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Towards gravitational QNM spectrum from quantum spacetime

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Abstract. The effective potential for the axial mode of gravitational wave on noncommutative Schwarzschild background is presented. Noncommutativity is introduced via deformed Hopf algebra of diffeomorphisms by means of a semi-Killing Drinfeld twist. The analysis is performed up to the first order in perturbation of the metric and noncommutativity parameter. This results in a modified Regge-Wheeler potential with the strongest differences in comparison to the classical Regge-Wheeler potential being near the horizon.

1. Introduction

One of the greatest puzzles of modern physics revolves around reconciliation of two highly successful theories - quantum mechanics and general relativity. Many different approaches have been launched to this day in order to confront this awe-inspiring but at the same time incredibly challenging task. They include string theory, loop quantum gravity, noncommutative (NC) gravity, causal set theory and others. Most of them propose that the classical notion of spacetime needs to be modified at small scales, leading to a concept of quantum spacetime. Moreover, they are associated with a notion of Lorentz symmetry breaking which appears to crop up in their low-energy limit as a remnant of quantum gravity. The Lorentz invariance violation therefore presents one of the most striking signals of quantum structure of spacetime and indirectly of quantum gravity.

It therefore doesn't come as surprise that the search for a quantum nature of spacetime included proposals for novel physical phenomena arising due to the effects of quantum gravity in the low-energy regime involving Lorentz violation as a central part. A suitable framework for carrying out this kind of study has either been provided by the Lorentz Invariance Violation (LIV) models [1],[2], or by the Doubly Special Relativity (DSR) theories which preserve the relativity principle, but implement it by means of deformed Lorentz transformations [3],[4].

Over a past few decades four major channels of cosmic messengers have been crystallized as suitable avenues for testing and identifying the quantum nature of spacetime. They include cosmic neutrinos, ultra high energy cosmic rays, gamma ray bursts and gravitational waves [5]. It is the latter channel that we shall focus on in the present work. In particular, we shall outline the main steps in constructing NC theory of gravity in a bottom-up approach by utilising the framework of Hopf algebra deformation by means of a Drinfeld twist [6],[7],[8],[9]. The construction will be applied to the linearized gravitation perturbation theory of the Schwarzschild black hole background, thus setting the ground for determining the corrections in the black hole QNM spectrum due to quantum spacetime.

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2. Hopf algebra deformation by twist and NC differential geometry

Let us assume that the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ of diffeomorphisms may be organized into a Hopf algebra, an algebraic structure that besides the multiplication and unital maps also involves additional structural maps like coproduct Δ, antipode S and counit ϵ, defined for each element of the Hopf algebra. We further assume that there is an invertible element $\mathcal{F}$ in $\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g})$. This element that we call the twist and its inverse may respectively be written as $\mathcal{F} = f^\alpha \otimes f_\alpha$, and $\mathcal{F}^{-1} = \bar{f}^\alpha \otimes \bar{f}_\alpha$, with the summation over α being understood. The twist element may be used to deform the initial Hopf algebra $\mathcal{U}(\mathfrak{g})$ by means of the transformations

$$\Delta^{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \quad S^{\mathcal{F}}(h) = \chi S(h)\chi^{-1}, \quad \epsilon^{\mathcal{F}}(h) = \epsilon(h), \tag{1}$$

where $\chi = S(\bar{f}^\alpha)\bar{f}_\alpha$. In this way the Hopf algebra $\mathcal{U}(\mathfrak{g})$ goes into a new, deformed structure that may be denoted as $\mathcal{U}^{\mathcal{F}}(\mathfrak{g})$, and that will also be a Hopf algebra if (by adopting the Sweedler's notation) the twist element satisfies $f^\beta f^\alpha_{(1)} \otimes f_\beta f^\alpha_{(2)} \otimes f_\alpha = f^\alpha \otimes f^\beta f_{\alpha(1)} \otimes f_\beta f_{\alpha(2)}$ and $\bar{f}^\alpha_{(1)}\bar{f}^\beta \otimes \bar{f}^\alpha_{(2)}\bar{f}_\beta \otimes \bar{f}_\alpha = \bar{f}^\alpha \otimes \bar{f}_{\alpha(1)}\bar{f}^\beta \otimes \bar{f}_{\alpha(2)}\bar{f}_\beta$, which is known as the 2-cocycle condition. The twist satisfying these conditions is called a Drinfeld twist. The deformation (1) induces also a deformation at the level of the module algebra of $\mathcal{U}(\mathfrak{g})$, which is $\mathcal{A} = C^\infty(\mathcal{M})$. In particular, the multiplication map is deformed from a pointwise multiplication to a $\star$-multiplication,

$$f \star g = \mu \circ \mathcal{F}^{-1}(f \otimes g) = \bar{f}^\alpha(f)\bar{f}_\alpha(g), \tag{2}$$

for any two functions f and g in the algebra $C^\infty(\mathcal{M})$.

Another important concept is that of the quasitriangular Hopf algebra and the R-matrix, $R = R^\alpha \otimes R_\alpha \in \mathcal{U}^{\mathcal{F}}(\mathfrak{g}) \otimes \mathcal{U}^{\mathcal{F}}(\mathfrak{g})$, $R^{-1} = \bar{R}^\alpha \otimes \bar{R}_\alpha$. The R-matrix, besides satisfying the quasitriangularity conditions $R^\alpha_{(1)} \otimes R^\alpha_{(2)} \otimes R_\alpha = R^\alpha \otimes R^\beta \otimes R_\alpha R_\beta$, and $\bar{R}^\alpha_{(1)} \otimes \bar{R}^\alpha_{(2)} \otimes \bar{R}_\alpha = \bar{R}^\alpha \otimes \bar{R}^\beta \otimes \bar{R}_\beta \bar{R}_\alpha$, may also be related to the Drinfeld twist $\mathcal{F}$ as $R = \mathcal{F}_{op}\mathcal{F}^{-1}$, where $\mathcal{F}_{op} = \sigma \circ \mathcal{F}$ and σ is the exchange operator defined as $\sigma(a \otimes b) = b \otimes a$.

For the details of the NC gravity construction, we refer the reader to [8, 9, 10]. In the rest of this section we briefly introduce the most important notions. To begin with, let $\mathcal{A}_\star, \chi_\star, \Omega_\star, \mathcal{T}^{(p,q)}_\star$ respectively denote the vector spaces of NC functions, vector fields, 1-forms and general tensor fields. The $\star$-pairing, $\langle\ ,\ \rangle_\star : \chi_\star \times \Omega_\star \to \mathcal{A}_\star$ of vector fields and one-forms is then introduced by

$$\langle u, \omega \rangle_\star = \langle \bar{f}^\alpha(u), \bar{f}_\alpha(\omega) \rangle. \tag{3}$$

Curiously, it appears that conditions of cocyclicity and quasitriangularity both give rise to the following properties of the $\star$-pairing:

$$\begin{aligned} \langle h \star u, \omega \star \tilde{h} \rangle_\star &= h \star \langle u, \omega \rangle_\star \star \tilde{h}, \qquad h, \tilde{h} \in \mathcal{A}_\star,\ u \in \chi_\star,\ \omega \in \Omega_\star, \\ \langle u, h \star \omega \rangle_\star &= \bar{R}^\alpha(h) \star \langle \bar{R}_\alpha(u), \omega \rangle_\star, \\ \langle \omega \otimes_\star u, \tau \rangle_\star &= \langle \omega, \langle u, \tau \rangle_\star \rangle_\star \qquad u \in \chi_\star,\ \omega \in \mathcal{T}^{(0,p)}_\star,\ \tau \in \mathcal{T}^{(q,s)}_\star, \quad (q > p). \end{aligned} \tag{4}$$

NC connection is introduced as a linear map, $\nabla^\star : \chi_\star \longrightarrow \Omega_\star \otimes_\star \chi_\star$, satisfying the standard Leibniz rule, $\nabla^\star(h \star v) = dh \otimes_\star v + h \star \nabla^\star v$, for all $h \in \mathcal{A}_\star$ and $v \in \chi_\star$. The notion of the covariant derivative is closely related to that of the connection. In particular, a covariant derivative $\nabla^\star_u$ along the vector field u, for any $u \in \chi_\star$, may be defined as $\nabla^\star_u v = \langle u, \nabla^\star v \rangle_\star$, for all $v \in \chi_\star$. Interestingly, the set of axioms that is usually required for the covariant derivative,

$$\begin{aligned} \nabla^\star_{u+v} z &= \nabla^\star_u z + \nabla^\star_v z, \\ \nabla^\star_{g \star u} v &= g \star \nabla^\star_u v, \\ \nabla^\star_u (g \star v) &= \mathcal{L}^\star_u(g) \star v + \bar{R}^c(g) \star \nabla^\star_{\bar{R}_c(u)} v, \end{aligned} \tag{5}$$

for all $u, v, z \in \chi_\star$ and $g \in \mathcal{A}_\star$, does not need to be postulated here, since it follows naturally as a mere consequence of the $\star$-pairing relations (4). Here $\mathcal{L}^\star_u(g) = \mathcal{L}_{\bar{f}^\alpha(u)}(\bar{f}_\alpha(g))$.

Consider now a $\star$-dual basis, $\langle \partial_\mu, dx^\nu \rangle_\star = \delta_\mu{}^\nu$, consisting of two mutually dual local frames, that of vector fields $\{\partial_\mu\}$ and that of 1-forms $\{dx^\nu\}$. The coefficients of affine connection $\Gamma^{\star\mu}{}_{\nu\lambda}$ in noncommutative theory are consistently given as

$$\nabla^\star_{\partial_\mu}\partial_\nu \equiv \nabla^\star_\mu \partial_\nu = \Gamma^{\star\lambda}{}_{\mu\nu} \star \partial_\lambda, \quad \nabla^\star_{\partial_\mu} dx^\nu \equiv \nabla^\star_\mu dx^\nu = -\Gamma^{\star\nu}{}_{\mu\lambda} \star dx^\lambda. \tag{6}$$

Other crucial concepts such as the NC torsion, Riemann curvature and Ricci tensors are then straightforwardly introduced for all $u, v, z \in \chi_\star$,

$$\begin{aligned}
T^\star(u,v) &= \nabla^\star_u v - \nabla^\star_{\bar{R}^c(v)} \bar{R}_c(u) - [u,v]_\star \equiv \langle u \otimes_\star v, T^\star \rangle_\star, \\
R^\star(u,v,z) &= \nabla^\star_u \nabla^\star_v z - \nabla^\star_{\bar{R}^c(v)} \nabla^\star_{\bar{R}_c(u)} z - \nabla^\star_{[u,v]_\star} z \equiv \langle u \otimes_\star v \otimes_\star z, R^\star \rangle_\star. \\
Ric^\star(u,v) &= \langle dx^\alpha, R^\star(\partial_\alpha, u, v) \rangle_\star.
\end{aligned}$$

and the components $T^{\star\lambda}{}_{\mu\nu}$ and $R^\star_{\mu\nu\sigma}{}^\lambda$ are given by

$$T^{\star\lambda}{}_{\mu\nu} = \langle dx^\lambda, T^\star(\partial_\mu, \partial_\nu) \rangle_\star, \quad R^\star_{\mu\nu\sigma}{}^\lambda = \langle dx^\lambda, R^\star(\partial_\mu, \partial_\nu, \partial_\sigma) \rangle_\star \tag{7}$$

It can be checked that the Riemann and torsion tensors are R-antisymmetric in the first two slots:

$$R^\star(u,v,z) = -R^\star(\bar{R}^c(v),\ \bar{R}_c(u),\ z), \qquad T^\star(u,v) = -T^\star(\bar{R}^c(v),\ \bar{R}_c(u)).$$

However, NC Riemann tensor is not R-antisymmetric in the second and the third slots, implying that NC Ricci tensor is neither R-symmetric nor symmetric.

Finally, the metric g is an element of $\Omega_\star \otimes_\star \Omega_\star$ and may be written as

$$g = g_{\mu\nu} \star dx^\mu \otimes_\star dx^\nu = g^a \otimes_\star g_a \in \Omega_\star \otimes_\star \Omega_\star,$$

where the sum over a is understood. Analogously, the inverse metric $g^{-1} \equiv g^\star$ is an element of $\chi_\star \otimes_\star \chi_\star$, $g^{-1} = g^{-1\,b} \otimes g^{-1}_b \in \chi_\star \otimes_\star \chi_\star$. These two are related by

$$\begin{aligned}
\langle \langle v, g \rangle_\star, g^{-1} \rangle_\star &= \langle v, g^a \rangle_\star \star \langle g_a, g^{-1\,b} \rangle_\star \star g^{-1}_b = v, \qquad \text{for all} \quad v \in \chi_\star \\
\langle \langle \omega, g^{-1} \rangle_\star, g \rangle_\star &= \langle \omega, g^{-1\,b} \rangle_\star \star \langle g^{-1}_b, g^a \rangle_\star \star g_a = \omega, \qquad \text{for all} \quad \omega \in \Omega_\star
\end{aligned} \tag{8}$$

The fact that the deformed Ricci tensor is not symmetric will have a consequence on finding a consistent definition of the NC Einstein manifold. From this reason, in this paper we shall be concerned with a definition of the NC Einstein manifold that amounts not to a solution of the equation $Ric^\star(\partial_\mu, \partial_\nu) = 0$, but rather to a solution to its symmetrized form, which in the case of nice basis corresponds to the symmetrized NC Ricci tensor

$$Ric^\star_{(\mu\nu)} = \frac{1}{2}\Big(Ric^\star(\partial_\mu, \partial_\nu) + Ric^\star(\partial_\nu, \partial_\mu)\Big) = 0. \tag{9}$$

In the case of deformation with a most general Drinfeld twist, the actual NC Einstein equation has a more elaborate structure which incorporates R-symmetry. Its precise form can be found in [10], as well as other details. However, for the special family of Drinfeld twists that we consider here (starting from the next section, see (11)), it reduces to the special form (9). In a situation that the deformation is carried by the same special family of twists, it might be tempting to contemplate about generalizing the Einstein tensor to a noncommutative case. Namely, if we symmetrize the Ricci tensor as above and define the Ricci scalar as $R^\star = g^{\star\mu\nu} \star Ric^\star_{(\mu\nu)}$, then postulating the vacuum Einstein equation in the form

$$Ric^\star_{(\mu\nu)} - \frac{1}{2} g_{\mu\nu} \star R^\star = 0 \tag{10}$$

straightforwardly implies that $Ric^\star_{(\mu\nu)} = 0$, as can be clearly seen by $\star$-multiplying (10) with $g^{\star\rho\mu}$ from the left. While this might be an indicator that defining the Einstein tensor as in (10) proofs correct, it yet remains to see if this object is divergentless with respect to the $\star$-connection.

3. Linearized noncommutative gravity on the Schwarzschild background

To induce spacetime noncommutativity we will use the twist of the form

$$\mathcal{F} = \exp\Big(- i\frac{a}{2}(K \otimes X - X \otimes K)\Big), \tag{11}$$

where X is some vector field and K is a Killing field of the background metric. It is a Drinfeld twist, i.e. satisfies the cocycle conditions. Our Lie algebra of diffeomorphisms is therefore two-dimensional, being generated by X and K. In addition, we require these two fields to commute, thus $\mathcal{F}$ is an Abelian twist. We will use the standard spherical coordinates (t, r, θ, φ) and demand that basis consisting of their generators $(\partial_t, \partial_r, \partial_\theta, \partial_\varphi)$ commutes with X and K. A basis of vector fields of this kind is referred to as a nice basis. All of these properties are satisfied for $K = \alpha\partial_t + \beta\partial_\varphi, \quad X = \partial_r$ on the Schwarschild background in spherical basis. Because of the nice basis and Abelian twist, many formulas look like their commutative counterparts with $\star$-product replacing the usual pointwise product. The $\star$-product according to (2) is

$$f \star g = fg + i\frac{a}{2}\Big(K(f)X(g) - X(f)K(g)\Big) + O(a^2).$$

We use the twist (11) to study a noncommutative deformation of a linearized gravitational perturbation theory. The background metric $\mathring{g}_{\mu\nu}$ is Schwarzschild:

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1-2M/r}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

To study perturbations of the Schwarzschild spacetime we split the metric into background $\mathring{g}$ and perturbation h,

$$g_{\mu\nu} = \mathring{g}_{\mu\nu} + h_{\mu\nu}, \qquad \mathring{g}^{\mu\nu}\mathring{g}_{\nu\lambda} = \delta^{\mu}{}_{\lambda}.$$

As already said, K is the Killing vector field for the background (unperturbed) metric $\mathring{g}_{\mu\nu}$,

$$K(\mathring{g}_{\mu\nu}) = \mathcal{L}_K(\mathring{g}_{\mu\nu}) = 0, \quad K(\mathring{g}^{\mu\nu}) = \mathcal{L}_K(\mathring{g}^{\mu\nu}) = 0.$$

The metric h is assumed to be small relative to $\mathring{g}$ and thus we do our calculations up to the first order in h. When switching on a deformation, which is controlled by the parameter of deformation a, we also keep only first order correction terms in a. It will turn out that perturbation and deformation parts come in pairs, and are thus always coupled.

Due to relative simplicity of the twist (11) and the nice basis, the conditions (8) for the metric inverse simplify significantly and reduce to a rather familiar form, except only for the pointwise multiplication being replaced by a $\star$ multiplication: $g^{\star\sigma\rho} \star g_{\rho\nu} = \delta^{\sigma}{}_{\nu}$ and $g_{\nu\rho} \star g^{\star\rho\sigma} = \delta_{\nu}{}^{\sigma}$. Restricting to the linear perturabations h and a, the solutions to the above stated conditions give for the inverse metric $g^{\star\mu\nu} = \mathring{g}^{\mu\nu} - \mathring{g}^{\mu\alpha} \star h_{\alpha\beta} \star \mathring{g}^{\beta\nu}$. The first order part can be written as

$$g^{\star\mu\nu} = \mathring{g}^{\mu\nu} - h^{\mu\nu} + \tilde{g}^{\mu\nu} = g^{\mu\nu} - g^{\mu\rho}g_{\rho\lambda} \wedge g^{\lambda\nu}, \tag{12}$$

where the following abbreviation was used

$$f \wedge g = i\frac{a}{2}\Big(K(f)X(g) - X(f)K(g)\Big).$$

The form of the inverse manifestly shows that a deformation, when present, is always coupled to perturbation $h_{\mu\nu}$. It turns out that such $\star$-inverse metric is not symmetric but Hermitian.

From the metric compatibility condition and relations (7) it then follows [8],[9] that coefficients of the Levi-Civita connection in the nice basis, torsion, and Riemann curvature tensor are in local coordinates given by

$$\Gamma^{\star\mu}{}_{\nu\rho} \equiv \Omega^{\mu}{}_{\nu\rho} = \frac{1}{2}g^{\star\mu\alpha} \star (\partial_\nu g_{\rho\alpha} + \partial_\rho g_{\nu\alpha} - \partial_\alpha g_{\nu\rho}), \qquad T^{\star\mu}{}_{\nu\rho} = \Omega^{\mu}{}_{\nu\rho} - \Omega^{\mu}{}_{\rho\nu},$$
$$R^{\star}_{\mu\nu\rho}{}^{\sigma} = \partial_\mu\Omega^{\sigma}{}_{\nu\rho} - \partial_\nu\Omega^{\sigma}{}_{\mu\rho} + \Omega^{\beta}{}_{\nu\rho} \star \Omega^{\sigma}{}_{\mu\beta} - \Omega^{\beta}{}_{\mu\rho} \star \Omega^{\sigma}{}_{\nu\beta},$$

which gives rise to the explicit form for the corrections to these quantities, all up to the linear order in h and a:

$$\begin{aligned}
\Omega^{\mu}{}_{\nu\rho} &= \mathring{\Gamma}^{\mu}{}_{\nu\rho} + \delta\Gamma^{\mu}{}_{\nu\rho} + (\mathring{g}^{\mu\sigma} \wedge \delta\Gamma^{\lambda}{}_{\nu\rho})\mathring{g}_{\sigma\lambda} - (h^{\mu\sigma} \wedge \mathring{\Gamma}^{\lambda}{}_{\nu\rho})\mathring{g}_{\sigma\lambda} \equiv \Gamma^{\mu}{}_{\nu\rho} + \tilde{\Omega}^{\mu}{}_{\nu\rho},\\
T^{\star\mu}{}_{\nu\rho} &= \Omega^{\mu}{}_{\nu\rho} - \Omega^{\mu}{}_{\rho\nu} = 0,\\
R^{\star}{}_{\mu\nu\rho}{}^{\sigma} &= \mathring{R}_{\mu\nu\rho}{}^{\sigma} + \delta R_{\mu\nu\rho}{}^{\sigma} + \partial_{\mu}\tilde{\Omega}^{\sigma}{}_{\nu\rho} - \partial_{\nu}\tilde{\Omega}^{\sigma}{}_{\mu\rho} + \mathring{\Gamma}^{\beta}{}_{\nu\rho} \wedge \delta\Gamma^{\sigma}{}_{\mu\beta} + \delta\Gamma^{\beta}{}_{\nu\rho} \wedge \mathring{\Gamma}^{\sigma}{}_{\mu\beta}\\
&+ \mathring{\Gamma}^{\beta}{}_{\nu\rho}\tilde{\Omega}^{\sigma}{}_{\mu\beta} + \tilde{\Omega}^{\beta}{}_{\nu\rho}\mathring{\Gamma}^{\sigma}{}_{\mu\beta} - \mathring{\Gamma}^{\beta}{}_{\mu\rho} \wedge \delta\Gamma^{\sigma}{}_{\nu\beta} - \delta\Gamma^{\beta}{}_{\mu\rho} \wedge \mathring{\Gamma}^{\sigma}{}_{\nu\beta} - \mathring{\Gamma}^{\beta}{}_{\mu\rho}\tilde{\Omega}^{\sigma}{}_{\nu\beta} - \tilde{\Omega}^{\beta}{}_{\mu\rho}\mathring{\Gamma}^{\sigma}{}_{\nu\beta},
\end{aligned}$$

and consequently leads to the corrections within the same order in the vaccuum Einstein equations. Here $\delta\Gamma^{\mu}{}_{\nu\rho}$ and $R_{\mu\nu\rho}{}^{\sigma}$ are just the h-linear parts without the noncommutativity corrections. Taking advantage of the explicit form for the twist (11), and the above relation for deformed Riemann tensor, the NC Ricci tensor follows as

$$\begin{aligned}
Ric^{\star}{}_{\nu\rho} &= R^{\star}{}_{\mu\nu\rho}{}^{\mu} = \mathring{R}_{\mu\nu\rho}{}^{\mu} + \delta R_{\mu\nu\rho}{}^{\mu} + \partial_{\mu}\tilde{\Omega}^{\mu}{}_{\nu\rho} - \partial_{\nu}\tilde{\Omega}^{\mu}{}_{\mu\rho} + i\frac{a}{2}\Big[K(\delta\Gamma^{\beta}{}_{\nu\rho})X(\mathring{\Gamma}^{\mu}{}_{\mu\beta}) - X(\mathring{\Gamma}^{\beta}{}_{\nu\rho})K(\delta\Gamma^{\mu}{}_{\mu\beta})\Big]\\
&- i\frac{a}{2}\Big[K(\delta\Gamma^{\beta}{}_{\mu\rho})X(\mathring{\Gamma}^{\mu}{}_{\nu\beta}) - X(\mathring{\Gamma}^{\beta}{}_{\mu\rho})K(\delta\Gamma^{\mu}{}_{\nu\beta})\Big] + \mathring{\Gamma}^{\beta}{}_{\nu\rho}\tilde{\Omega}^{\mu}{}_{\mu\beta} + \tilde{\Omega}^{\beta}{}_{\nu\rho}\mathring{\Gamma}^{\mu}{}_{\mu\beta} - \mathring{\Gamma}^{\beta}{}_{\mu\rho}\tilde{\Omega}^{\mu}{}_{\nu\beta} - \tilde{\Omega}^{\beta}{}_{\mu\rho}\mathring{\Gamma}^{\mu}{}_{\nu\beta},
\end{aligned}$$

where

$$\tilde{\Omega}^{\mu}{}_{\nu\rho} = i\frac{a}{2}\Big[K(-h^{\mu\sigma})X(\mathring{\Gamma}^{\lambda}{}_{\nu\rho}) - X(\mathring{g}^{\mu\sigma})K(\delta\Gamma^{\lambda}{}_{\nu\rho})\Big]\mathring{g}_{\sigma\lambda}.$$

Before writing down the equations describing NC gravitational perturbations, we introduce the parameter λ as an eigenvalue of the Killing field when acting on the perturbation:

$$\begin{aligned}
h_{\mu\nu} \propto e^{-i\omega t}e^{im\varphi} &\implies \mathcal{L}_K h_{\mu\nu} = i\lambda\, h_{\mu\nu},\\
\text{for} \quad K = \alpha\partial_t + \beta\partial_{\varphi}, &\quad \lambda = -\alpha\omega + \beta m.
\end{aligned}$$

Note that in case of the vanishing deformation, the equation (9) reduces to the standard linear gravitational perturbation equation

$$Ric^{\star}_{\mu\nu} \xrightarrow{a\to 0} \delta Ric_{\mu\nu} = \nabla_{\rho}\,\delta\Gamma^{\rho}{}_{\mu\nu} - \nabla_{\nu}\,\delta\Gamma^{\rho}{}_{\mu\rho} = 0.$$

The structure of the perturbation matrix under rotations on the 2-sphere is given by

$$h_{\mu\nu} = \begin{pmatrix} S & S & V & V\\ S & S & V & V\\ V & V & T & T\\ V & V & T & T \end{pmatrix}.$$

Since not all components transform like scalars, it is necessary to introduce the tensor spherical harmonics (generalized spherical harmonics) and expand perturbations $h_{\mu\nu}$ over them:

$$h_{\mu\nu}(t,r,\theta,\varphi) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\sum_{n=1}^{10} c_{n,\ell,m}(t,r)(\mathcal{Y}^{(n)}_{\ell m})_{\mu\nu}.$$

There are 10 different types of them,

$$\mathcal{Y}^{(1)}_{\ell m} = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} Y_{\ell m}, \quad \mathcal{Y}^{(2)}_{\ell m} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} Y_{\ell m}, \quad \mathcal{Y}^{(3)}_{\ell m} = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix} Y_{\ell m},$$

$$\mathcal{Y}^{(4)}_{\ell m} = \frac{ir}{\sqrt{2\ell(\ell+1)}}\begin{pmatrix} 0 & 0 & \partial_\theta Y_{\ell m} & \partial_\varphi Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ \partial_\theta Y_{\ell m} & 0 & 0 & 0 \\ \partial_\varphi Y_{\ell m} & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^{(5)}_{\ell m} = \frac{r}{\sqrt{2\ell(\ell+1)}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta Y_{\ell m} & \partial_\varphi Y_{\ell m} \\ 0 & \partial_\theta Y_{\ell m} & 0 & 0 \\ 0 & \partial_\varphi Y_{\ell m} & 0 & 0 \end{pmatrix},$$

$$\mathcal{Y}^{(6)}_{\ell m} = \frac{r}{\sqrt{2\ell(\ell+1)}}\begin{pmatrix} 0 & 0 & \frac{1}{\sin\theta}\partial_\varphi Y_{\ell m} & -\sin\theta\partial_\theta Y_{\ell m} \\ 0 & 0 & 0 & 0 \\ \frac{1}{\sin\theta}\partial_\varphi Y_{\ell m} & 0 & 0 & 0 \\ -\sin\theta\partial_\theta Y_{\ell m} & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{Y}^{(9)}_{\ell m} = \frac{r^2}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sin^2\theta \end{pmatrix} Y_{\ell m},$$

$$\mathcal{Y}^{(7)}_{\ell m} = \frac{ir}{\sqrt{2\ell(\ell+1)}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sin\theta}\partial_\varphi Y_{\ell m} & -\sin\theta\partial_\theta Y_{\ell m} \\ 0 & \frac{1}{\sin\theta}\partial_\varphi Y_{\ell m} & 0 & 0 \\ 0 & -\sin\theta\partial_\theta Y_{\ell m} & 0 & 0 \end{pmatrix},$$

$$\mathcal{Y}^{(8)}_{\ell m} = \frac{-ir^2}{\sqrt{2\ell(\ell^2-1)(\ell+2)}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{\sin\theta}\Big(\partial_\theta\partial_\varphi - \cot\theta\partial_\varphi\Big)Y_{\ell m} & \sin\theta\Big(\partial_\theta^2 - \cot\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial_\varphi^2\Big)Y_{\ell m} \\ 0 & 0 & \sin\theta\Big(\partial_\theta^2 - \cot\theta\partial_\theta - \frac{1}{\sin^2\theta}\partial_\varphi^2\Big)Y_{\ell m} & 2\sin\theta\Big(\partial_\theta\partial_\varphi - \cot\theta\partial_\varphi\Big)Y_{\ell m} \end{pmatrix},$$

$$\mathcal{Y}^{(10)}_{\ell m} = \frac{r^2}{\sqrt{2\ell(\ell+1)(\ell-1)(\ell+2)}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \partial_\theta^2 Y_{\ell m} & \Big(\partial_\theta\partial_\varphi - \cot\theta\partial_\varphi\Big)Y_{\ell m} \\ 0 & 0 & \Big(\partial_\theta\partial_\varphi - \cot\theta\partial_\varphi\Big)Y_{\ell m} & \Big(\partial_\varphi^2 + \sin\theta\cos\theta\partial_\theta\Big)Y_{\ell m} \end{pmatrix},$$

Tensor spherical harmonics behave differently under the parity transformation

$$\mathbf{P} : \mathbf{r} \longrightarrow -\mathbf{r} \quad [(\theta,\varphi) \longrightarrow (\pi-\theta, \pi+\varphi)].$$

They all fall into one of the two classes: axial perturbations $[(-1)^{l+1}]$ or polar perturbations $[(-1)^l]$. $\mathcal{Y}^{(i)}, i = 6,7,8$ are axial, the rest are polar. In what follows we focus only on axial perturbations. Another important point is that there exists a freedom in choosing the local coordinates, which means that the change in coordinates $x'^\alpha = x^\alpha - \xi^\alpha(x)$ amounts to gauging. Under this gauge transformation, the perturbation tensor changes as

$$h^{old}_{\mu\nu} \longrightarrow h^{new}_{\mu\nu} = h^{old}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu,$$

with the appropriate gauge parameter ξ^α. Here the covariant derivative is withe respect to the background (Schwarzschild) connection. The choice of the gauge

$$\xi^\mu = \Big(0, 0, \Lambda(t,r)\frac{1}{\sin\theta}\partial_\varphi Y_{\ell m}, -\Lambda(t,r)\frac{1}{\sin\theta}\partial_\theta Y_{\ell m}\Big),$$

which is known as the Regge-Wheeler gauge [11], significantly simplifies the form of the perturbation tensor

$$h_{\mu\nu} = \begin{pmatrix} 0 & 0 & -h_{0a}(t,r)\frac{1}{\sin\theta}\partial_\varphi Y_\ell^m & h_{0a}(t,r)\sin\theta\partial_\theta Y_\ell^m \\ 0 & 0 & -h_{1a}(t,r)\frac{1}{\sin\theta}\partial_\varphi Y_\ell^m & h_{1a}(t,r)\sin\theta\partial_\theta Y_\ell^m \\ -h_{0a}(t,r)\frac{1}{\sin\theta}\partial_\varphi Y_\ell^m & -h_{1a}(t,r)\frac{1}{\sin\theta}\partial_\varphi Y_\ell^m & 0 & 0 \\ h_{0a}(t,r)\sin\theta\partial_\theta Y_\ell^m & h_{1a}(t,r)\sin\theta\partial_\theta Y_\ell^m & 0 & 0 \end{pmatrix}.$$

Plugging this ansatz into perturbation equation (9) leads to 10 partial differential equations. While 3 among these 10 equations are trivially satisfied, the radial parts of the remaining 7 are proportional to the radial parts of one of these three components:

$$\begin{aligned}
Ric^\star_{(r\varphi)} &= \Bigg[\frac{1}{1-\frac{2M}{r}}\Big(\partial_t^2 h_{1a} + \frac{2}{r}\partial_t h_{0a} - \partial_r\partial_t h_{0a}\Big) + \frac{h_{1a}}{r^2}\Big(\ell(\ell+1)-2\Big) \\
&+ \lambda a\Big(\frac{r-4M}{r(r-2M)^2}\partial_t h_{0a} - \frac{M}{(r-2M)^2}\partial_t^2 h_{1a} - \frac{\ell(\ell+1)+12}{r^3}h_{1a} + \frac{9M}{r^4}h_{1a} \\
&+ \frac{M}{(r-2M)^2}\partial_r\partial_t h_{0a} - \frac{r-2M}{r^3}\partial_r h_{1a}\Big)\Bigg]\sin\theta\partial_\theta Y_{\ell m}(\theta,\varphi) = 0, \\
Ric^\star_{(\theta\varphi)} &= \Bigg[\partial_t h_{0a} - \frac{2M}{r^2}\Big(1-\frac{2M}{r}\Big)h_{1a} + \Big(\frac{4M}{r} - \frac{4M^2}{r^2} - 1\Big)\partial_r h_{1a} - \lambda a\Big(1-\frac{2M}{r}\Big)\Big(\frac{M}{(r-2M)^2}\partial_t h_{0a} \\
&+ \frac{3}{r^2}\Big(1-\frac{2M}{r}\Big)h_{1a} + \frac{M}{r^2}\partial_r h_{1a}\Big)\Bigg]\Big(\sin\theta\partial_\theta^2 - \cos\theta\partial_\theta - \frac{1}{\sin\theta}\partial_\varphi^2\Big)Y_{\ell m}(\theta,\varphi) = 0, \\
Ric^\star_{(t\varphi)} &= \Bigg[\Big(1-\frac{2M}{r}\Big)\Big(\frac{1}{r}\partial_t h_{1a} + \frac{1}{2}\partial_r\partial_t h_{1a} - \frac{1}{2}\partial_r{}^2 h_{0a}\Big) + \Big(\ell(\ell+1) - \frac{4M}{r}\Big)\frac{h_{0a}}{2r^2} \\
&+ \frac{\lambda a}{4r^4}\Big(-\Big(2\ell(\ell+1)r + 2M\Big)h_{0a} + 2r(2r-3M)\partial_t h_{1a} - 2r(2r-5M)\partial_r h_{0a} \\
&+ 2Mr^2\Big(\partial_r\partial_t h_{1a} - \partial_r^2 h_{0a}\Big)\Big)\Bigg]\sin\theta\partial_\theta Y_{\ell m}(\theta,\varphi) = 0.
\end{aligned}$$

The radial parts of these equations are not all independent. As shown in [10], only two of them are independent. Combining first two radial equations and substituting $h_{1a}(r) = \frac{r^2}{r-2M}Q(r)$ leads to the second-order differential equation for $Q(r)$,

$$\begin{aligned}
&\lambda a\Big(\frac{2\ell(\ell+1)r(r-2M)^2 - 2(3r-4M)(r-M)(r-4M) - 2r^4M\omega^2}{2r^3(r-2M)^3}Q + \frac{(r-4M)(3r-4M)}{r^2(r-2M)^2}\partial_r Q \\
&+ \frac{M}{r(r-2M)}\partial_r^2 Q\Big) + \frac{(r-2M)(6M-\ell(\ell+1)r) + r^4\omega^2}{r^2(r-2M)^2}Q + \frac{2M}{r(r-2M)}\partial_r Q + \partial_r^2 Q = 0. \qquad (13)
\end{aligned}$$

In order to further reduce (13) to a Schrödinger-type equation, we introduce the modified tortoise coordinate r_* and reshape the function $Q(r)$ as

$$\begin{aligned}
&\frac{dr}{dr_*} = 1 - \frac{2M}{r} + \lambda a\frac{M}{r^2} \implies r_* = r + 2M\log\frac{r-2M}{2M} + \frac{\lambda a}{2}\frac{2M}{r-2M}, \\
&Q(r) = \Big(1 + \frac{\lambda a}{2}\Big(\frac{3}{r} - \frac{1}{r-2M} + \frac{1}{2M}\log\frac{r}{r-2M}\Big)\Big)W(r).
\end{aligned}$$

This leads to crystallizing the equation that keeps control over NC gravitational perturbations,

$$\begin{aligned}
&\frac{d^2W}{dr_*^2} + \Big(\omega^2 - V(r)\Big)W = 0, \\
&V(r) = \frac{(r-2M)(\ell(\ell+1)r-6M)}{r^4} + \lambda a\frac{\ell(\ell+1)(6M-2r)r + 2M(5r-16M)}{2r^5}.
\end{aligned}$$

The first part of the potential is the Regge-Wheeler potential governing the axial perturbations of the Schwarzschild black hole and the second part is the correction coming from the spacetime noncommutativity. Plot of the potential for $\ell = 2, 3, 4$ and several values of λa is shown in the Figure 1.

Potentials on Fig.1 exhibit two interesting features, one being a Zeeman-like splitting, similar to that observed in case of a charged scalar field in the vicinity of Reissner-Nordström black hole [12],[13] and the other being the smearing of the horizon [10]. Smearing of the horizon is visible on the Fig.1 as differing zeros of the potential (located at $2M - \lambda a/2$), but also from the tortoise coordinate's dependence on λa. Quasinormal mode frequencies will be calculated in the future work [14].

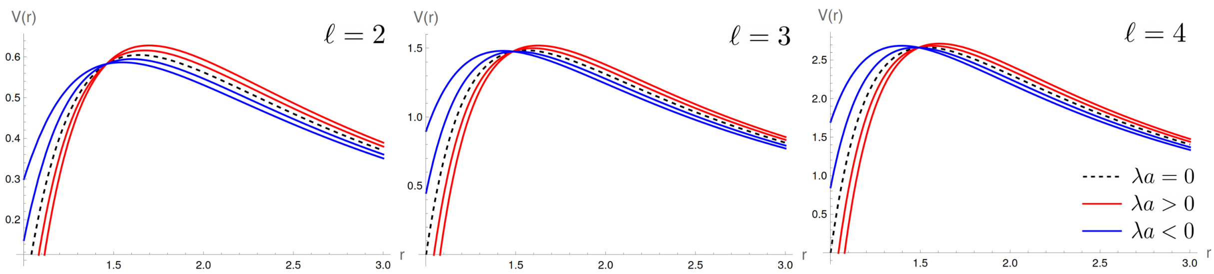

Figure 1. Plot of the potential with respect to the radial coordinate r for $\ell = 2, 3$ and 4. The blue lines correspond to $\lambda a = 0.1,\ 0.2$ and the red lines to $\lambda a = -0.1,\ -0.2$. Schwazschild radius is at $2M = 1$ and the dashed line is potential without the noncommutativity corrections.

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