



Quantum perturbative solutions of extended Snyder and Yang models with spontaneous symmetry breaking

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ABSTRACT

We propose $\hbar$ -expansions as perturbative solutions of quantum extended Snyder and Yang models, with $\hbar$ -independent classical zero-th order terms responsible for the spontaneous breaking of $D = 4$ and $D = 5$ de Sitter symmetries. In such models, with algebraic basis spanned by $\hat{o}(D, 1)$ Lie algebra generators, we relate the vacuum expectation values (VEV) of the spontaneously broken generators with the Abelian set of ten (Snyder, $D = 4$) or fifteen (Yang, $D = 5$) antisymmetric tensorial generalized coordinates, which are also used as zero order input for obtaining the perturbative solutions of quantum extended Snyder and Yang models. In such a way we will attribute to these Abelian generalized coordinates the physical meaning of the order parameters describing spontaneous symmetry breaking (SSB). It appears that the consecutive terms in $\hbar$ -power series can be calculated explicitly if we supplement the SSB order parameters by the dual set of tensorial commutative momenta.

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1. Introduction

Snyder and Yang models, proposed in the first half of XX-th century [1], [2], were precursors of modern noncommutative geometry. They are based on the idea of identifying the quantum space-time with the operators of a noncommutative algebra. In Snyder and Yang models the commutation relation between coordinates is proportional to the curvature of quantum positions [1], [2] and Yang model also includes quantum noncommuting momenta [2]. The quantum structure of space-time in these models permits to introduce non-trivial commutators between the components of quantum position and quantum momenta operators without explicitly breaking Lorentz invariance. Such modifications of phase space commutation relations are expected in the algebraic description of quantum gravity (QG) and are important for investigating possible physical effects arising as quantum gravitational corrections.

The main aim of this paper is to present a novel approach to the perturbative solutions of extended Snyder and Yang models and disclose in them the spontaneous symmetry breaking (SSB) effects. We construct the operator-valued perturbative solutions

expressed as a power series in the Planck constant $\hbar$, and explain the role of classical, commutative parts of such solutions as providing spontaneous symmetry breaking (SSB). The use of a perturbative $\hbar$ -expansion of the solutions (see e.g. [3,4]) permits to distinguish the classical, commuting terms obtained in the limit $\hbar = 0$, from the remaining $\hbar$ -dependent quantum parts. When the quantum model is formulated in terms of Lie algebra symmetry generators, it is known that the presence of their classical parts leads to SSB effects (see e.g. [5–11]). In particular $D = 4$ extended Snyder model [12–17], with its algebraic formulation described by ten independent $\hat{o}(4, 1)$ symmetry generators ($D = 4$ de Sitter algebra) fits very well in such a framework.

It is well known that in quantum theories one can consider two ways of breaking symmetries. The first, explicit symmetry breaking, leads to modified basic symmetry properties of the algebraic structure in the quantum models under consideration, e.g. one obtains the modified action integrals, quantum equations of motion etc. The second way, SSB, does not change the basic symmetries of algebraic structures, but provides the solutions as quantum states with broken symmetries. In such a case the SSB effects have been considered in Quantum Mechanics (QM) and Quantum Field Theory (QFT) models, and in particular in Standard Model (SM), which describes the theory of elementary particles by the tools of QFT (see e.g. [18,19]). We recall that in SM the suitable SSB of local

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gauge symmetries leads to the Higgs mechanism [20,21] generating the mass parameters which are necessary for the comparison with experiment.

In this paper we consider the presence of SSB in the description of quantum space-times and quantum deformed phase spaces. By using the tools of noncommutative (NC) geometry (see e.g. [22–25]), various NC models (see e.g. [1,2,26–28]) describing $D = 4$ quantum space-times, quantum deformed phase spaces, as well as quantum symmetry groups have been obtained. We recall that the first NC models with preserved $D = 4$ relativistic covariance were introduced as early as in 1947 by Snyder [1] and Yang [2]. These models and their generalizations were subsequently considered in numerous papers (see e.g. [12,13,16,29–32]), where however the appearance of SSB effects had not been pointed out.¹ In this paper, we will show that the introduction of explicit $\hbar$ -dependence and the use of perturbation theory described by $\hbar$ -power series² permits to provide the SSB interpretation of the obtained results.

The plan of our paper is the following. In next Sec. 2 we present a short algebraic description of quantum Snyder model with two deformation parameters: the elementary length l , often identified with Planck length l_P , and the universal Planck constant $\hbar$ characterizing quantum theories.³ The $\hbar$ -dependent $D = 4$ dS algebra basis of Snyder algebra can be treated as the relativistic dS extension from $\hat{o}(3)$ to $\hat{o}(4, 1)$ of the $\hbar$ -dependent nonrelativistic $D = 3$ angular momentum algebra, which is well-known from basic textbooks on QM. Further, we describe in the presence of SSB, the reducible structure of Hilbert spaces of states with irreducible components labelled by the order parameters which characterize the spontaneously broken solutions. For simplicity, in Snyder model, we will consider the degeneracy of quantum states parametrized by the four-vector parameter a_μ describing the curved translations in the coset $\hat{o}(4, 1)/\hat{o}(3, 1)$. It appears that the vacuum state $||0\rangle\rangle$ in Snyder model is degenerate, given by the direct integral of irreducible vacua $||0; a_\mu\rangle$.⁴ In Sec. 3 we consider the $\hbar$ -perturbative solution of spontaneously broken Snyder model with explicit formulae providing the first- and second-order terms. In Sec. 4 we deal with quantum $D = 4$ Yang model which is described algebraically by $D = 5$ dS algebra. Very recently such models were studied and generalized (see [43–45]) with the participation of the authors of the present paper. Further in Sec. 4 we consider the $\hbar$ -perturbative solutions of Yang model and present explicitly the leading term linear in $\hbar$. Finally, Sec. 5 contains a short discussion of results and provides new suggestions about the continuation of present research and proposes the possible generalizations.

2. Quantum $D = 4$ Snyder model and spontaneously broken $D = 4$ dS symmetries

a) Algebraic description of quantum $D = 4$ Snyder model

The algebraic $D = 4$ Snyder model is determined by the quantum

NC space-time position generators $\hat{x}_\mu$ and Lorentz-algebra generators $\hat{M}_{\mu\nu}$ satisfying the following relation⁵

$$[\hat{x}_\mu, \hat{x}_\nu] = i \frac{l^2}{\hbar} \hat{M}_{\mu\nu}, \quad (1)$$

where l is an elementary length, $\mu, \nu = 0, 1, 2, 3$, and

$$[\hat{M}_{\mu\nu}, \hat{M}_{\rho\tau}] = i\hbar(\eta_{\mu\rho}\hat{M}_{\nu\tau} - \eta_{\mu\tau}\hat{M}_{\nu\rho} + \eta_{\nu\tau}\hat{M}_{\mu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\tau}), \quad (2)$$

$$[\hat{M}_{\mu\nu}, \hat{x}_\rho] = i\hbar(\eta_{\mu\rho}\hat{x}_\nu - \eta_{\nu\rho}\hat{x}_\mu). \quad (3)$$

Relations (3) show that Snyder quantum space-time coordinates $\hat{x}_\mu$ describe a Lorentz-covariant four-vector. Using the Compton length formula, $l = \frac{\hbar}{Mc}$, after setting $c = 1$ in relation (1), we are led to the following form⁶

$$[\hat{x}_\mu, \hat{x}_\nu] = i \frac{\hbar}{M^2} \hat{M}_{\mu\nu}, \quad (4)$$

where M is an elementary mass (e.g. Planck mass). If we introduce

$$\hat{M}_{4\mu} = M\hat{x}_\mu, \quad (5)$$

one can describe the relations (2)–(4) as providing $D = 4$ de Sitter algebra

$$[\hat{M}_{AB}, \hat{M}_{CD}] = i\hbar(\eta_{AC}\hat{M}_{BD} - \eta_{AD}\hat{M}_{BC} + \eta_{BD}\hat{M}_{AC} - \eta_{BC}\hat{M}_{AD}), \quad (6)$$

with $A, B, = 0, 1, 2, 3, 4$.

Originally, the Snyder model (2)–(4) was introduced by adding to Snyder quantum space-time $\hat{x}_\mu$ the commuting four-momenta p_μ , what leads to the description of Snyder quantum phase space [1], [46]. In such a case the generators $\hat{M}_{\mu\nu}$ can be expressed in terms of the quantum phase space coordinates $(\hat{x}_\mu, p_\mu)$, which after the use of relations (2)–(4) and Jacobi identities lead to the set of quantum-deformed Lorentz-covariant Heisenberg algebras [31,46,47]. The structure constants of the algebra (6) are proportional to the Planck constant $\hbar$. Special realization of algebras (2) and (6) in symmetric ordering can be written as power series in structure constants i.e. power series in $\hbar$ [34]. All other realizations could be obtained using similarity transformations from special realization in symmetric ordering, for extended Snyder model see e.g. [14]. In the limit $\hbar \rightarrow 0$, the de Sitter algebra (6) becomes an Abelian algebra. In the limit $M \rightarrow \infty$ Snyder algebra (4) reduces to $[x_\mu, x_\nu] = 0$, where x_μ are the commutative coordinates.

b) Spontaneous breaking of $D = 4$ dS symmetries

In this paper we investigate the class of models with independent generators $\hat{x}_\mu, \hat{M}_{\mu\nu}$ satisfying eqs. (2)–(4), which define⁷ a ten-dimensional independent algebraic basis of $D = 4$ Snyder model. Such models were studied during the last twenty years and were named alternative [12] or extended [13–17,33]. They were solved perturbatively in terms of tensorial canonical quantum phase space coordinates $(x_{AB}; p_{AB}) \equiv (x_{\mu\nu}, Mx_\mu; p_{\mu\nu}, p_\mu)$, where⁸

¹ It should be recalled, however, that in [14,15,32,33] the Snyder-type models were solved perturbatively as embedded in the canonical vectorial and tensorial Heisenberg algebras [34,35] but without introducing the SSB interpretation.

² For the discussion of $\hbar$ -power expansions in quantum theories, see e.g. [4].

³ If we follow the standard description of quantum theory defined by the passage from Poisson brackets to quantum commutators (see e.g. [36]), the limit $\hbar \rightarrow 0$ of the quantum commutators describes the transition from quantum to classical theory. In this paper we study the $\hbar \rightarrow 0$ limit of perturbative $\hbar$ expansion as describing the Abelian SSB parameters. However, it should be mentioned that recently such prescription has been challenged. In particular in quantum gravity were studied models with quantum solutions which contain “side by side” both classical and quantum parts (see e.g. [37–39]).

⁴ The direct integrals of Hilbert spaces for reducible quantum fields satisfying Wightman axioms were first considered by Borchers [40]; see also Haag [41], who considered the spontaneously broken quantum states in BCS model [42] describing superconductivity.

⁵ For simplicity we put $c = 1$, because our paper is not aimed at the consideration of relativistic corrections, characterized by inverse powers of c . We stress, however, that the quantum nature of the model considered here is underlined by the explicit dependence on the Planck constant $\hbar$, in agreement with the historic formulation of the Snyder model [1].

⁶ If $c = 1$, the length l and the mass M are related by the “quantum” mass-length relation $l = \hbar/M$, see e.g. [4].

⁷ We will consider the most physical $D = 4$ case, but the results can be extended in a straightforward way to any dimension $D \geq 2$, with the Snyder algebras spanned by generators of $\hat{o}(D, 1)$.

⁸ In some of our papers (see e.g. [14], [16], [34]) we used the tensorial canonical Heisenberg algebras, but we did not consider their explicit $\hbar$ -dependence. In most of our earlier papers, related with Snyder models, [13–17,33] we considered the Heisenberg algebra relations (7) with $\hbar = 1$.

$$\begin{aligned} [x_{AB}, x_{CD}] &= [p_{AB}, p_{CD}] = 0, \\ [x_{AB}, p_{CD}] &= i\hbar(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}). \end{aligned} \quad (7)$$

The novelty of this paper is to express the generators $\hat{M}_{AB}$ as an algebraic $\hbar$ -power series and consider the zero-th order terms $\hat{M}_{AB}^{(0)} = x_{AB}$ as representing Nambu-Goldstone (NG) modes which describe the spontaneous symmetry breaking of $D = 4$ de Sitter symmetry. One can study the following two particular choices:

i) $x_\mu \neq 0$, $x_{\mu\nu} = 0$. In such a case the Lorentz symmetry is not broken, and the NG modes are determined by the parametrization of the coset $\hat{o}(4, 1)/\hat{o}(3, 1)$, which describes the curved de Sitter translations (see e.g. [48]).

ii) $x_\mu = 0$, $x_{\mu\nu} \neq 0$. This case corresponds to spontaneously broken Lorentz symmetries (see e.g. [6], [10], [11], [52]).

The canonical coordinates x_{AB} (see (7)) are given by the classical $\hbar$ -independent part of the $\hbar$ -expansions,

$$\hat{x}_\mu = x_\mu^{(cl)} + \hat{x}_\mu^{(q)}, \quad \hat{M}_{\mu\nu} = x_{\mu\nu}^{(cl)} + \hat{x}_{\mu\nu}^{(q)}, \quad (8)$$

where x_μ and $x_{\mu\nu}$ describe the zero-th order in the $\hbar$ -power expansions and describe the classical parts of the generators $\hat{M}_{AB}$. In quantum models with preserved $D = 4$ dS symmetries the classical parts of $\hbar$ -power series vanish and one can introduce a unique (invariant under symmetries) cyclic vacuum state $|0\rangle$ ($\langle 0|0\rangle = 1$), which satisfies the relations

$$\hat{x}_\mu^{(q)}|0\rangle = 0, \quad \hat{x}_{\mu\nu}^{(q)}|0\rangle = 0. \quad (9)$$

In general case, if $x_{AB} \neq 0$, one should introduce the degenerate continuous set of vacua $|0; x_{AB}\rangle$. The commuting coordinates x_{AB} define the order parameters which describe spontaneously broken rotations in the planes (A, B) ⁹ which are the Abelian subgroups of spontaneously broken $D = 4$ de Sitter symmetry.

Let us discuss the $D = 4$ Snyder model with spontaneous symmetry breaking, which is generated by the curved $D = 4$ dS translations, parametrized by a constant four-vector a_μ (see e.g. [48,49]).¹⁰ In such a case the SSB is generated by the action of nonlinear unitary representation $U(a_\mu)$ on the NC curved space-time coordinates $\hat{x}_\mu$, which results in the following inhomogeneous nonlinear formulae [48,49]:

$$\hat{x}(a_\mu)_\mu = U^{-1}(a_\mu)\hat{x}_\mu U(a_\mu) = \hat{x}_\mu + a_\mu + O(\hat{x}_\mu; a_\mu) \quad (10)$$

where $O(\hat{x}_\mu; a_\mu)$ contains higher powers of $\hat{x}_\mu$. In such a case one can introduce the continuous sets of degenerate vacuum states $|0; a_\mu\rangle$ and the a_μ -dependent Hilbert spaces $\mathcal{H}(a_\mu)$, with different values of a_μ linked by the unitary representation $U(a_\mu)$ as follows:

$$|0; a_\mu\rangle \rightarrow |0; a_\mu + a'_\mu\rangle = U(a'_\mu)|0; a_\mu\rangle \quad (11)$$

and

$$\mathcal{H}(a_\mu) \rightarrow \mathcal{H}(a_\mu + a'_\mu) = U(a'_\mu)\mathcal{H}(a_\mu), \quad (12)$$

i.e. to each Hilbert space $\mathcal{H}(a_\mu)$ there exists an associated spontaneously broken set of vacua states (11). The total Hilbert space $\mathcal{H}$ describing whole spontaneously broken quantum system can be described by the direct integral of Hilbert spaces $\mathcal{H}(a_\mu)$,¹¹ with

⁹ If $A = 0$ the rotational symmetry is $\hat{o}(1, 1)$, if $A = 1, \dots, 4$ we deal with SSB of $\hat{o}(2)$ rotations.

¹⁰ In [48,49], the analogous case of nonlinear curved $D = 4$ AdS translations is considered.

¹¹ The direct integrals of Hilbert spaces and degenerated vacua for reducible quantum fields satisfying Wightman axioms were considered in [40]. In [41] was considered the quantum BCS model [51], and in [50] was studied a toy model of QFT with degenerate vacuum and reducible Hilbert spaces.

all possible values of the numerical parameters a_ν described by a classical manifold V with Lebesgue measure $d\mu$

$$\mathcal{H} = \int_V^{\oplus a_\nu} d\mu \mathcal{H}(a_\nu). \quad (13)$$

Analogously, the reducible degenerated vacuum $|0\rangle$ covariant under the spontaneously broken curved translation symmetries can be defined by the formula

$$|0\rangle = \int_V^{\oplus a_\nu} d\mu |0; a_\nu\rangle. \quad (14)$$

3. $\hbar$ -perturbative solutions of spontaneously broken $D = 4$ Snyder model

We firstly apply the scheme of perturbative $\hbar$ -expansions to the extended Snyder model (see, e.g. (1)-(5)), with the algebra described by $\hat{o}(4, 1)$ generators $\hat{M}_{AB} = (\hat{M}_{\mu\nu}, \hat{M}_{4\mu} = M\hat{x}_\mu)$. We expand the generators $\hat{M}_{AB}$ in the following $\hbar$ -power series:

$$\hat{M}_{AB} = M_{AB}^{(0)} + \hbar \hat{M}_{AB}^{(1)} + \hbar^2 \hat{M}_{AB}^{(2)} + \dots \quad (15)$$

where

$$M_{AB}^{(0)} \equiv M_{AB}^{(cl)} = \lim_{\hbar \rightarrow 0} \hat{M}_{AB} = \langle 0 | \hat{M}_{AB} | 0 \rangle \quad (16)$$

or equivalently (see (9))

$$M_{AB}^{(0)} \equiv x_{AB} = (x_{\mu\nu}, Mx_\mu) \quad (17)$$

where x_{AB} are the order parameters describing the SSB of the ten one-dimensional (pseudo-) orthogonal symmetries generated by $\hat{M}_{AB}$ on all planes (A, B) , where $(A, B = 0, 1, 2, 3, 4)$, of $D = 5$ space-time with signature $\eta_{AB} = \text{diag}(-1, 1, 1, 1, 1)$.

One can deduce from the relations (1)-(5) the iterated set of algebraic equations determining the perturbative quantum terms $\hat{M}_{AB}^{(n)}$ ($n = 1, 2, 3, \dots$) as functions of x_{AB} , describing Nambu-Goldstone (NG) degrees of freedom x_{AB} and dual momenta p_{AB} , which satisfy together the generalized canonical quantum phase space relations (7). The most general case, when all $x_{AB} \neq 0$, describes the situation when all the $D = 4$ deS symmetries are spontaneously broken.

a) *Perturbative $\hbar$ -expansion: first order in $\hbar$*

From relation (6) one gets:

$$\begin{aligned} [x_{AB}, \hat{M}_{CD}^{(1)}] + [\hat{M}_{AB}^{(1)}, x_{CD}] \\ = i(\eta_{AC}x_{BD} - \eta_{AD}x_{BC} - \eta_{BC}x_{AD} + \eta_{BD}x_{AC}) \end{aligned} \quad (18)$$

and relations (3), (4) lead to

$$\begin{aligned} [x_{\mu\nu}, \hat{M}_{\rho\sigma}^{(1)}] - [x_{\rho\sigma}, \hat{M}_{\mu\nu}^{(1)}] \\ = i(\eta_{\mu\rho}x_{\nu\sigma} - \eta_{\mu\sigma}x_{\nu\rho} - \eta_{\nu\rho}x_{\mu\sigma} + \eta_{\nu\sigma}x_{\mu\rho}), \end{aligned} \quad (19)$$

$$[x_\mu, \hat{M}_{\rho\sigma}^{(1)}] - [x_{\rho\sigma}, \hat{x}_\mu^{(1)}] = i(\eta_{\mu\sigma}x_\rho - \eta_{\mu\rho}x_\sigma), \quad (20)$$

$$[x_\mu, \hat{x}_\nu^{(1)}] - [x_\nu, \hat{x}_\mu^{(1)}] = \frac{i}{M^2} x_{\mu\nu}. \quad (21)$$

In order to solve the relations (18)-(21) we employ the generalized momenta $p_{AB} = (p_{\mu\nu}, p_\mu)$ (see (7)). From (19) and (20) one can obtain a particular solution, given by

$$\hbar \hat{M}_{\mu\nu;S}^{(1)} = \frac{1}{2} (x_\mu^\rho p_{\nu\rho} - x_\nu^\rho p_{\mu\rho}) + x_\mu p_\nu - x_\nu p_\mu \quad (22)$$

and in consistency with (21)

$$\hbar \hat{x}_{\mu;S}^{(1)} = -\frac{1}{2M^2} x_{\mu\rho} p^\rho. \quad (23)$$

The general first order solution depends on one free parameter [14] and can be obtained by a suitable choice of similarity transformations of the particular solutions (22), (23).

b) *Perturbative $\hbar$ -expansion: second order in $\hbar$*

The second order counterpart of relation (18) looks as follows:

$$\begin{aligned} & [x_{AB}, \hat{M}_{CD}^{(2)}] - [x_{CD}, \hat{M}_{AB}^{(2)}] + [\hat{M}_{AB}^{(1)}, \hat{M}_{CD}^{(1)}] \\ & = i(\eta_{AC} \hat{M}_{BD}^{(1)} + \eta_{BD} \hat{M}_{AC}^{(1)} - \eta_{BC} \hat{M}_{AD}^{(1)} - \eta_{AD} \hat{M}_{BC}^{(1)}) \end{aligned} \quad (24)$$

which leads to:

$$\begin{aligned} & [x_{\mu\nu}, \hat{M}_{\rho\sigma}^{(2)}] - [x_{\rho\sigma}, \hat{M}_{\mu\nu}^{(2)}] \\ & = i(\eta_{\mu\rho} \hat{M}_{\nu\sigma}^{(1)} + \eta_{\nu\sigma} \hat{M}_{\mu\rho}^{(1)} - \eta_{\nu\rho} \hat{M}_{\mu\sigma}^{(1)} - \eta_{\mu\sigma} \hat{M}_{\nu\rho}^{(1)}) \\ & \quad - [\hat{M}_{\mu\nu}^{(1)}, \hat{M}_{\rho\sigma}^{(1)}], \end{aligned} \quad (25)$$

$$[x_\mu, \hat{M}_{\rho\sigma}^{(2)}] - [x_{\rho\sigma}, \hat{x}_\mu^{(2)}] = i(\eta_{\mu\sigma} \hat{x}_\rho^{(1)} - \eta_{\mu\rho} \hat{x}_\sigma^{(1)}) - [\hat{x}_\mu^{(1)}, \hat{M}_{\rho\sigma}^{(1)}], \quad (26)$$

$$[x_\mu, \hat{x}_\sigma^{(2)}] - [x_\sigma, \hat{x}_\mu^{(2)}] = \frac{i}{M^2} \hat{M}_{\mu\sigma}^{(1)} - [\hat{x}_\mu^{(1)}, \hat{x}_\sigma^{(1)}]. \quad (27)$$

Substituting in (25)–(27) the solutions (22), (23) one gets the particular solution, to second order in $\hbar$ ¹²:

$$\hbar^2 \hat{M}_{\mu\nu;S}^{(2)} = -\frac{1}{12} (x_{\mu\rho} p^{\rho\sigma} p_{\nu\sigma} - x_{\nu\rho} p^{\rho\sigma} p_{\mu\sigma} - 2x^{\rho\sigma} p_{\mu\rho} p_{\nu\sigma}), \quad (28)$$

$$\hbar^2 \hat{x}_{\mu;S}^{(2)} = \frac{1}{M^2} \left(x_\rho p^\rho p_\mu + \frac{1}{4} (x_{\mu\rho} p_{\rho\sigma} p_\sigma + x^{\rho\sigma} p_\rho p_{\mu\sigma}) \right). \quad (29)$$

General solutions in the second $\hbar$ -order can be obtained from the formulae (28), (29) by performing suitable similarity transformations. One can also show that, in the perturbative n -th order in $\hbar$, the solutions $(\hat{x}_{\mu;S}^{(n)}, \hat{M}_{\mu\nu;S}^{(n)})$ are n -linear in momenta $p_\mu, p_{\mu\nu}$ (see also [14], [16]).

4. Quantum $D = 4$ Yang model and spontaneously broken algebra $\hat{o}(5, 1)$

In the following we will apply our method to $D = 4$ Yang model (see e.g. [2], [53–55]), algebraically described by fifteen generators of $D = 5$ dS algebra $\hat{o}(5, 1)$ ($K, L = 0, 1, 2, 3, 4, 5$)

$$\hat{M}_{KL} = (\hat{M}_{\mu\nu}, \hat{M}_{4\mu} = M\hat{x}_\mu, \hat{M}_{5\mu} = R\hat{q}_\mu, \hat{M}_{45} = MR\hat{r}) \quad (30)$$

satisfying the following relation

$$[\hat{M}_{KL}, \hat{M}_{PR}] = i\hbar(\eta_{KP} \hat{M}_{LR} - \eta_{KR} \hat{M}_{LP} + \eta_{LR} \hat{M}_{KP} - \eta_{LP} \hat{M}_{KR}). \quad (31)$$

The $D = 4$ Yang model describes a $D = 4$ Lorentz-covariant quantum-deformed relativistic Heisenberg algebra with two deformation parameters (M, R) of length dimensions $[M] = L^{-1}, [R] = L$

¹² The subscript S denotes the Snyder case. The factor $\hbar$ on the left hand side in (22), (23) and $\hbar^2$ in (28), (29) reflect the property that we deal with quantum-mechanical momenta satisfying the relations (7), proportional to $\hbar$ (one can recall the space-time realization $p_\mu = -i\hbar\partial_\mu$). Relation (22) describes generalized angular momentum, in space-time realization, given by the $\hbar$ -independent formula $M_{\mu\nu}^{(1)} = i(x_{[\mu}\partial_{\nu]} + \frac{1}{2}x_{[\mu}{}^\rho\partial_{\nu]}\rho)$. In the general case, the coefficients $M_{AB}^{(n)}$ in (15) are proportional to n -th powers of the canonical momenta (7) and are $\hbar$ -independent.

and one dimensionless scalar Abelian $\hat{o}(2)$ generator $\hat{r}$. In the general case one can introduce in the Yang model fifteen Abelian NG modes $x_{KL} = -x_{LK}$, which break spontaneously the $\hat{o}(5, 1)$ symmetry

$$x_{KL} = (x_{\mu\nu}, Mx_\mu, Rq_\mu, MRr). \quad (32)$$

In order to solve the Yang model by using a perturbative $\hbar$ -expansion one should introduce fifteen canonically conjugated commuting NG momenta

$$p_{KL} = (p_{\mu\nu}, p_\mu, k_\mu, s). \quad (33)$$

The variables (32), (33) satisfy $D = 5$ extension of the canonical commutation relations (7), with the following Lorentz-covariant additional relations

$$[q_\mu, k_\nu] = i\hbar\eta_{\mu\nu}, \quad [r, s] = i\hbar. \quad (34)$$

Using the variables (32), (33) we present below the first order $\hbar$ -perturbative solution of the Yang model.

a) *Algebraic description of $D = 4$ Yang model*

The Yang model was obtained in [2] as a group-theoretic extension by momentum sector of the extended Snyder model. Such an extension can be obtained by the Born map applied to the Snyder model generators $\hat{x}_\mu \rightarrow \hat{p}_\mu$, $\hat{M}_{\mu\nu} \rightarrow \hat{M}_{\mu\nu}$ and adding the Born map-invariant scalar generator $\hat{r}$.

In the Yang model we extend the relations (2)–(4) by the following set of algebraic equations¹³

$$[\hat{q}_\mu, \hat{q}_\nu] = i\frac{\hbar}{R^2} \hat{M}_{\mu\nu}, \quad (35)$$

$$[\hat{x}_\mu, \hat{q}_\nu] = i\frac{\hbar}{MR} \eta_{\mu\nu} \hat{M}_{45}, \quad \hat{M}_{45} = MR \cdot \hat{r}, \quad (36)$$

$$[\hat{M}_{\mu\nu}, \hat{q}_\rho] = i\hbar(\eta_{\mu\rho} \hat{q}_\nu - \eta_{\nu\rho} \hat{q}_\mu), \quad (37)$$

$$[\hat{r}, \hat{x}_\mu] = \frac{i\hbar}{M^2} \hat{q}_\mu, \quad (38)$$

$$[\hat{r}, \hat{q}_\mu] = -\frac{i\hbar}{R^2} \hat{x}_\mu \quad (39)$$

where $\hat{q}_\mu = \hat{q}_\mu^{(cl)} + \hat{q}_\mu^{(q)}$, $\hat{r} = \hat{r}^{(cl)} + \hat{r}^{(q)}$. It should be added that in the Yang model the original phase space variables $(\hat{x}_\mu, \hat{q}_\mu)$ represent the generalized set of quantum coordinates, which can be doubled by Hopf-algebraic duality relations $(\hat{x}_\mu, \hat{q}_\mu \rightarrow \hat{x}_\mu, \hat{q}_\mu; \hat{p}_\mu, \hat{k}_\mu)$. In the limit $R \rightarrow \infty$, the Yang model becomes the Snyder model, while for $M \rightarrow \infty$, we obtain the inhomogeneous $D = 4$ de Sitter algebra in momentum space. When both $M \rightarrow \infty$ and $R \rightarrow \infty$ Yang model gives rise to the semidirect product of Poincaré algebra and commutative four-momenta, supplemented by a scalar variable.

b) *$\hbar$ -perturbative expansion of Yang model - linear terms*

We obtain the first order $\hbar$ -approximation of the algebraic solutions of the Yang model if in the $\hbar$ -expansions of the solutions (2)–(4) and (35)–(39) we consider the linear $\hbar$ -terms. Besides (19)–(21) one gets

$$[q_\mu, \hat{q}_\nu^{(1)}] - [q_\nu, \hat{q}_\mu^{(1)}] = \frac{i}{R^2} x_{\mu\nu}, \quad (40)$$

$$[x_\mu, \hat{q}_\nu^{(1)}] - [q_\nu, \hat{x}_\mu^{(1)}] = i\eta_{\mu\nu}, \quad (r \equiv \hat{r}^{(0)}), \quad (41)$$

¹³ In Yang model we denote curved noncommutative momenta by $\hat{q}_\mu$, while $\hat{q}_\mu^{(0)} = q_\mu$ describe their classical commutative limit. The canonically dual coordinates are k_μ (see (34)) which are different from x_μ . Obviously, we assume that $[\hat{r}, \hat{M}_{\mu\nu}] = 0$.

$$[x_{\mu\nu}, \hat{q}_\rho^{(1)}] + [\hat{M}_{\mu\nu}^{(1)}, q_\rho] = i(\eta_{\mu\rho}q_\nu - \eta_{\nu\rho}q_\mu), \quad (42)$$

$$[r, \hat{x}_\mu^{(1)}] + [\hat{r}^{(1)}, x_\mu] = \frac{i}{M^2}q_\mu, \quad (43)$$

$$[r, \hat{q}_\mu^{(1)}] + [\hat{r}^{(1)}, q_\mu] = -\frac{i}{R^2}x_\mu. \quad (44)$$

For the extended Snyder model, in the first order, we obtained the formulas (22), (23). In Yang model, due to the presence of additional coordinates (q_μ, r) and momenta (k_μ, s) , see (32), (33), one should extend the formulae (22), (23) by terms which are linear in momenta (k_μ, s) , see (33). We get (a, b, c, d) are numerical constants):

$$\hbar \hat{M}_{\mu\nu}^{(1)} = \frac{1}{2}(x_\mu^\rho p_{\nu\rho} - x_\nu^\rho p_{\mu\rho}) + x_\mu p_\nu - x_\nu p_\mu - q_\mu k_\nu + q_\nu k_\mu, \quad (45)$$

$$\hbar \hat{x}_\mu^{(1)} = -\frac{1}{2M^2}x_{\mu\rho}p^\rho + ax_{\mu\rho}k^\rho + brk_\mu + cq_\mu s, \quad (46)$$

and add the following formulae for the first order solutions of $\hat{q}_\mu$ and $\hat{r}$:

$$\hbar \hat{q}_\mu^{(1)} = -\frac{1}{2R^2}x_{\mu\rho}k^\rho + \tilde{a}x_{\mu\rho}p^\rho + \tilde{b}rp_\mu + \tilde{c}x_\mu s, \quad (47)$$

$$\hbar \hat{r}^{(1)} = dq^\rho p_\rho + fx^\rho k_\rho, \quad (48)$$

which depend on five additional numerical constants $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ and f . The equations (40)–(44) impose the following constraints on the eight parameters in (46)–(48):

$$a + \tilde{a} = 0, \quad \tilde{b} = b + 1, \quad c - d = \frac{1}{M^2}, \quad \tilde{c} - f = -\frac{1}{R^2} \quad (49)$$

and, in formulae (46)–(48), imply the absence of terms proportional to $p_{\mu\nu}$. We see therefore that the solutions of equations (40)–(44) which are linear in $\hbar$ contain four unconstrained numerical parameters a, b, c, f .

The above calculation can be extended to higher orders in $\hbar$, what we plan to present in a forthcoming publication.

5. Outlook and final remarks

The basic idea of Snyder and Yang models relies on the use of $D = 4$ and $D = 5$ de Sitter algebras for the algebraic description of, respectively, relativistic noncommutative quantum space-times and quantum phase spaces with noncommutative four-momenta. In this paper, the quantum nature of Snyder and Yang models has been underlined by considering their explicit dependence on the Planck constant $\hbar$, in agreement with the first historical formulations of both models [1], [2]. By using $\hbar$ as an expansion parameter in the perturbative solutions, we were able to interpret the generalized tensorial coordinates, introduced in our earlier papers (see e.g. [14–17], [34]) and we present them here as appearing due to the effects of spontaneous symmetry breaking of $D = 4$ and $D = 5$ de Sitter symmetries. We should also add that Snyder and Yang models can be considered as providing examples of the noncommutative space-times and quantum deformed phase spaces which are considered in quantum gravity studies as the physics related applications of noncommutative geometry.

In our future work we plan to study the generalizations and modifications of the models considered here, in particular:

i) The κ -deformed extended Snyder models (see [13], [16], [56]) were obtained by adding to the basic deformation parameter M the second parameter κ with mass-like dimension, in a way which leads in the limit $M \rightarrow \infty$ to the well-known κ -deformed quantum Minkowski space-time (see e.g. [57], [58]). Similarly in Yang model

with the pair of basic deformation parameters M and R (see Sec. 4) one can add a pair of parameters $(\kappa, \tilde{\kappa})$ and introduce doubly κ -deformed Yang models with κ -deformed coordinate sector and $\tilde{\kappa}$ -deformed momenta [59].

ii) Snyder and Yang models are obtained by quantum group-theoretic constructions, exploiting the $D = 4$ and $D = 5$ de Sitter algebras. However, several Yang-like models, describing quantum deformed Lorentz-covariant phase spaces were introduced by direct algebraic methods as well, based e.g. on the use of Jacobi identities (see e.g. [28], [60], [61]).

iii) The relativistic Snyder and Yang models are described algebraically in an equivalent way by $D = 4$ and $D = 5$ dS algebras. It is interesting to ask the fate of this equivalence if we consider quantum dS algebras as Hopf algebras with *nonprimitive* coalgebra sector. In such a case we can introduce the corresponding quantum-deformed Snyder models if the quantum Lorentz algebra is (in the Hopf-algebraic sense) the quantum subalgebra of properly chosen quantum dS algebras (for the choice of such quantum dS algebras, see e.g. [62]).

iv) In the Hopf-algebraic framework of quantum groups the generalized quantum phase spaces can be obtained as the Heisenberg double algebra $\mathcal{H} = \mathbb{H} \rtimes \tilde{\mathbb{H}}$ (see e.g. [35,63,64]), where $\mathbb{H}$ describes quantum-deformed algebra with Hopf symmetries, $\tilde{\mathbb{H}}$ is the quantum Hopf group dual (in Hopf sense) to $\mathbb{H}$, and $\rtimes$ represents the so-called smash product (see e.g. [65]). In such a scheme the Planck constant $\hbar$ appears as introduced in the Hopfian dualization procedure.¹⁴

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

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¹⁴ The Heisenberg double algebra for extended $D = 4$ Snyder model has been explicitly calculated in [15].

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