Letter

Exact results for scaling dimensions of neutral operators in scalar conformal field theories

Oleg Antipin,[1,*] Jahmall Bersini,[2,†] and Francesco Sannino[3,4,5,6,‡]

[1]*Rudjer Boskovic Institute, Division of Theoretical Physics, Bijenička 54, 10000 Zagreb, Croatia*
[2]*Kavli IPMU (WPI), UTIAS, The University of Tokyo, Kashiwa, Chiba 277-8583, Japan*
[3]*Quantum Theory Center (ħQTC) at IMADA and D-IAS, Southern Denmark University, Campusvej 55, 5230 Odense M, Denmark*
[4]*Department of Physics E. Pancini, Università di Napoli Federico II, via Cintia, 80126 Napoli, Italy*
[5]*INFN sezione di Napoli, via Cintia, 80126 Napoli, Italy*
[6]*Scuola Superiore Meridionale, Largo S. Marcellino, 10, 80138 Napoli, Italy*

(Received 29 August 2024; revised 20 October 2024; accepted 24 January 2025; published 12 February 2025)

We determine the scaling dimension Δ_n for the class of composite operators ϕ^n in the $\lambda\phi^4$ theory in $d = 4 - \epsilon$ taking the double scaling limit $n \to \infty$ and $\lambda \to 0$ with fixed λn via a semiclassical approach. Our results resum the leading power of n at any loop order. In the small λn regime we reproduce the known diagrammatic results and predict the infinite series of higher-order terms. For intermediate values of λn we find that Δ_n/n increases monotonically approaching a $(\lambda n)^{1/3}$ behavior in the $\lambda n \to \infty$ limit. We further generalize our results to neutral operators in the ϕ^4 in $d = 4 - \epsilon$, ϕ^3 in $d = 6 - \epsilon$, and ϕ^6 in $d = 3 - \epsilon$ theories with $O(N)$ symmetry.

DOI: 10.1103/PhysRevD.111.L041701

Introduction. Critical behavior of quantum field theories plays a crucial role in our understanding of phase transitions in Nature across all realms of physics from condensed matter to high energy physics and cosmology. Such behavior is encoded in scaling exponents associated with correlation functions of different operators of the underlying conformal field theory (CFT) employed to describe a given physical process.

Since the pioneering work of Wilson [1,2], scalar field theories have been used as primary models to unveil universal behavior in phase transitions. Despite the fifty-year-long effort to solve these models much is still left to be understood. Perhaps one of the greatest challenges is the investigation of composite operators. For these, perturbation theory, very quickly shows its limitations. Beyond perturbation theory a number of methodologies have been employed from the use of larger symmetries, such as supersymmetry, to large charge [3] and/or spin [4] expansions, bootstrap [5], and numerical approaches.

In this work, we develop a novel way to determine the scaling dimensions Δ_n of neutral operators, schematically ϕ^n, in scalar CFTs in the double-scaling limit of large n, small self coupling λ, and generic λn. We will employ the methodology to tackle various scalar CFTs living in different space-time dimensions d. We discover that, for all the models, the large λn behavior is of the type: $\Delta_n \propto n^{d/(d-1)}$. We therefore conjecture that this leading large n scaling holds nonperturbatively away from the small λ limit. Interestingly, this behavior mimics the one found for large charge operators with charge n [3,6].

Methodology and ϕ^4 theory in $d = 4 - \epsilon$. A cornerstone example of CFT is the critical $\lambda\phi^4$ theory in $d = 4 - \epsilon$ dimensions. We use this model to introduce the semiclassical framework to determine the scaling dimensions Δ_n of the ϕ^n composite operator at the Wilson-Fisher infrared fixed point stemming from the Lagrangian below

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\lambda}{4}\phi^4. \tag{1}$$

The two-loop fixed-point coupling value is

$$\lambda^* = \frac{8\pi^2}{9}\epsilon + \frac{136\pi^2}{243}\epsilon^2 + \mathcal{O}(\epsilon^3). \tag{2}$$

For general n, we compute the three-loop value of Δ_n and obtain

*Contact author: oantipin@irb.hr
†Contact author: jahmall.bersini@ipmu.jp
‡Contact author: sannino@qtc.sdu.dk

$$\Delta_n = n\left(1-\frac{\epsilon}{2}\right)+\frac{n}{6}(n-1)\epsilon-\frac{\epsilon^2}{324}\left(17n^3-67n^2+47n\right) + \frac{n}{34992}\left(1125n^3-7433n^2+15034n-8399 + 2592(n-3)(n-1)\zeta(3)\right)\epsilon^3+\mathcal{O}(\epsilon^4). \tag{3}$$

The two-loop result has been previously determined in [7]. In general, ϕ^n mixes with other scalar operators containing derivatives and, therefore, Δ_n appears as a diagonal entry in the anomalous dimension matrix evaluated at the fixed point.

The perturbative expansion of Δ_n has the general form

$$\Delta_n = n\left(\frac{d-2}{2}\right)+\sum_{l=1}^{\infty}P_l(n)\epsilon^l, \tag{4}$$

where the l-loop coefficient $P_l(n)$ is a polynomial of degree $l+1$ in n

$$P_l(n)=\sum_{k=0}^{l}c_{kl}n^{l+1-k}. \tag{5}$$

Determining Δ_n at arbitrary orders in perturbation theory is a highly demanding task. Within the path integral formalism, calculating Δ_n amounts to perform the following functional integration

$$\langle\phi^n(x_f)\phi^n(x_i)\rangle = \int\mathcal{D}\phi\,\phi^n(x_f)\phi^n(x_i)e^{i\int d^dx\mathcal{L}}. \tag{6}$$

Upon exponentiating the field insertion and rescaling the field as $\phi\to\sqrt{n}\phi$ we observe that n becomes a counting parameter. As a consequence, the above correlator can be estimated semiclassically around the saddle point of the following action:

$$n\left[\int d^dx\left(\frac{1}{2}(\partial\phi)^2-\frac{\lambda n}{4}\phi^4\right)-i\left(\log\phi(x_f)+\log\phi(x_i)\right)\right]. \tag{7}$$

Further employing the double scaling limit $n\to\infty$, $\lambda\to 0$ with fixed λn yields the following expansion for Δ_n

$$\Delta_n = n\sum_{i=0}\frac{C_i(\lambda n)}{n^i}, \tag{8}$$

where the coefficients C_i arise from the ith order of the semiclassical expansion. A similar approach has been used to determine multiparticle scattering amplitudes and decay rates [8,9], and also to compute scaling dimensions of large charge composite operators in theories with continuous symmetries [3,6].

For CFTs the computation is efficiently performed by conformal mapping flat space into a cylinder $\mathbb{R}\times S^{d-1}$ with unit radius. According to Cardy's state-operator correspondence [10] a given scaling dimension becomes the energy on the cylinder of the corresponding state. On the cylinder the Lagrangian reads

$$\mathcal{L}=\frac{1}{2}(\partial\phi)^2-\frac{(d-2)R}{(d-1)8}\phi^2-\frac{\lambda}{4}\phi^4, \tag{9}$$

with the Ricci curvature $R=(d-1)(d-2)$. In this work, we will determine the leading coefficient C_0 of the semiclassical expansion which is given by the classical energy on the cylinder. To the leading order in the expansion (8), one can set $d=4$ since the classical theory is scale invariant.

To compute the energy on the cylinder we solve the following time-dependent equation of motion (EOM)

$$\frac{d^2\phi}{dt^2}+\phi+\lambda\phi^3=0, \tag{10}$$

assuming a spatially homogeneous field configuration supplemented by the Bohr-Sommerfeld condition

$$2\pi^2\int_0^{\mathcal{T}}\left(\frac{d\phi}{dt}\right)^2dt=2\pi n, \tag{11}$$

needed to select the appropriate state in the theory. Here $\mathcal{T}$ is the period of the solution which depends on the product λn. The leading order of the semiclassical expansion C_0 can now be obtained by evaluating the energy on the solution of the equation of motion. This procedure yields

$$\frac{n}{2\pi^2}C_0=T_{00}=\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2+\frac{1}{2}\phi^2+\frac{\lambda}{4}\phi^4, \tag{12}$$

with $T_{\mu\nu}$ the stress-energy tensor of the theory and the $2\pi^2$ factor being the volume of S^3. C_0 resums the terms with the leading power of n at any loop order, i.e., the c_{0l} coefficients appearing in Eq. (5). The general solution found in [11] is

$$\phi(t)=\sqrt{n}x_0\mathrm{cn}(\omega t|m), \tag{13}$$

where $\mathrm{cn}(\omega t|m)$ denotes the Jacobi elliptic function with the frequency and the initial position given by

$$x_0=\sqrt{\frac{2m}{\lambda n(1-2m)}},\qquad \omega=\frac{1}{\sqrt{1-2m}}. \tag{14}$$

The corresponding energy yields the leading order in the semiclassical expansion for Δ_n which reads

$$C_0(\lambda n)=\frac{2\pi^2m(1-m)}{\lambda n(1-2m)^2}, \tag{15}$$

where $0\le m\le 1/2$ is a function of λn which is determined by solving the Bohr-Sommerfeld condition with $\mathcal{T}=4\mathcal{K}/\omega$ where $\mathcal{K}(m)$ is the complete elliptic integral of the first kind. Naturally $\mathcal{T}$ is the period of $\mathrm{cn}(\omega t|m)$ and therefore of the solution $\phi(t)$. We obtain

$$\lambda n = \frac{8\pi}{3(1-2m)^{3/2}}\left[(2m-1)\mathcal{E}(m) + (1-m)\mathcal{K}(m)\right]. \quad (16)$$

Here $\mathcal{E}$ denotes the complete elliptic integral of the second kind. Equation (15) supplemented by (16) constitutes our main result. To build some intuition let us consider first the limit $m \to 0$ where one has the known solution of the harmonic oscillator with unit frequency. This is the trivial free-field theory limit $\lambda = 0$ discussed in [12] for which $\Delta_n = n$. This result is obtained by noting that for $\lambda n \ll 1$ one has $m \sim \frac{\lambda n}{2\pi^2}$. In fact, in this regime, the solution to the EOM reduces to

$$\phi(t) = \frac{\sqrt{n}}{\pi}\cos(t) + \mathcal{O}(\lambda n), \quad (17)$$

and has period $\mathcal{T} = 2\pi$. The $\lambda n \ll 1$ limit maps into ordinary perturbation theory and will be discussed later in the text.

When the anharmonic term dominates, for $\lambda n \gg 1$, we observe that m approaches $1/2$ from below, and for $m = 1/2$, one obtains the interesting solution of the pure quartic anharmonic oscillator. The transcendental equation in (16) can be solved numerically for any λn with its solution given graphically in Fig. 1. Here it is clear that m grows monotonically with λn achieving asymptotically the value $m = 1/2$. In the other panel of Fig. 1 we plot the leading order value for Δ_n/n in the semiclassical expansion. Its behavior can be summarized as follows:

(i) In the $\lambda n \to \infty$ limit m reads

$$m = \frac{1}{2} - \pi\left(\frac{\Gamma\left(\frac{1}{4}\right)}{6\Gamma\left(\frac{3}{4}\right)}\right)^{2/3}\left(\frac{1}{\lambda n}\right)^{2/3} + \mathcal{O}\left((\lambda n)^{-4/3}\right), \quad (18)$$

leading to

$$\Delta_n = \left(\frac{3\Gamma\left(\frac{3}{4}\right)}{2^{5/4}\Gamma\left(\frac{1}{4}\right)}\right)^{4/3}\lambda^{1/3}n^{4/3} + \mathcal{O}(n^{2/3}\lambda^{-1/3}). \quad (19)$$

We deduce a leading $n^{4/3}$ dependence in the large λn limit. This is the same scaling observed for the scaling dimension of large charge operators, with their charge playing the role of n [3,6].

(ii) For intermediate λn we observe a smooth increase with λn.

(iii) At small λn we recover both the free field theory limit as well as the conventional diagrammatic expansion as we will detail momentarily.

The loop expansion is obtained by expanding Eq. (15) around $\lambda n = 0$. We adopt the notation $C_0 = \sum_{l=0} a_l (\frac{\lambda n}{\pi^2})^l$ and list the first 13 coefficients below

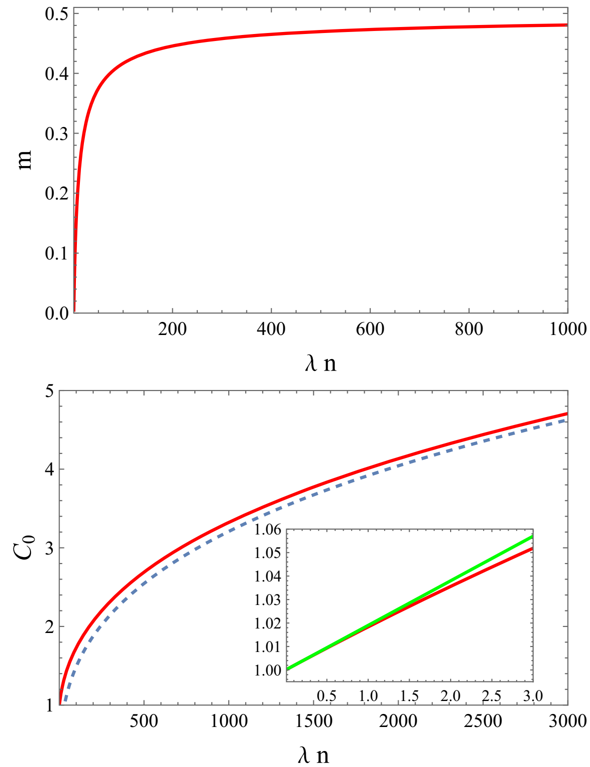

FIG. 1. The parameter m (top) and the leading order scaling dimension C_0 (bottom) as a function of λn. The dashed line denotes the leading large λn behavior of C_0 given by Eq. (19). The inset plot shows a detail of C_0 in the small λn regime along with the one-loop approximation (in green).

$$\begin{aligned}
&a_0 = 1, \quad a_1 = \frac{3}{16}, \quad a_2 = -\frac{17}{256}, \quad a_3 = \frac{375}{8192},\\
&a_4 = -\frac{10689}{262144}, \quad a_5 = \frac{87549}{2097152}, \quad a_6 = -\frac{3132399}{67108864},\\
&a_7 = \frac{238225977}{4294967296}, \quad a_8 = -\frac{18945961925}{274877906944},\\
&a_9 = \frac{194904116847}{2199023255552}, \quad a_{10} = -\frac{8240234242929}{70368744177664},\\
&a_{11} = \frac{11128512976035}{70368744177664}, \quad a_{12} = -\frac{15671733036451359}{72057594037927936},\\
&a_{13} = \frac{87535900033269525}{288230376151711744}.
\end{aligned} \quad (20)$$

Inserting the Wilson-Fisher fixed point value Eq. (2), the above yields the c_{0l} coefficients in Eq. (5). One can check that the first four coefficients agree with the diagrammatic result in Eq. (3). Note that the a_l coefficients also match the known anomalous dimension of the ϕ^n operator in $d = 4$. In fact, the small λn expansion of C_0 yields results valid also away from the fixed point. Similarly, one can now predict the terms with the leading power of n to arbitrarily high loop orders.

The $O(N)$ model. We now extend our analysis to the $O(N)$ model in $d = 4 - \epsilon$ dimensions. The fixed point value to two-loop order is

$$\lambda^* = \frac{8\pi^2}{(N+8)}\epsilon + \frac{24\pi^2(3N+14)}{(N+8)^3}\epsilon^2 + \mathcal{O}(\epsilon^3), \qquad (21)$$

and

$$\Delta_n = n\left(1-\frac{\epsilon}{2}\right) + \frac{n}{2(N+8)}(3n+N-4)\epsilon - \left[\frac{17}{4(N+8)^2}n^3 - \frac{604+(10-11N)N}{4(N+8)^3}n^2 + \frac{576-N(118+35N)}{4(N+8)^3}n\right]\epsilon^2 + \mathcal{O}(\epsilon^3), \qquad (22)$$

is the two-loop value of Δ_n for the singlet operator $(\phi_a\phi_a)^{n/2}$ with even n and $a = 1, \ldots, N$ [7].[1]

By recognizing that by an $O(N)$ rotation the modulus coincides with one of the scalar field directions, the EOM reduces to the one of the $N = 1$ case discussed above. The dependence on N, to the leading order in $1/n$, appears via the fixed point value of the coupling shown above. Therefore C_0 will be again given by Eqs. (15) and (16) with λ the $O(N)$ fixed point coupling in Eq. (21). Note that in the $O(N)$ case, n has to be an even integer.

The ϕ^3 theory in $d = 6 - \epsilon$. To illustrate the generality of the approach we further consider the $O(N)\phi^3$ theory in $d = 6 - \epsilon$ dimensions. Besides being a textbook example of quantum field theory, the $N = 0$ case is physically relevant being related to the Lee-Yang edge singularity and percolation problems [13,14]. For large enough N the theory features a perturbative infrared Wilson-Fisher fixed point that is believed to provide a UV completion to the quartic $O(N)$ model in more than four dimensions [15]. Intriguingly, it has also been proposed that the d dimensional $O(N)$ CFT has a dual holographic description in terms of Vasiliev higher-spin theories in AdS_{d+1} [16]. However, this CFT is nonperturbatively unstable due to instanton solutions which give rise to a nonzero imaginary part in the CFT data [17]. The Lagrangian reads

$$\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 + \frac{1}{2}(\partial\eta)^2 - \frac{g}{2}\eta(\phi_a)^2 - \frac{\lambda}{3}\eta^3, \qquad (23)$$

where ϕ_a with $a = 1, \ldots, N$ is a $O(N)$ vector while η is a singlet. For our investigation, we just need to know the one-loop fixed point value of λ which is

$$\lambda^* = 3\sqrt{\frac{6\epsilon(4\pi)^3}{N}}\left(1 + \frac{162}{N} + \frac{68766}{N^2} + \cdots + \mathcal{O}(\epsilon)\right). \qquad (24)$$

Equipped with the above, we proceed by computing the scaling dimension $\Delta_{n,\eta}$ for the family of composite operators η^n in the same double scaling limit considered for the quartic $O(N)$ theory resulting in a semiclassical expansion analogous to Eq. (8)

$$\Delta_n = n\sum_{i=0}\frac{H_i(\lambda^2 n)}{n^i}. \qquad (25)$$

To this end, we again map our theory on $\mathbb{R}\times S^{d-1}$ and consider a homogeneous field configuration for η and a vanishing expectation value for ϕ_a. The resulting EOM reads

$$\frac{d^2\eta}{dt^2} + 4\eta + \lambda\eta^2 = 0, \qquad (26)$$

and admits the following nontrivial solution

$$\eta(t) = \frac{1}{\lambda}\left(\frac{6m\,\mathrm{cn}\left(\frac{t}{((m-1)m+1)^{1/4}}\middle|m\right)^2 - 4m + 2}{\sqrt{(m-1)m+1}} - 2\right), \qquad (27)$$

with $0 \le m \le 1$. By inserting the above into the following expression for the classical ground state energy

$$T_{00} = \frac{n}{\pi^3}H_0 = \frac{1}{2}\left(\frac{\partial\eta}{\partial t}\right)^2 + 2\eta^2 + \frac{\lambda}{3}\eta^3, \qquad (28)$$

we obtain the following leading coefficient H_0 of the expansion (25)

$$H_0 = \frac{8\pi^3}{3\lambda^2 n}\left(\frac{-2m^3 + 3m^2 + 3m - 2}{((m-1)m+1)^{3/2}} + 2\right), \qquad (29)$$

where m is a nontrivial function of the product $\lambda^2 n$ which is determined by the Bohr-Sommerfeld condition as follows

$$\frac{2((m-1)m+1)\mathcal{E}(m) - (m-2)(m-1)K(m)}{5((m-1)m+1)^{5/4}} = \frac{\lambda^2 n}{48\pi^2}. \qquad (30)$$

As for the ϕ^4 theory, H_0 resums the leading powers of n at any order of the perturbative expansion for Δ_n. Their coefficients can be read off by expanding H_0 around $\lambda^2 n = 0$. In parallel with our previous analysis, we adopt the notation $H_0 = \sum_{l=0} b_l(\frac{\lambda^2 n}{\pi^3})^l$ and provide the first 8 coefficients below

[1]We employed the perturbative results of [7] for $(\phi_a\phi_a)^s$ and wrote them in terms of $n = 2s$.

$$
\begin{aligned}
&b_0=2, \quad b_1=-\frac{5}{192}, \quad b_2=-\frac{235}{221184},\\
&b_3=-\frac{38585}{509607936}, \quad b_4=-\frac{2663129}{391378894848},\\
&b_5=-\frac{156934505}{225434243432448} \quad b_6=-\frac{13400341405}{173133498956120064},\\
&b_7=-\frac{7275692993855}{797799163189801254912}.
\end{aligned}
\tag{31}
$$

The coefficients b_0 and b_1 match the results in [15,18]. In fact, by requiring consistency between the one-loop results for η and η^2 obtained in [15,18] and Eq. (4), one can determine the c_{01} and c_{11} coefficients in Eq. (5) hence obtaining the $\mathcal{O}(\epsilon)$ scaling dimension of η^n for general n[2]

$$
\Delta_n = n\left(2-\frac{\epsilon}{2}\right) - \left[n^2\left(\frac{90}{N}+\frac{29160}{N^2}+\dots\right) - n\left(\frac{130}{N}+\frac{35960}{N^2}+\dots\right)\right]\epsilon + \mathcal{O}(\epsilon^2). \tag{32}
$$

We checked that the terms of order n^2 in the above are reproduced by evaluating $\frac{b_1\lambda^2 n^2}{\pi^3}$ at the fixed point (24). The b_l coefficients with $l>1$ are new predictions for the term with the leading power of n in the lth loop orders of the perturbative expansion of Δ_n.

Interestingly, the complementary limit of large $\lambda^2 n$ reveals the unstable nature of the theory. In fact, Eq. (30) admits real solutions only for $\frac{\lambda^2 n}{(4\pi)^2}\le\frac{6}{5}$ where the equality is attained for $m=1$. An analogous behavior has been previously observed in the large charge sector of the theory in [19,20] where it has been discovered the existence of a critical value of the charge above which the scaling dimension of the operators carrying the $O(N)$ charges becomes complex. A possible link to instantonic solutions has been later discussed in [21]. In the large $\lambda^2 n$ limit we, therefore, obtain

$$
m \sim e^{\frac{\pm i\pi}{3}} - \left(e^{\frac{\pm i9\pi}{10}} 3^{7/10}\left(\frac{16\pi^2}{5}K\left(e^{\frac{\pm i\pi}{3}}\right)\right)^{4/5}\right)\left(\frac{1}{\lambda^2 n}\right)^{4/5}, \tag{33}
$$

leading to

$$
H_0 = \frac{e^{\mp\frac{i\pi}{10}}}{3^{13/10}}\left(\frac{5\sqrt{\pi}}{2^{3/2}K(e^{\frac{\pm i\pi}{3}})}\right)^{6/5}(\lambda^2 n)^{1/5} + \mathcal{O}\left((\lambda^2 n)^{-1/5}\right). \tag{34}
$$

The two complex conjugate solutions correspond to a pair of complex CFTs. We again note the asymptotic $\Delta_n \sim n^{\frac{d}{d-1}}$ large n behavior previously observed in the large charge sector of the theory [19].

The ϕ^6 theory in $d=3-\epsilon$. For our last example, we determine Δ_n for the singlet operators $(\phi_a\phi_a)^{n/2}$ with $a=1,\dots,N$ in the critical $(\phi_a\phi_a)^3$ theory in $d=3-\epsilon$. The Lagrangian reads

$$
\mathcal{L} = \frac{1}{2}(\partial\phi_a)^2 - \frac{\lambda^2}{6}(\phi_a\phi_a)^3. \tag{35}
$$

This theory has a perturbative Wilson-Fisher fixed point whose two-loop value is

$$
\frac{\lambda^{*2}}{(4\pi)^2} = \frac{\epsilon}{4(22+3N)}, \tag{36}
$$

while the one-loop beta function vanishes identically in $d=3$. In the same double scaling limit considered for the quartic theory, the scaling dimension of the $(\phi_a\phi_a)^{n/2}$ operators takes the form of Eq. (8). The complete solution of the equation of motion is quite cumbersome and will be treated in a separate work. Here, we limit ourselves to the perturbative small λn regime where the solution of the time-dependent EOM of the model on $\mathbb{R}\times S^{d-1}$ reads

$$
\begin{aligned}
\frac{\phi_a(t)}{\sqrt{n}} = {}& \frac{\cos(\frac{t}{2})}{\sqrt{\pi}} + \frac{(\lambda n)^2}{96\pi^{5/2}}\left(-60t\sin\left(\frac{t}{2}\right) - 60\cos\left(\frac{t}{2}\right) + 15\cos\left(\frac{3t}{2}\right) + \cos\left(\frac{5t}{2}\right)\right)\\
&+ \frac{(\lambda n)^4}{18432\pi^{9/2}}\left(-14280\cos\left(\frac{3t}{2}\right) - 440\cos\left(\frac{5t}{2}\right) + 95\cos\left(\frac{7t}{2}\right) + 3\cos\left(\frac{9t}{2}\right)\right.\\
&\left.+ 10\left((4019-360t^2)\cos\left(\frac{t}{2}\right) + 3864t\sin\left(\frac{t}{2}\right) - 60t\left(9\sin\left(\frac{3t}{2}\right) + \sin\left(\frac{5t}{2}\right)\right)\right)\right) + \mathcal{O}((\lambda n)^6).
\end{aligned}
\tag{37}
$$

Replacing the above into the stress-energy tensor one obtains the first few terms of the small λn expansion for C_0

$$
C_0 = \frac{1}{2} + \frac{5\lambda^2 n^2}{24\pi^2} - \frac{131\lambda^4 n^4}{384\pi^4} + \mathcal{O}((\lambda n)^6). \tag{38}
$$

[2]Since η^2 mixes with the $\phi_a\phi_a$ operator one has to consider the corresponding diagonal entry in the anomalous dimension matrix.

To test the solution we insert the fixed point value Eq. (36) into the above, and see that it reproduces the leading n term in the 2-loop expression for Δ_n which reads [22]

$$\Delta_n = \frac{1}{6(22+3N)} n(n-2)(5n+3N-8)\epsilon. \quad (39)$$

To summarize our work we have developed a semiclassical framework to compute the scaling dimensions for the class of neutral composite operators in several time-honored CFTs. This has been achieved by considering the double scaling limit $n \to \infty$ and $\lambda \to 0$ with a fixed value of the product λn and employing a saddle point evaluation. We tested our findings at small λn with known diagrammatic results and have been able to predict the infinite series of higher-order terms. Additionally, our leading semiclassical results hold true for various gauge-Yukawa models because fermion and gauge degrees of freedom have vanishing classical backgrounds. Noteworthy examples include the Abelian Higgs and Gross-Neveu-Yukawa models in $d = 4 - \epsilon$.

Additionally, our results constitute a strong asset to boost perturbative computations. In fact, one can combine our semiclassical expansion with known perturbative results, at fixed n, to determine novel complete higher loop expressions. Importantly, our findings provide an infinite series of checks for future diagrammatic computations.

We plan to go beyond this initial investigation by determining the next semiclassical order C_1 stemming from the determinant of the quantum fluctuations around the classical solution. Therefore, the computations for determining C_1 resemble the ones related to computing the leading correction around an instantonic background. As we have shown in this letter, our framework can be extended to several physically relevant quantum field theories in various space-time dimensions.

Acknowledgments. The work of F. S. is partially supported by the Carlsberg Foundation, semper ardens Grant No. CF22-0922. The work of J. B. was supported by the World Premier International Research Center Initiative (WPI Initiative), MEXT, Japan; and also supported by the JSPS KAKENHI Grant No. JP23K19047. O. A. and J. B. thank the Quantum Theory Center and the Danish Institute for Advanced Study at the University of Southern Denmark for their hospitality and partial support while this work was completed.

[1] K. G. Wilson and J. B. Kogut, The renormalization group and the epsilon expansion, Phys. Rep. **12**, 75 (1974).

[2] K. G. Wilson, The renormalization group: Critical phenomena and the Kondo problem, Rev. Mod. Phys. **47**, 773 (1975).

[3] S. Hellerman, D. Orlando, S. Reffert, and M. Watanabe, On the CFT operator spectrum at large global charge, J. High Energy Phys. 12 (2015) 071.

[4] Z. Komargodski and A. Zhiboedov, Convexity and liberation at large spin, J. High Energy Phys. 11 (2013) 140.

[5] R. Rattazzi, V. S. Rychkov, E. Tonni, and A. Vichi, Bounding scalar operator dimensions in 4D CFT, J. High Energy Phys. 12 (2008) 031.

[6] G. Badel, G. Cuomo, A. Monin, and R. Rattazzi, The epsilon expansion meets semiclassics, J. High Energy Phys. 11 (2019) 110.

[7] S. E. Derkachov and A. N. Manashov, On the stability problem in the $O(N)$ nonlinear sigma model, Phys. Rev. Lett. **79**, 1423 (1997).

[8] L. S. Brown, Summing tree graphs at threshold, Phys. Rev. D **46**, R4125 (1992).

[9] D. T. Son, Semiclassical approach for multiparticle production in scalar theories, Nucl. Phys. **B477**, 378 (1996).

[10] J. L. Cardy, Conformal invariance and universality in finite-size scaling, J. Phys. A **17**, L385 (1984).

[11] A. Martín Sánchez and J. Díaz Bejarano, Quantum anharmonic symmetrical oscillators using elliptic functions, J. Phys. A **19**, 887 (1986).

[12] G. Cuomo, L. Rastelli, and A. Sharon, Moduli spaces in CFT: Large charge operators, J. High Energy Phys. 09 (2024) 185.

[13] M. E. Fisher, Yang-Lee edge singularity and ϕ^3 field theory, Phys. Rev. Lett. **40**, 1610 (1978).

[14] O. F. de Alcantara Bonfim, J. E. Kirkham, and A. J. McKane, Critical exponents for the percolation problem and the Yang-Lee edge singularity, J. Phys. A **14**, 2391 (1981).

[15] L. Fei, S. Giombi, and I. R. Klebanov, Critical $O(N)$ models in $6-\epsilon$ dimensions, Phys. Rev. D **90**, 025018 (2014).

[16] I. R. Klebanov and A. M. Polyakov, AdS dual of the critical $O(N)$ vector model, Phys. Lett. B **550**, 213 (2002).

[17] S. Giombi, R. Huang, I. R. Klebanov, S. S. Pufu, and G. Tarnopolsky, The $O(N)$ Model in $4 < d < 6$: Instantons and complex CFTs, Phys. Rev. D **101**, 045013 (2020).

[18] L. Fei, S. Giombi, I. R. Klebanov, and G. Tarnopolsky, Three loop analysis of the critical $O(N)$ models in 6-ε dimensions, Phys. Rev. D **91**, 045011 (2015).

[19] O. Antipin, J. Bersini, F. Sannino, Z. W. Wang, and C. Zhang, More on the cubic versus quartic interaction equivalence in the $O(N)$ model, Phys. Rev. D **104**, 085002 (2021).

[20] S. Giombi and J. Hyman, On the large charge sector in the critical $O(N)$ model at large N, J. High Energy Phys. 09 (2021) 184.

[21] M. Watanabe, Stability analysis of a non-unitary CFT, J. High Energy Phys. 11 (2023) 042.

[22] P. Basu and C. Krishnan, ϵ-expansions near three dimensions from conformal field theory, J. High Energy Phys. 11 (2015) 040.