. We first use an approximate model of L in order to compute an approximate list of representatives $\{v_1, \dots, v_m\}$. The orthogonal lattice $v^\perp$ satisfies $r(v^\perp) \geqslant 5$, and so, we can apply Theorem 4.5 to the class of lattices $v_i^\perp$.

If $\beta \neq 0$, then the strategy of Theorem 4.5 works to test equivalence and so reduces the approximate list to an exact list.

If $\beta = 0$, then $v_1^\perp$ is a lattice of dimension $n-1$ that contains v_1. Thus, the lattice $v_1^\perp$ is degenerate. By using Theorem 4.6, we can test for isomorphisms among the lattices $v_i^\perp$. By Lemma 4.7, those isomorphisms can be lifted to isomorphisms of the associated $\mathbb{Q}$-vector spaces, and by **Alg 2**, we can actually check if an integral isomorphism can be obtained. In this way, we can decide which of the v_i are isomorphic. □

In order to compute the Tits building, we must also deal with isotropic planes. For this reason, we now turn more generally to higher-dimensional isotropic k-planes where the situation is considerably more complicated.

Theorem 4.9. *Let $k \geqslant 1$ be an integer and L an indefinite non-degenerate lattice.*

(i) *Given an isotropic k-plane Is, we can compute the stabiliser* $\mathrm{Stab}(L, Is)$ *of Is in the isometry group* $\mathrm{O}(L)$ *of L. We can also compute a finite set $(g_i)_{1\leqslant i\leqslant m}$ of elements of* $\mathrm{O}(Is^{\perp})$ *such that*

$$\mathrm{O}(Is^{\perp}) = \cup_{i=1}^{m} g_i \,\mathrm{Stab}(L, Is)_{Is^{\perp}}$$

with $\mathrm{Stab}(L, Is)_{Is^{\perp}}$ *the restriction of* $\mathrm{Stab}(L, Is)$ *to $Is^{\perp}$.*

(ii) *Given two isotropic k-planes Is_1 and Is_2, we can test whether there is an isometry of L mapping Is_1 to Is_2.*

Proof.

(i) Let us take a basis $(e_{k+1}, \dots, e_{2k})$ of Is. We have $Is \subset Is^{\perp}$ and so we can complete this to a basis $(e_{k+1}, \dots, e_n)$ of $Is^{\perp}$. We then complete this to a basis of L by finding suitable vectors $(e_1, \dots, e_k)$.

The matrix of scalar products is expressed in this basis as

$$B = \begin{pmatrix} H & J & K \\ J^T & 0 & 0 \\ K^T & 0 & A \end{pmatrix}$$

with J a non-degenerate $k \times k$-matrix and A a non-degenerate symmetric matrix of size $(n-2k) \times (n-2k)$. The matrix of scalar product of $Is^{\perp}$ in the basis $(e_{k+1}, \dots, e_n)$ is

$$C = \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

Let us take an isometry Q of $Is^{\perp}$. It will preserve Is and its expression in $(e_{k+1}, \dots, e_n)$ is

$$Q = \begin{pmatrix} Q_1 & 0 \\ Q_2 & Q_3 \end{pmatrix}$$

with $Q_3 A Q_3^T = A$. Here, we recall that we use the action on row vectors from the right.

If the isometry Q has an extension P to $L \otimes \mathbb{Q}$, then this extension satisfies $PBP^T = B$ and will necessarily be of the form

$$P = \begin{pmatrix} P_1 & P_2 & P_3 \\ 0 & Q_1 & 0 \\ 0 & Q_2 & Q_3 \end{pmatrix}.$$

When expanding the expression $PBP^T = B$, we obtain the equations

$$\begin{aligned} H &= P_1 H P_1^T + \{P_2 J^T P_1^T + P_1 J P_2^T\} + \{P_3 K^T P_1^T + P_1 K P_3^T\} + P_3 A P_3^T, \\ J &= P_1 J Q_1^T, \\ K &= P_1 J Q_2^T + P_1 K Q_3^T + P_3 A Q_3^T. \end{aligned}$$

The second equation determines $P_1 \in \mathrm{GL}_k(\mathbb{Q})$ uniquely. Then the third equation will determine $P_3 \in M_{k,n-2k}(\mathbb{Q})$ uniquely. However, the first equation will leave P_2 underdetermined which is a major complication in the case $k > 1$.

Let us take $G_1 = \mathrm{O}(Is^\perp)$. We have $J^T = Q_1 J^T P_1$ which implies

$$P_1^{-1} = (J^T)^{-1} Q_1 J^T.$$

This implies, in turn, that if we force Q_1 to preserve the lattice L_{J^T} spanned by the rows of the matrix J^T, then P_1 is integral. By applying a conjugacy transformation and back, we can apply **Alg 1** to the lattice L_{J^T} instead of $\mathbb{Z}^k$. So, we obtain a finite index subgroup G_2 of G_1 that preserves L_{J^T}. Also using **Alg 3**, we can obtain the cosets of G_2 in G_1.

The equation

$$P_3 = (K - P_1 J Q_2^T - P_1 K Q_3^T)(Q_3^T)^{-1} A^{-1}$$

implies that there exists a denominator d_3 such that $P_3 \in \frac{1}{d_3} M_{k,n-2k}(\mathbb{Z})$, for example, $d_3 = |\det(A)|$. The equation for P_2 that we obtain is

$$(P_1 J P_2^T)^T + P_1 J P_2^T = H - P_1 H P_1^T - P_3 A P_3^T - \{P_3 K^T P_1^T + P_1 K P_3^T\}. \tag{13}$$

We interpret this as a system of linear equations for P_2. Since P_1 and J are non-degenerate, we can equivalently interpret this as linear for $P_1 J P_2^T$. The right-hand side of this system of equations is symmetric. Since any equation of the form $X^T + X = M$ with M symmetric obviously has a solution, for example, $X = M/2$, it follows that Equation (13) has a solution P_2. The kernel of this linear system has dimension $k(k-1)/2$. We can find a denominator d_2 such that for any $Q \in G_1$, there exists a solution P_2 in $\frac{1}{d_2} M_{k,k}(\mathbb{Z})$. To be more precise, a possible denominator of the right-hand side of Equation (13) is d_3^2. So, a possible denominator of $P_1 J P_2^T$ is $2d_3^2$ and so a denominator of P_2 is $2d_3^3$. Define d as the lowest common multiple of d_2 and d_3. We define the sublattice

$$L_3 = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k + \mathbb{Z}de_{k+1} + \cdots + \mathbb{Z}de_n \subset L.$$

Any solution of Equation (13) in $\frac{1}{d} M_{k,k}(\mathbb{Z})$ will preserve L_3.

We define the group H_2 of matrices $P \in \mathrm{GL}_n(\mathbb{Q})$ which preserve L_3 and $Is^\perp$ and whose restriction to $Is^\perp$ belongs to G_2. Thus, the natural mapping $\phi : H_2 \to G_2$ is surjective. By applying **Alg 1**, we can get a finite index subgroup $H_3 \subset \mathrm{GL}_n(\mathbb{Z})$ of H_2. The group H_3 is the group $\mathrm{Stab}(L, Is)$, which is the group of isometric transformation of L preserving Is.

By applying **Alg 3**, we can obtain a coset decomposition of H_3 in H_2. We also have a coset decomposition of G_2 in G_1:

$$H_2 = \cup_{u \in U} u H_3,\ G_1 = \cup_{v \in V} v G_2 \text{ with } U \subset H_3, V \subset G_1 \text{ and } U, V \text{ finite.}$$

By applying ϕ to the first decomposition and substituting, we obtain

$$G_1 = \cup_{u \in U, v \in V} v \phi(u) G_3,$$

which is the required finite coset covering. It is only a covering and not a decomposition since some of the cosets may coincide.

(ii) The process works similarly. We compute the equivalence for the spaces $L_{J_1^T}$ and $L_{J_2^T}$. If they are not equivalent, then the spaces are not equivalent. Otherwise we map the equivalence, build the corresponding spaces and then use **Alg 2** to conclude. □

The algorithm used in this construction is relatively complex. It would have been simpler if we had a sublattice L' of L such that for any $f \in O(Is^{\perp})$, there exists an extension that preserves L'. Unfortunately, we could not find a universal construction of such a lattice. However, in all the cases we considered, a practical algorithm allowed us to solve this problem.

In Theorem 4.8, we established an algorithm for computing isotropic lines. We shall now extend this to arbitrary dimension.

Theorem 4.10. *There exists an algorithm for computing the orbits of isotropic k-planes of indefinite lattices L.*

Proof. The algorithm is constructed by induction on the dimension k of the isotropic spaces starting with $k = 1$, which is Theorem 4.8. Suppose that we know some orbit representatives of isotropic $k-1$-dimensional planes. For each such representative Is, we compute the lattice $Is^{\perp}$ which we decompose as a lattice sum $Is + K$. This is actually also an orthogonal decomposition since $K \subset Is^{\perp}$. We enumerate the orbits of isotropic primitive vectors in K for the group $\mathrm{O}(K)$ using Theorem 4.8 and obtain some representatives $v_1, \ldots, v_l$. Those can also be interpreted as isotropic k-planes $Is + \mathbb{Z}v_i$ in $Is + K$ for the group $\mathrm{O}(Is + K)$.

By using Theorem 4.9 (i), we can compute the stabiliser $\mathrm{Stab}(L, Is)$ of $Is^{\perp}$ in L. We can further compute a covering of the cosets of $\mathrm{Stab}(L, Is)$ restricted to $Is + K$ in $\mathrm{O}(Is + K)$. If the cosets are g_1, ..., g_m, then this gets us candidates $g_j(Is + \mathbb{Z}v_i)$ for the isotropic k-planes containing Is covering all orbits.

We then apply Theorem 4.9 (ii) to compute a complete list of mutually non-equivalent isotropic k-planes. □

We also note that the algorithm can be extended to enumerating flags of isotropic spaces. We simply need to replace the group $\mathrm{GL}_{\dim \ker(A)}(\mathbb{Z})$ in Theorem 4.6 by the integral stabiliser of the flag which is isomorphic to a group of invertible triangular matrices.

4.5 | Relationship with work by Dawes

Dawes [11] also developed algorithms for orthogonal groups, in particular the computation of the Tits buildings. His work is not concerned with moduli problems of polarised Enriques surfaces, which were the starting point of our investigations. Here, we want to comment on similarities and differences in our approaches. Some of Dawes' techniques are similar to ours. His Algorithms 2.1 and 2.2 use the same strategy as the one we implemented. However, Dawes does not have our integral group algorithms, and so, he is forced to iterate over group elements, which can be expensive. Instead, the author uses an alternative approach: he uses the fact that some genera are known to have only one class (see Theorem 2.3) which allows him to prove some isomorphisms relatively easily. However, genus theory, while computationally much easier, does not provide explicit isomorphisms and does not give a generating set of the automorphism group of a lattice. Another idea used in [11] is to use Vinberg's algorithm. This can be done, provided that the lattice is reflective, which is clearly a substantial restriction. In Algorithm 3.1, Dawes' approach seems needlessly complicated, since he does not use the notion of double coset, which is exactly what one needs when splitting orbits. This forces him to use iteration over group elements to find the matching cosets.

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ORCID

Mathieu Dutour Sikirić https://orcid.org/0000-0001-7641-4785
Klaus Hulek https://orcid.org/0000-0002-8824-7942

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