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E A Bergshoeff[1] ![ORCID], **K van Helden**[1,2] ![ORCID], **J Lahnsteiner**[1,6,*] ![ORCID],
L Romano[1,3] ![ORCID] **and J Rosseel**[4,5] ![ORCID]

[1] Van Swinderen Institute, University of Groningen, Nijenborgh 4, 9747 AG Groningen, The Netherlands
[2] Bernoulli Institute, University of Groningen, Nijenborgh 9, 9747 AG Groningen, The Netherlands
[3] Departamento de Electromagnetismo y Electrónica, Universidad de Murcia, Campus de Espinardo, Murcia 30100, Spain
[4] Faculty of Physics, University of Vienna, Boltzmanngasse 5, 1090 Wien, Austria
[5] Division of Theoretical Physics, Rudjer Bošković Institute, Bijenička 54, 10000 Zagreb, Croatia

E-mail: j.m.lahnsteiner@outlook.com

Abstract

We discuss the generalized Newton–Cartan geometries that can serve as gravitational background fields for particles and strings. In order to enable us to define affine connections that are invariant under all the symmetries of the structure group, we describe torsionful geometries with independent torsion tensors. A characteristic feature of the non-Lorentzian geometries we consider is that some of the torsion tensors are so-called 'intrinsic torsion' tensors. Setting some components of these intrinsic torsion tensors to zero leads to constraints on the geometry. For both particles and strings, we discuss various such constraints that can be imposed consistently with the structure group symmetries. In this way, we reproduce several results in the literature.

[6] Affiliation after 1 November 2022: Nordita, KTH Royal Institute of Technology and Stockholm University, Hannes Alfvéns väg 12, SE-106 91 Stockholm, Sweden.
* Author to whom any correspondence should be addressed.

1. Introduction

One of the cornerstones of Einstein's description of general relativity is its underlying semi-Riemannian geometry giving a geometrical interpretation to the gravitational force. What is less known is that also Newtonian gravity can be given a geometrical interpretation using a degenerate foliated geometry. Its proper formulation was given eight years after Einstein's formulation by Élie Cartan [1, 2]. This generalization of Newtonian gravity is valid in any coordinate system and is called Newton–Cartan (NC) gravity with an underlying geometry that is called NC geometry. This is the correct geometry to describe the coupling of gravity to massive non-relativistic particles and field theories.

Recently, there has been a growing interest in other non-Lorentzian[7], gravity models and corresponding geometries. One key example is an extension of Newtonian gravity including so-called 'twistless torsion' that was shown to occur in Lifshitz holography where it was realized as a background geometry of the boundary conformal field theory [3]. This is a natural extension since the twistless torsion condition is invariant under (anisotropic) local dilatations, as it should for a Lifshitz conformal field theory, whereas the zero torsion condition describing a Newtonian space-time is not. For a useful review on general non-Lorentzian holography, see [4]. Another interesting non-Lorentzian geometry is Carroll geometry, which appears as the natural geometry of null surfaces (see for instance [5]); see also the recent paper [6] and references therein.

Another way to generalize NC geometry is to go beyond particles and consider the gravitational coupling to extended objects such as strings. Whereas any extended object can be coupled to general relativity, in the non-Lorentzian case each extended object requires a different non-Lorentzian geometry with a foliation that is determined by the spatial extension of the object: particles require a foliation with leaves of codimension one, but strings require a foliated geometry where the leaves are submanifolds of codimension two, that describe the dimensions transversal to the string. This geometry is not only relevant to describe the coupling of non-Lorentzian gravity to a classical cosmic string but can also be used to formulate the sigma model describing non-relativistic string theory in a general curved background[8]. Originally, non-relativistic string theory was only formulated for a flat non-Lorentzian space-time [7, 8] or special backgrounds [9]. Only recently a formulation for a generic background has been given [10, 11]. This opens the way to study essential features of non-relativistic string theory as a candidate theory of non-relativistic quantum gravity, independent of the relativistic case. The geometry underlying non-relativistic string theory has natural torsion tensors

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[7] We will generically call any gravity theory with a structure group that differs from the Lorentz group 'non-Lorentzian'. However, for historic reasons we will instead sometimes use the denomination 'non-relativistic' for NC gravity and its generalization to strings. In this context we will also use the wording 'non-relativistic string theory' as a candidate theory of 'non-relativistic quantum gravity'.

[8] By a non-relativistic string we mean a string with a relativistic worldsheet that propagates in a non-relativistic target space.

that are constrained by requiring that the quantum effective action remains non-relativistic [12] and/or requiring supersymmetry [13]. For recent reviews on non-relativistic string theory and non-Lorentzian geometries with more references, see [14, 15].

An important feature of NC gravity and its generalization to strings is that its coupling to particles and/or strings is described by additional fields beyond the usual (co)frame fields. In the case of particles this extra field is a 1-form, called m_μ.[9] This 1-form field is needed due to the fact that, unlike in the relativistic case, mass and energy are two distinct conserved quantities in the non-relativistic case. It has a clear algebraic interpretation as the gauge field associated with the central extension that distinguishes the Galilei from the Bargmann algebra. It couples to a particle via a Wess–Zumino term. In the case of (bosonic) string theory these extra fields are the non-relativistic Kalb–Ramond 2-form $b_{\mu\nu}$ and the dilaton ϕ. Like in the particle case, the 2-form $b_{\mu\nu}$ couples to a non-relativistic string via a Wess–Zumino term. Both m_μ and $b_{\mu\nu}$ have in common that they are part of the geometric fields in the sense that they vary under boost transformations and, in fact, are needed to write down boost-invariant actions describing the coupling to particles and/or strings.

When discussing torsionful geometries it is important to distinguish between the relativistic and non-Lorentzian case. In the relativistic case, the torsion tensor of a metric-compatible affine connection can be arbitrarily specified without imposing any constraints on the metric structure; in particular one may always consider a torsion-free affine connection (the Levi–Civita connection). This is no longer the case in non-Lorentzian geometry. There, part of the torsion consists of so-called intrinsic torsion tensor components that form an obstruction to defining a metric compatible and torsionless connection, without imposing differential constraints on the metric structure [17]. In the physics literature, these intrinsic torsion tensors are sometimes introduced as dependent tensors that are expressed in terms of other (e.g. geometric) fields of the model [18]. It is the purpose of this work to introduce torsion tensors as independent fields in the spirit of [19–22] and demonstrate which of these tensors can be set to zero consistently with the symmetries of the structure group. A benefit of this approach is that in this way we are always able to define a proper affine connection that is invariant under all the symmetries of the structure group. Only afterwards we will try to express some of the torsion tensors as dependent tensors in terms of the other (geometric) fields of the model. Both for particles and strings we will give explicit examples of such dependent intrinsic torsion tensors.

This work is organized as follows. In section 2 we review, using the Cartan frame formulation, standard NC geometry with torsion. In particular, we show how to introduce independent torsion tensors and how these tensors can be used to define spin connections and an invariant affine connection. We discuss various constraints that can be imposed on these torsion tensors without breaking the symmetries of the structure group. We end this section by giving several examples of dependent torsion tensors that have appeared in the literature. In section 3 we extend all calculations of section 2 from particles to strings. Apart from getting slightly more

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[9] The addition of just this field applies to the standard NC geometry framework and suffices to write down the equations of motion of NC gravity. An action principle for these equations of motion requires a departure from standard NC geometry, in which extra geometric fields are introduced; see, e.g. [16]. A similar remark holds for the generalization of NC geometry to strings, where the addition of $b_{\mu\nu}$ suffices to write down equations of motion of the common sector of effective field theories of non-relativistic string theories, but is not sufficient to formulate actions. In this paper, we will only concern ourselves with the standard NC geometry framework and its stringy generalization.

involved the structure of this section is very similar to that of section 2. Finally, in section 4 we give our conclusions and present an outlook for future extensions.

2. Torsionful NC geometry in a Cartan formulation

NC geometry refers to the geometry of D-dimensional manifolds, called NC manifolds, that are equipped with a degenerate metric structure that reduces the local structure group to the homogeneous Galilei group in D dimensions. The latter is given by the semi-direct product $\mathrm{SO}(D-1) \ltimes \mathbb{R}^{D-1}$, where $\mathrm{SO}(D-1)$ is physically interpreted as the group of local spatial rotations and $\mathbb{R}^{D-1}$ as that of local Galilean boosts. A convenient way to introduce the metric and metric compatible affine connection structures of NC geometry is in terms of frame, 'extended coframe' and structure group connection fields, all to be defined below, in analogy to the Cartan formulation of Lorentzian geometry.

Such a Cartan formulation of NC geometry, for both torsionless and torsionful affine connections, has been developed in a number of [3, 18, 19, 23–27]. Here, we will review it to facilitate generalization to the case of string NC (SNC) geometry. We will first introduce the frame and extended coframe fields and ensuing metric structure in section 2.1. In section 2.2, we will discuss how torsionful, metric compatible affine connections, that are completely determined in terms of the frame, extended coframe and suitable torsion tensor fields, can be defined on a NC manifold. We will at first treat the torsion in a general manner, by introducing it as an extra, independent ingredient. Special cases and examples of torsionful NC geometry that appear in the context of e.g. Lifshitz holography [3, 18, 25–27], will be discussed in section 2.3.

2.1. Frame fields, extended coframe fields and metric structure

A local frame[10] on a D-dimensional NC manifold $\mathcal{M}$, whose local coordinates are denoted by x^μ, is given by a collection $\{\tau^\mu, e_a{}^\mu\}$ of D vector fields, referred to as the frame fields. Here, the index a takes values $1, \ldots, D-1$. In what follows, this 'spatial index' a will be freely raised and lowered using Kronecker deltas δ_{ab} and δ^{ab}. The structure group $\mathrm{SO}(D-1) \ltimes \mathbb{R}^{D-1}$ acts on the frame fields τ^μ and $e_a{}^\mu$ according to the following infinitesimal local transformation rules:

$$\delta\tau^\mu = -\lambda^a e_a{}^\mu, \qquad\qquad \delta e_a{}^\mu = -\lambda_a{}^b e_b{}^\mu. \qquad\qquad (1)$$

Here, $\lambda^{ab} = -\lambda^{ba}$ denote the parameters of $\mathrm{SO}(D-1)$ spatial rotations, whereas λ^a refer to the parameters of $\mathbb{R}^{D-1}$ Galilean boosts. Given a local frame $\{\tau^\mu, e_a{}^\mu\}$, a local coframe on $\mathcal{M}$ is a collection $\{\tau_\mu, e_\mu{}^a\}$ of D one-forms, called coframe fields, that are dual to the frame fields in the sense that the following relations hold:

$$\tau^\mu \tau_\mu = 1, \qquad\qquad \tau^\mu e_\mu{}^a = 0, \qquad\qquad e_a{}^\mu \tau_\mu = 0,$$
$$e_\mu{}^a e_b{}^\mu = \delta_b^a, \qquad\qquad \tau_\mu \tau^\nu + e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu. \qquad\qquad (2)$$

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[10] The geometries of this paper can be aptly described in the language of principal fiber bundles. A frame is then a section of the frame bundle of $\mathcal{M}$, a principal fiber bundle, whose $\mathrm{GL}(D,\mathbb{R})$ structure group is reduced to $\mathrm{SO}(D-1) \ltimes \mathbb{R}^{D-1}$ due to the presence of 2 invariant metric tensors of rank 1 and rank $D-1$. We will not use this language much in this paper, since we are only concerned with a local description of the geometries under consideration. This terminology is however required when dealing with global issues, when it becomes imperative to know how frames in different intersecting local patches are related to each other via structure group transformations on overlaps of patches.

In addition to the coframe fields τ_μ and $e_\mu{}^a$, NC geometry also includes an extra one-form m_μ as part of its geometric data[11]. The one-forms τ_μ, $e_\mu{}^a$ and m_μ[12] transform in a reducible, indecomposable representation of the structure group, according to the following infinitesimal local transformation rules:

$$\delta\tau_\mu = 0, \qquad\qquad \delta e_\mu{}^a = -\lambda^{ab} e_{\mu b} + \lambda^a \tau_\mu, \qquad\qquad \delta m_\mu = \lambda^a e_{\mu a}. \tag{3}$$

We will call this representation space, i.e. the collection of one-forms $\{\tau_\mu, e_\mu{}^a, m_\mu\}$, the 'extended coframe'. In the following, we will refer to the extended coframe fields m_μ, τ_μ and $e_\mu{}^a$ as the 'mass form', 'time-like Vielbein'[13] and 'spatial Vielbein' respectively. Since (2) expresses that the square matrices $\begin{pmatrix} \tau_\mu & e_\mu{}^a \end{pmatrix}$ and $\begin{pmatrix} \tau^\mu \\ e_a{}^\mu \end{pmatrix}$ are each other's inverse, we will then (with slight abuse of terminology) refer to τ^μ as the 'inverse time-like Vielbein' and to $e_a{}^\mu$ as the 'inverse spatial Vielbein'[14].

Using the frame and coframe fields, one can construct two degenerate symmetric (covariant and contravariant) two-tensors that are invariant under local rotations and boosts:

$$\tau_{\mu\nu} \equiv \tau_\mu \tau_\nu, \qquad\qquad h^{\mu\nu} \equiv e_a{}^\mu e_b{}^\nu \delta^{ab}. \tag{4}$$

These define the degenerate metric structure on the NC manifold $\mathcal{M}$. The covariant metric $\tau_{\mu\nu}$ has rank 1 and is referred to as the 'time-like metric', whereas the contravariant metric $h^{\mu\nu}$ has rank $D-1$ and is often called the 'spatial cometric'. Note that τ_μ is in the kernel of the spatial cometric, i.e. $h^{\mu\nu}\tau_\nu = 0$, as a consequence of (2).

The local causal structure of a NC manifold can be viewed as a limit of that of a Lorentzian manifold, in which the speed of light in a local inertial reference frame is sent to infinity. In local Minkowskian coordinates $\{x^0, x^a\}$, this can be achieved by rescaling x^0 with a (dimensionless) parameter ω and taking the limit $\omega \to \infty$. In this limit the local lightcone $\omega^2 (x^0)^2 = x^a x_a$ flattens out and degenerates into the $x^0 = 0$ hyperplane. With respect to such a local inertial reference frame, vectors can be classified as time-like future-/past-directed, when they have a strictly positive/negative x^0-component and as spatial when their x^0-component is zero. This can be phrased covariantly, using the time-like Vielbein τ_μ, by saying that a vector X^μ[15] is time-like future-directed when $\tau_\mu X^\mu > 0$, time-like past-directed when $\tau_\mu X^\mu < 0$ and spatial when $\tau_\mu X^\mu = 0$.

As their names suggest, the symmetric two-tensors $\tau_{\mu\nu}$ and $h^{\mu\nu}$ allow one to compute time intervals and spatial distances in NC geometry in a way that is analogous to how the metric is used to calculate lengths of curves in Riemannian geometry [28, 29]. Time intervals in the NC manifold $\mathcal{M}$ are defined along any curve $\gamma : t \in [0,1] \to x^\mu(t) \in \mathcal{M}$, whose tangent

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[11] In approaches to define NC geometry as a gauging of the Bargmann algebra, i.e. the centrally extended Galilei algebra, m_μ corresponds to the gauge field associated with the central extension [23, 24]. For this reason, it is often called the 'central charge gauge field' in the literature.

[12] As will be seen in (19), m_μ also acts as an abelian gauge field, i.e. as a connection on a principal U(1)-bundle. In principle, the full structure group then contains an additional U(1) factor. In this paper, we will mostly be interested in affine connections that are connections on the frame bundle of the manifold. We will then use the term 'structure group' as shorthand for the structure group of the frame bundle of $\mathcal{M}$ (with sections $\{\tau^\mu, e_a{}^\mu\}$), as this is the part of the full structure group that is relevant for us. Our usage of the term 'structure group' thus does not include the extra U(1) factor.

[13] The time-like Vielbein τ_μ is also often called the 'clock form' in the literature.

[14] In calling frame fields inverse Vielbeine and coframe fields Vielbeine, we conform to the physics literature. In the mathematics literature, one usually reserves the term Vielbeine for a section of the frame bundle, i.e. for the frame fields.

[15] Here, it is understood that X^μ is such that $\tau_\mu X^\mu$ is invariant under the structure group.

vectors $\dot{x}^\mu(t) \equiv \mathrm{d}x^\mu(t)/\mathrm{d}t$ are time-like future-directed for all $t \in (0,1)$.[16] Such a curve models the motion of a non-relativistic physical particle or observer between two points with local coordinates $x^\mu(0)$ and $x^\mu(1)$. Given the time-like metric $\tau_{\mu\nu}$, the time interval measured by the particle/observer to traverse the curve γ is then computed by the following integral

$$\int_0^1 \mathrm{d}t\, \sqrt{\dot{x}^\mu \dot{x}^\nu \tau_{\mu\nu}} = \int_0^1 \mathrm{d}t\, \dot{x}^\mu \tau_\mu = \int_\gamma \mathrm{d}x^\mu \tau_\mu. \tag{5}$$

To define spatial distances in an analogous way, one needs an inverse of the spatial cometric $h^{\mu\nu}$. The latter is not invertible when viewed as a map from one-forms to vectors, since it has a non-trivial kernel spanned by τ_μ. It does however give rise to a well-defined map between the space of equivalence classes $[\alpha_\mu] = \{\alpha_\mu + f\tau_\mu | f \in C^\infty(\mathcal{M})\}$ of one-forms that differ by a multiple of τ_μ, i.e. $f\tau_\mu$, and the space of spatial vectors, where $h^{\mu\nu}$ maps $[\alpha_\mu]$ to $h^{\mu\nu}[\alpha_\nu] \equiv h^{\mu\nu}\alpha_\nu$. When viewed like this, $h^{\mu\nu}$ is invertible and its inverse is given by

$$h_{\mu\nu} = e_\mu{}^a e_\nu{}^b \delta_{ab}. \tag{6}$$

Here, $h_{\mu\nu}$ is interpreted as a map that assigns the equivalence class $[h_{\mu\nu}X^\nu]$ to each spatial vector X^μ. Using (2), one sees that

$$h^{\mu\rho}h_{\rho\nu} = \delta_\nu^\mu - \tau^\mu \tau_\nu, \tag{7}$$

from which it follows that $h^{\mu\rho}h_{\rho\nu}$ and $h_{\nu\rho}h^{\rho\mu}$ act as the identity δ_ν^μ on spatial vectors X^ν, resp. equivalence classes $[\alpha_\mu]$. The two-tensors $h^{\mu\nu}$ and $h_{\mu\nu}$ are thus indeed each other's inverse, when viewed as maps between the space of equivalence classes of one-forms that are equal up to a multiple of τ_μ and the space of spatial vectors. It is also worth mentioning that $h_{\mu\nu}$ (unlike $h^{\mu\nu}$) is not invariant under local boosts: $\delta h_{\mu\nu} = 2\lambda^a \tau_{(\mu} e_{\nu)a}$.[17] It thus does not give a covariant metric on the full space of vectors. Note however that $X^\mu Y^\nu h_{\mu\nu}$ is boost invariant when X^μ and Y^ν are spatial[18], so that $h_{\mu\nu}$ constitutes a covariant metric (with Euclidean signature) on the space of spatial vectors. With these remarks in mind, spatial distances can be defined along any curve $\tilde{\gamma} : s \in [0,1] \to x^\mu(s)$, whose tangent vectors $x'^\mu(s) \equiv \mathrm{d}x^\mu(s)/\mathrm{d}s$ are spatial for all $s \in (0,1)$. The length of such a curve is defined in terms of the metric $h_{\mu\nu}$ on spatial vectors as:

$$\int_0^1 \mathrm{d}s\, \sqrt{x'^\mu x'^\nu h_{\mu\nu}}. \tag{8}$$

The notion that one can only measure lengths between simultaneous events is formalized by the fact that spatial distances can only be defined along curves whose tangent vectors are spatial.

The above discussion shows that the coframe fields τ_μ and $e_\mu{}^a$ can be viewed as a non-relativistic analogue of the Vielbein of Lorentzian geometry in the Cartan formulation. The extended coframe field m_μ has no analogue in Lorentzian geometry. It is not needed to specify the metric structure of NC geometry. It however plays an important role in defining metric compatible affine connections that are fully expressed in terms of frame, extended coframe and torsion tensor fields, as we will review in the next section.

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[16] The restriction to curves whose tangent vectors are everywhere time-like future-directed is motivated by viewing them as worldlines of particles that move at finite speed. The time interval (5) is however also well-defined for curves that have sections where their tangent vectors are spatial. Such sections then do not contribute to the time interval integral (5).

[17] So, while $h_{\mu\nu}$ is not a well-defined boost invariant object, an equivalence class $[h_{\mu\nu}] = \{h_{\mu\nu} + 2f_{(\mu}\tau_{\nu)}\}$ with f_μ an arbitrary one-form, is boost invariant.

[18] A slightly stronger statement is that the equivalence class $[h_{\mu\nu}X^\nu]$ is boost invariant, when X^μ is spatial.

2.2. Torsionful, metric compatible connection

In the Cartan formulation of NC geometry, metric compatible affine connections are defined by introducing a structure group connection one-form Ω_μ that takes values in the Lie algebra of the homogeneous Galilei group in D dimensions:

$$\Omega_\mu = \frac{1}{2}\omega_\mu{}^{ab}J_{ab} + \omega_\mu{}^a G_a, \tag{9}$$

where $J_{ab} = -J_{ba}$ and G_a are generators of the Lie algebra of $SO(D-1)$ (spatial rotations) and $\mathbb{R}^{D-1}$ (Galilean boosts). We will refer to $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ and $\omega_\mu{}^a$ as the spin connections for spatial rotations and Galilean boosts respectively. Their infinitesimal local structure group transformations are given by

$$\delta\omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} - 2\lambda^{[a|c|}\omega_{\mu c}{}^{b]}, \qquad\qquad \delta\omega_\mu{}^a = \partial_\mu\lambda^a + \omega_\mu{}^{ab}\lambda_b - \lambda^{ab}\omega_{\mu b}. \tag{10}$$

To introduce an affine connection $\Gamma^\rho_{\mu\nu}$ that is compatible with the NC metric structure, one then considers the following 'Vielbein postulates':

$$\partial_\mu\tau_\nu - \Gamma^\rho_{\mu\nu}\tau_\rho = 0, \qquad\qquad \partial_\mu e_\nu{}^a + \omega_\mu{}^{ab}e_{\nu b} - \omega_\mu{}^a\tau_\nu - \Gamma^\rho_{\mu\nu}e_\rho{}^a = 0. \tag{11}$$

These postulates immediately lead to the metric compatibility conditions

$$\nabla_\mu\tau_{\nu\rho} \equiv \partial_\mu\tau_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\tau_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\tau_{\nu\sigma} = 0, \qquad \nabla_\mu h^{\nu\rho} \equiv \partial_\mu h^{\nu\rho} + \Gamma^\nu_{\mu\sigma}h^{\sigma\rho} + \Gamma^\rho_{\mu\sigma}h^{\nu\sigma} = 0. \tag{12}$$

The form of the Vielbein postulates (11) is furthermore motivated by the requirement that the affine connection $\Gamma^\rho_{\mu\nu}$ should be invariant under the local structure group for the above given transformation rules of (extended co)frame fields and spin connections. Indeed, using (11) to express $\Gamma^\rho_{\mu\nu}$ in terms of the spin connections $\omega_\mu{}^{ab}$, $\omega_\mu{}^a$ and the time-like and spatial Vielbeine τ_μ, $e_\mu{}^a$, one obtains:

$$\Gamma^\rho_{\mu\nu} = \tau^\rho\partial_\mu\tau_\nu + e_a{}^\rho\left(\partial_\mu e_\nu{}^a + \omega_\mu{}^{ab}e_{\nu b} - \omega_\mu{}^a\tau_\nu\right). \tag{13}$$

One readily checks that this expression for $\Gamma^\rho_{\mu\nu}$ is invariant under the local rotation and boost transformations (1), (3) and (10) (and that it has the appropriate transformation law under general coordinate transformations).

So far, we have not imposed any restrictions on the torsion $2\Gamma^\rho_{[\mu\nu]}$ of the affine connection $\Gamma^\rho_{\mu\nu}$. In this section, we will keep the torsion completely arbitrary and view it as an extra independent geometric ingredient. It is then convenient to decompose it in two *a priori* independent tensors $T_{\mu\nu}$ and $T_{\mu\nu}{}^a$ as follows:

$$2\Gamma^\rho_{[\mu\nu]} = \tau^\rho T_{\mu\nu} + e_a{}^\rho T_{\mu\nu}{}^a \qquad \Leftrightarrow \qquad T_{\mu\nu} \equiv 2\Gamma^\rho_{[\mu\nu]}\tau_\rho \quad \text{and} \quad T_{\mu\nu}{}^a = 2\Gamma^\rho_{[\mu\nu]}e_\rho{}^a. \tag{14}$$

We will refer to $T_{\mu\nu}$ and $T_{\mu\nu}{}^a$ as the time-like and spatial torsion respectively. They transform under infinitesimal local rotations and boosts as:

$$\delta T_{\mu\nu} = 0, \qquad\qquad \delta T_{\mu\nu}{}^a = -\lambda^a{}_b T_{\mu\nu}{}^b + \lambda^a T_{\mu\nu}. \tag{15}$$

By antisymmetrizing the Vielbein postulates (11), one obtains the following equations that are covariant with respect to local spatial rotations and Galilean boosts:

$$2\partial_{[\mu}\tau_{\nu]} = T_{\mu\nu}, \tag{16a}$$

$$2\partial_{[\mu}e_{\nu]}{}^a + 2\omega_{[\mu}{}^{ab}e_{\nu]b} - 2\omega_{[\mu}{}^a\tau_{\nu]} = T_{\mu\nu}{}^a. \tag{16b}$$

The second of these should be viewed as an identity that exhibits that some components of $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ are not independent fields. Viewing it as a system of algebraic equations for $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$, one can solve it to express some of the spin connection components in

terms of (co)frame fields and the torsion tensor $T_{\mu\nu}{}^a$. Note however that this procedure does not allow to express all components of $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ in this way. Indeed, the fields $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$ contain $D(D-1)(D-2)/2 + D(D-1) = D^2(D-1)/2$ components, whereas (16*b*) only contains $D(D-1)^2/2$ equations. Consequently, $D(D-1)/2$ spin connection components cannot be solved in terms of (inverse) Vielbeine and $T_{\mu\nu}{}^a$ from (16*b*) alone.

Whereas in Lorentzian geometry all spin connection components are completely determined in terms of the (inverse) Vielbeine and torsion by the Lorentzian analogue of equations (16), the preceding paragraph indicates that this is not the case in NC geometry. It is however possible to express all NC spin connection components as functions of torsion tensors, (inverse) Vielbeine and the mass form m_μ. To do this, we introduce an extra independent two-form $T_{\mu\nu}^{(m)}$ that we call the 'mass torsion tensor' and equate it to the properly covariantized (with respect to Galilean boosts) exterior derivative of the mass form field m_μ:

$$2\partial_{[\mu} m_{\nu]} - 2\omega_{[\mu}{}^a e_{\nu]a} = T_{\mu\nu}^{(m)}. \tag{17}$$

Spatial rotations leave the left-hand side of this equation inert, while Galilean boosts transform it to the left-hand side of (16*b*). The right-hand sides of (17) and (16*b*) should then have the same transformation behavior to ensure that the set of equations (16) and (17) is invariant under the structure group. This tells us that the mass torsion tensor is invariant under spatial rotations and transforms under Galilean boosts as follows:

$$\delta T_{\mu\nu}^{(m)} = \lambda_a T_{\mu\nu}{}^a. \tag{18}$$

Note that the left-hand side of (17) is also invariant under an abelian gauge transformation (with parameter σ) of m_μ:

$$\delta m_\mu = \partial_\mu \sigma. \tag{19}$$

This transformation is often called the 'central charge transformation'[19]. As we will allude to at the end of this section, it plays an important role in physical applications of NC geometry, where it is realized as a symmetry. Although it can be treated as part of the structure group, we will use the latter term to only include transformations that act on the frame fields and we thus do not view (19) as a structure group transformation (see also footnote 6).

The two equations (16*b*) and (17) constitute a system of $D(D-1)^2/2 + D(D-1)/2 = D^2(D-1)/2$ algebraic equations for as many components of $\omega_\mu{}^{ab}$ and $\omega_\mu{}^a$. Solving these equations then leads to the following expressions for the spin connections in terms of the frame and extended coframe fields and torsion tensors $T_{\mu\nu}{}^a$, $T_{\mu\nu}^{(m)}$:

$$\omega_\mu{}^a = \tau_\mu \tau^\nu e^{a\rho} \partial_{[\nu} m_{\rho]} + e^{a\nu} \partial_{[\mu} m_{\nu]} + e_{\mu b} e^{a\nu} \tau^\rho \partial_{[\nu} e_{\rho]}{}^b + \tau^\nu \partial_{[\mu} e_{\nu]}{}^a$$
$$- \tau_\mu \tau^\nu e^{a\rho} T_{\nu\rho}^{(m)} + e_{\mu b} \tau^\nu e^{(a|\rho|} T_{\nu\rho}{}^{b)} - \frac{1}{2} e_{\mu b} e^{b\nu} e^{a\rho} T_{\nu\rho}^{(m)},$$

$$\omega_\mu{}^{ab} = 2e^{[a|\nu|} \partial_{[\mu} e_{\nu]}{}^{b]} - e_{\mu c} e^{a\nu} e^{b\rho} \partial_{[\nu} e_{\rho]}{}^c + \tau_\mu e^{a\nu} e^{b\rho} \partial_{[\nu} m_{\rho]}$$
$$- \frac{1}{2} \tau_\mu e^{a\nu} e^{b\rho} T_{\nu\rho}^{(m)} - e^{[a|\nu|} T_{\mu\nu}{}^{b]} + \frac{1}{2} e_{\mu c} e^{a\nu} e^{b\rho} T_{\nu\rho}{}^c. \tag{20}$$

Plugging these expressions in (13), one can express $\Gamma^\rho_{\mu\nu}$ in terms of the NC metric structure, m_μ and torsion tensors:

$$\Gamma^{\rho}_{\mu\nu} = \tau^{\rho}\partial_{\mu}\tau_{\nu} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}h_{\sigma\nu} + \partial_{\nu}h_{\mu\sigma} - \partial_{\sigma}h_{\mu\nu}\right) + h^{\rho\sigma}\tau_{\mu}\partial_{[\sigma}m_{\nu]} + h^{\rho\sigma}\tau_{\nu}\partial_{[\sigma}m_{\mu]}$$

$$+ h^{\rho\sigma}\tau_{(\mu}T^{(m)}_{\nu)\sigma} - h^{\rho\sigma}e_{(\mu|a|}T_{\nu)\sigma}{}^{a} + \frac{1}{2}e_{a}{}^{\rho}T_{\mu\nu}{}^{a}. \tag{21}$$

One can explicitly check that the structure group transformations of the right-hand sides of (20), induced by (3), (1), (15) and (18), coincide with (10) and that the right-hand side of (21) is similarly invariant under the structure group. For Galilean boosts, this invariance is not manifest; one can however rewrite (21) in a form that exhibits manifest boost invariance as follows:

$$\Gamma^{\rho}_{\mu\nu} = \bar{\tau}^{\rho}\partial_{(\mu}\tau_{\nu)} + \frac{1}{2}h^{\rho\sigma}\left(\partial_{\mu}\bar{h}_{\sigma\nu} + \partial_{\nu}\bar{h}_{\mu\sigma} - \partial_{\sigma}\bar{h}_{\mu\nu}\right) + \frac{1}{2}\tau^{\rho}T_{\mu\nu}$$

$$+ h^{\rho\sigma}m_{(\mu}T_{\nu)\sigma} + h^{\rho\sigma}\tau_{(\mu}T^{(m)}_{\nu)\sigma} - h^{\rho\sigma}e_{(\mu|a|}T_{\nu)\sigma}{}^{a} + \frac{1}{2}e_{a}{}^{\rho}T_{\mu\nu}{}^{a}, \tag{22}$$

where

$$\bar{\tau}^{\mu} = \tau^{\mu} + h^{\mu\nu}m_{\nu}, \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - 2m_{(\mu}\tau_{\nu)}, \tag{23}$$

are boost invariant expressions.

Note that the equations (16) and (17) are invariant under the central charge transformation (19), if the torsion tensors $T_{\mu\nu}$, $T_{\mu\nu}{}^{a}$ and $T^{(m)}_{\mu\nu}$ are. As a consequence the spin and affine connection expressions (20), (21) are then also invariant under (19). Note however that there is no good *a priori* reason to expect that the connections are invariant under the central charge, since we do not consider the latter to be part of the structure group. The torsion tensors $T_{\mu\nu}$, $T_{\mu\nu}{}^{a}$ and/or $T^{(m)}_{\mu\nu}$ are thus allowed to have non-trivial central charge transformations, resulting in expressions for the spin and affine connections that are no longer obtained from central charge invariant equations and thus are no longer central charge invariant[20].

Taking the anti-symmetric part in $[\mu\nu]$ of (21), one explicitly sees that $T_{\mu\nu}$ and $T_{\mu\nu}{}^{a}$ constitute the components of the torsion $2\Gamma^{\rho}_{[\mu\nu]}$ of the affine connection, as in (14), with $T_{\mu\nu}$ given by $2\partial_{[\mu}\tau_{\nu]}$ as in (16*a*). The mass torsion tensor $T^{(m)}_{\mu\nu}$ has no Lorentzian analogue and does not appear in the affine connection torsion. We nevertheless still view it as a torsion tensor in a Cartan formulation sense, namely as a tensor that transforms covariantly under the structure group and that measures the non-vanishing of the exterior covariant derivative of an extended coframe field. As such, $T^{(m)}_{\mu\nu}$ forms a reducible, indecomposable structure group representation together with the affine connection torsion components $T_{\mu\nu}$ and $T_{\mu\nu}{}^{a}$ and one can thus view it as a component of a 'torsion tensor multiplet'. Its presence in NC geometry is crucial in ensuring that the transformation rule of the spin connection expressions (20), induced by (3), (1), (15) and (18), is consistent with (10).

Apart from featuring an extra torsion tensor $T^{(m)}_{\mu\nu}$, the structure of the torsion in NC geometry differs in another important way from its Lorentzian geometry counterpart. In Lorentzian geometry, all affine connection torsion components appear in the expressions for the spin connection in terms of the (inverse) Vielbeine and torsion tensor components (i.e. in the Lorentzian analogue of (20)). Setting torsion tensor components equal to zero then only amounts to choosing a particular connection (e.g. the Levi–Civita connection if all torsion components are

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[20] An analogous phenomenon occurs in supergravity, where the (supercovariant) Levi–Civita spin connection has a non-trivial transformation rule under supersymmetry, because it is obtained by solving a set of equations that is not invariant under supersymmetry. This also motivates the boost transformation rule (18) of $T^{(m)}_{\mu\nu}$, since with a different transformation rule the set of equations (16) and (17) would not be boost invariant and as a result the boost transformation rules of the right-hand sides of (20) would no longer coincide with those given in (10).

set to zero). By contrast, in NC geometry the explicit spin connection expressions (20) contain only the spatial components $T_{\mu\nu}{}^a$ and not the time-like components $T_{\mu\nu}$ of the affine connection torsion. Whereas setting the $T_{\mu\nu}{}^a$ components equal to zero only amounts to picking a particular connection, setting components of $T_{\mu\nu}$ equal to zero also leads to extra geometric constraints on the exterior derivative of τ_μ, as is seen from (16*a*). For this reason, $T_{\mu\nu}$ is also called the 'intrinsic torsion' of NC geometry [17].

Starting from the affine connection (21) and (22), one can construct the Riemann and Ricci tensors in the usual way. The metric structure (4) and affine connection (21) and (22) thus fully specify torsionful NC geometry in terms of the frame and extended coframe fields and torsion tensors $T_{\mu\nu}, T_{\mu\nu}{}^a, T_{\mu\nu}^{(m)}$.

Before discussing various special cases and examples that have appeared in the recent literature, let us remark that NC geometry is the natural framework to describe the mechanics of non-relativistic point particles. A point particle traces out a worldline in space-time and, as remarked in section 2.1, time intervals along such a worldline and spatial distances to it can be measured with the metrics $\tau_{\mu\nu}$ and $h^{\mu\nu}$. The mass form m_μ also has a natural particle interpretation. Unlike relativistic theories, non-relativistic theories exhibit mass conservation. The inclusion of m_μ among the extended coframe fields and the extra central charge transformation (19) then give an extra ingredient to implement the conservation of mass of a non-relativistic particle. Given a particle with mass m that moves along a worldline $\gamma : \mathbb{R} \ni t \to x^\mu(t) \in \mathcal{M}$, this can be done by introducing the following coupling to m_μ

$$m \int_\gamma \mathrm{d}x^\mu m_\mu = m \int_\mathbb{R} \mathrm{d}t\, \dot{x}^\mu m_\mu. \tag{24}$$

This coupling of m_μ to the particle's mass current is analogous to how an electrically charged relativistic particle couples to the electromagnetic gauge potential. Gauge invariance of the coupling (24) under the central charge transformation (19) then implies conservation of the particle's mass current, in analogy to how charge conservation is realized in electromagnetism. In NC gravity, the diffeomorphism covariant reformulation of Newtonian gravity [1, 2], one can choose adapted coordinates, in which only the time-like component of m_μ is non-vanishing. This time-like component then corresponds to the Newton potential and the coupling (24) reduces to the coupling of a particle to the Newton potential. We thus see that the presence of m_μ in NC geometry is natural both from the mathematical and physical point of view. Mathematically, m_μ is needed because metric compatibility no longer completely fixes the connection in terms of the metric and torsion, in case the metric structure is a degenerate non-relativistic one. Physically, it plays the role of a gauge field that couples to the Noether current that implements mass conservation and gives a diffeomorphism covariant generalization of the Newton potential of Newtonian gravity.

2.3. Special cases and examples

In the previous section, we saw that the specification of a generic torsionful affine connection that is compatible with the NC metric structure involves the introduction of time-like and spatial torsion tensors $T_{\mu\nu}$ and $T_{\mu\nu}{}^a$, as well as an extra mass torsion tensor $T_{\mu\nu}^{(m)}$. While we have thus far kept these tensors completely arbitrary, it is possible to consider special cases, in which some of their components are equal to zero. To do this, we will use the following notation to denote torsion tensor components:

$$T_{0a} = \tau^\mu e_a{}^\nu T_{\mu\nu}, \qquad\qquad T_{ab} = e_a{}^\mu e_b{}^\nu T_{\mu\nu}, \tag{25}$$

and similarly for components of $T_{\mu\nu}{}^a$ and $T_{\mu\nu}^{(m)}$.

(a) This diagram gives a schematic representation of the boost transformations of the different torsion components. Arrows display relations between the different torsion components under boost transformations. If the boost transformation of a torsion component contains another torsion component, an arrow points from the former component towards the latter, e.g. the upmost arrow represents the boost transformation $\delta T_{\mu\nu}^{(m)} = \lambda_a T_{\mu\nu}{}^a$.

(b) The colours in every rectangle of this diagram represent the parts of the corresponding torsion tensors that are unconstrained. In particular, if a colour is missing, it implies that this part of the torsion tensor is zero. In Figure 1a, we have indicated what colour corresponds to what part of the torsion tensors. Furthermore, arrows point from more special cases towards more general cases.

Figure 1. Classification of constraints on the torsion tensors (b) that are consistent with the local structure group transformations (a).

Since the components of $T_{\mu\nu}$, $T_{\mu\nu}{}^a$ and $T_{\mu\nu}^{(m)}$ transform non-trivially into each other under Galilean boosts, in a way that is summarized in figure 1(a), one cannot set their components equal to zero independently. Let us illustrate this by outlining several scenarios in which components of the torsion tensors are set to zero consistently. These possible truncations are displayed in figure 1(b). The cases displayed in figure 1(b) can be retrieved from figure 1(a) as follows: every possible scenario (a rectangle in figure 1(b)) corresponds to setting the torsion components whose color (indicated in figure 1(a)) is absent, to zero. For example, case 4 of figure 1(b) corresponds to setting T_{ab} equal to zero. For consistency, it is then required that torsion components that are set to zero point towards torsion components that are also put equal to zero in figure 1(a). E.g., since in case 2 of figure 1(b) $T_{\mu\nu}{}^a$ is set to zero, the components T_{0a} and T_{ab} also have to vanish.

A useful way to divide the different cases of figure 1(b) is according to the following list.

- Cases 1 and 2 in figure 1(b) correspond to the cases in which the affine connection has zero torsion ($T_{\mu\nu} = 0 = T_{\mu\nu}{}^a$). Case 1 is known as 'torsionless NC geometry' in the literature[21].
- Cases 1, 2 and 3 in figure 1(b) have zero time-like/intrinsic torsion: $T_{\mu\nu} = 0$. Of these, case 3 has unconstrained (spatial) torsion of $\Gamma^\rho_{\mu\nu}$ (i.e. has unconstrained components of $T_{\mu\nu}{}^a$). The vanishing of $T_{\mu\nu}$ means that the time-like Vielbein τ_μ is closed:

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[21] The existence of case 2 is related to a difference between NC and Lorentzian geometry, namely the fact that in the former $\Gamma^\rho_{\mu\nu}$ is not uniquely specified by metric compatibility (12) and $\Gamma^\rho_{[\mu\nu]} = 0$, but only up to an ambiguity parametrized by a two-form $K_{\mu\nu}$. In case 1, this two-form is taken to be exact and given by the exterior derivative of m_μ. The existence of case 2 indicates that $K_{\mu\nu}$ can also be a generic two-form that does not even have to be closed.
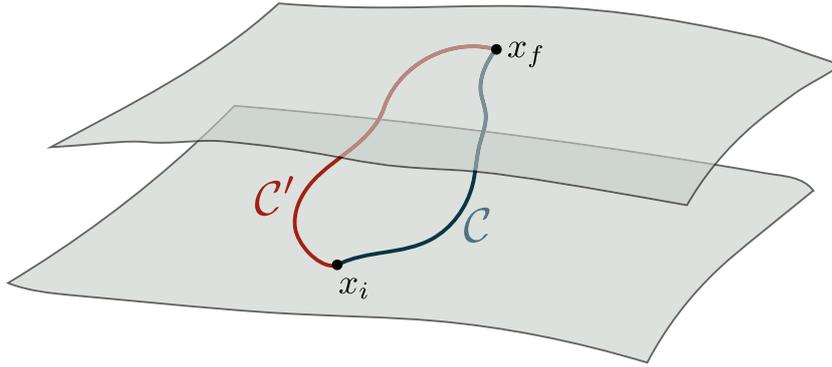
Figure 2. This figure depicts two events x_i, x_f in space-time that are connected by two distinct future-directed time-like curves $\mathcal{C}, \mathcal{C}'$. With $\tau = \tau_\mu \mathrm{d}x^\mu$, Stokes' theorem implies that $\int_{\mathcal{C}} \tau - \int_{\mathcal{C}'} \tau = \int_{\partial\Sigma} \tau = \int_{\Sigma} \mathrm{d}\tau$, where Σ is a surface enclosed by $\mathcal{C}$ and $\mathcal{C}'$. In case the time-like torsion $T_{\mu\nu}$ and thus $\mathrm{d}\tau$ are zero, one then finds that the time interval (5) is path-independent: $\int_{\mathcal{C}} \tau = \int_{\mathcal{C}'} \tau$.

$$\partial_{[\mu}\tau_{\nu]} = 0. \tag{26}$$

As illustrated in figure 2, Stokes' theorem then implies that the time interval (5) is independent of the curve that connects two particular events. Different physical observers that move along different curves between the same initial and final events thus measure the same time interval for their respective journeys. In other words, NC manifolds with vanishing intrinsic torsion admit a notion of absolute time. Locally, (26) implies that τ_μ is exact, i.e. $\tau_\mu = \partial_\mu t$, and the function t can be identified as an absolute time function.

- Case 4 has $T_{ab} = 0$ but T_{0a} unconstrained. These conditions are equivalent to stating that τ_μ is hypersurface orthogonal:

$$\tau_{[\mu}\partial_\nu\tau_{\rho]} = 0, \tag{27}$$

but not necessarily closed. This case is known as 'twistless torsionful NC geometry' [3, 25]. As can be seen from figure 1(b), consistency with Galilean boosts requires that both $T_{\mu\nu}{}^a$ and $T_{\mu\nu}^{(m)}$ cannot be set to zero in general. Unlike the previous cases, it is no longer possible to define an absolute time in twistless torsionful NC geometry, since the time interval (5) between two particular events now depends on the path that connects them. There however still is a notion of absolute simultaneity. This follows from Frobenius' theorem, according to which a NC manifold on which the hypersurface orthogonality condition (27) holds, can be foliated in $(D-1)$-dimensional spatial hypersurfaces, i.e. hypersurfaces of simultaneous events. Locally, τ_μ is only exact after multiplication with an integrating factor $\mathrm{e}^{-\phi}$, i.e. one can write $\tau_\mu = \mathrm{e}^\phi \partial_\mu t$. The $(D-1)$-dimensional leaves of the foliation are given by the $t = $ constant hypersurfaces. Twistless torsionful NC geometry then still has a notion of Newtonian causality in the sense that, given a spatial hypersurface $t = c_0$, one can distinguish its future, given by the collection of hypersurfaces $t = c_1$ with $c_1 > c_0$, from its past, given by the collection of hypersurfaces $t = c_2$ with $c_2 < c_0$.

- Case 5 leaves both T_{0a} and T_{ab} unconstrained. Consistency with boosts then requires that all torsion tensors are unconstrained. In this case, there is neither a notion of absolute time nor of absolute simultaneity and Newtonian causality.

Note that we have not given a complete classification of all possible scenarios in which the torsion components can be set to zero consistently. One could for instance split the torsion tensor $T_{\mu\nu}^{(m)}$ up into a part that (partially) projects on the longitudinal Vielbein $T_{0a}^{(m)}$ and a part that does not, $T_{ab}^{(m)}$, which would lead to a finer classification. We have also not considered cases, in which combinations of components of different torsion tensors are put equal to zero.

Torsionless NC geometry is the geometry underlying NC gravity [1, 2]. Torsionful NC geometry has appeared in recent applications. Let us mention two examples. The first example deals with supergravity versions of NC gravity, that have thus far only been constructed in three space-time dimensions [30–41]. These theories are based on torsionful NC geometry, where the torsion tensors $T_{\mu\nu}$, $T_{\mu\nu}{}^a$ and $T_{\mu\nu}^{(m)}$ are built out of fermionic gravitino fields. For example, the three-dimensional NC supergravity theory with four supercharges of [30] contains two gravitino fields $\psi_{\mu+}$ and $\psi_{\mu-}$ that are both Majorana vector-spinors. Their transformation rules under local spatial rotations and Galilean boosts are given by

$$\delta\psi_{\mu+} = \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu+}, \qquad\qquad \delta\psi_{\mu-} = \frac{1}{4}\lambda^{ab}\gamma_{ab}\psi_{\mu-} - \frac{1}{2}\lambda^a\gamma_{a0}\psi_{\mu+}. \qquad (28)$$

Here $\gamma_{ab} = \gamma_{[a}\gamma_{b]}$, $\gamma_{a0} = \gamma_a\gamma_0$ and $\{\gamma_0,\gamma_a|a=1,2\}$ constitute a set of three-dimensional gamma matrices (for a Clifford algebra with signature $(-++)$). The NC geometry used in [30] then belongs to case 5 of figure 1(b), with torsion tensors $T_{\mu\nu}$, $T_{\mu\nu}{}^a$ and $T_{\mu\nu}^{(m)}$ constructed out of $\psi_{\mu\pm}$ as follows:

$$T_{\mu\nu} = \frac{1}{2}\bar\psi_{[\mu+}\gamma^0\psi_{\nu]+}, \qquad\qquad T_{\mu\nu}{}^a = \bar\psi_{[\mu+}\gamma^a\psi_{\nu]-}, \qquad\qquad T_{\mu\nu}^{(m)} = \bar\psi_{[\mu-}\gamma^0\psi_{\nu]-}. \qquad (29)$$

Using the transformation rules (28) of $\psi_{\mu\pm}$, one finds that these torsion tensors satisfy the transformation rules (15) and (18) that ensure invariance of the affine connection $\Gamma^\rho_{\mu\nu}$ (21) under local rotations and boosts. Note however that $\Gamma^\rho_{\mu\nu}$ is not invariant under supersymmetry.

Our second example concerns NC geometry as it occurs in attempts to extend the AdS/CFT correspondence to describe non-relativistic conformal field theories (CFTs) [42–44]. In these proposals, non-relativistic CFTs live on the boundary of so-called Schrödinger or Lifshitz space-times that are vacuum solutions of matter coupled relativistic bulk gravity theories, and whose isometries form a non-relativistic conformal symmetry group. CFT quantities are then holographically encoded in bulk gravitational ones. While Schrödinger or Lifshitz space-times are relativistic in the bulk, their boundaries have a non-relativistic causal structure and are thus naturally described by NC geometry. It has in particular been shown that in holography around Lifshitz space-times, the relevant boundary geometry is that of torsionful NC geometry [3, 25] in which the intrinsic torsion is non-vanishing (as in cases 4 and 5 in figure 1(b)). The torsion tensors $T_{\mu\nu}$, $T_{\mu\nu}{}^a$ and $T_{\mu\nu}^{(m)}$ that occur are expressed in terms of the extended coframe fields τ_μ, $e_\mu{}^a$ and m_μ and possible choices are given by [18]:

$$T_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}, \qquad T_{\mu\nu}{}^a = 2e^{a\rho}m_\rho\partial_{[\mu}\tau_{\nu]}, \qquad T_{\mu\nu}^{(m)} = e^{a\rho}e_a{}^\sigma m_\rho m_\sigma\partial_{[\mu}\tau_{\nu]}, \qquad (30)$$

and

$$T_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}, \qquad T_{\mu\nu}{}^a = 2e^{a\rho}m_\rho\partial_{[\mu}\tau_{\nu]}, \qquad T_{\mu\nu}^{(m)} = -2\tau^\rho m_\rho\partial_{[\mu}\tau_{\nu]}. \qquad (31)$$

Using the rules (3) and (1), one sees that these tensors indeed transform under the structure group as in (15) and (18). Both torsion tensor choices of (30) and (31) are, however, not invariant under the central charge transformation (19). Consequently, the affine connections constructed using them are invariant under local rotations and boosts, but not under the central charge transformation. As a result, central charge invariance is usually only realized in a non-manifest manner in holographic descriptions of non-relativistic CFTs.

3. Torsionful SNC geometry in a Cartan formulation

In the previous section, we described NC geometry, which forms the natural differential geometric arena for non-relativistic particle mechanics. The framework of NC geometry can be generalized to manifolds, in which one can describe the movement of extended objects, such as strings and branes, in a degenerate limit that is akin to a non-relativistic one. Here we will focus on so-called non-relativistic strings [7, 8, 45] (see also [14] for a recent review). These are obtained from relativistic strings by sending the speed of light in the directions transversal to the strings to infinity, while leaving the relativistic character of the worldsheet untouched. Upon quantization, one then finds that this limit only retains vibrational modes with non-relativistic dispersion relations in the string spectrum. The target space-times that non-relativistic strings move in are referred to as SNC manifolds and their geometry is likewise called SNC geometry.

Similar to NC geometry, D-dimensional SNC manifolds have a degenerate metric structure that reduces the local structure group to

$$(SO(1,1) \times SO(D-2)) \ltimes \mathbb{R}^{2(D-2)}. \tag{32}$$

The Minkowskian worldsheet of a non-relativistic string at rest divides up the tangent space directions of a SNC manifold in two 'longitudinal' directions and $D-2$ 'transversal' ones. The $SO(1,1)$ and $SO(D-2)$ factors of the structure group then correspond to Lorentz transformations of the two longitudinal directions and rotations of the transversal directions, respectively. The $\mathbb{R}^{2(D-2)}$ factor represents boost transformations that can transform transversal directions into longitudinal ones, but not vice versa. We will refer to these as 'String Galilean boosts'. In the Lie algebra of (32) the generators of $\mathbb{R}^{2(D-2)}$ then transform in the $(\mathbf{2}, \mathbf{D-2})$ representation under the adjoint action of the Lie algebras of $SO(1,1)$ and $SO(D-2)$.

For the torsionless case, a Cartan formulation of SNC geometry was discussed from the viewpoint of space-time symmetry algebra gaugings and a particular limit of the Cartan formulation of Lorentzian geometry in [11, 46].[22] Recently, the relevance of including non-trivial torsion in SNC geometry has been pointed out in [13, 49–51]. In this section, we will present the metric and affine connection structure of torsionful SNC geometry, in the same spirit as our presentation of torsionful NC geometry of the previous section. We will first discuss the frame and extended coframe fields and resulting metric structure in section 3.1. Next, in section 3.2, we will discuss metric compatible affine connections by introducing suitable structure group spin connections and Vielbein postulates. As in the previous section, we will at first leave the torsion arbitrary and independent. We will see that, unlike what happens for NC geometry, the affine connection of SNC geometry can (for our choice of extended coframe fields) no longer be fully expressed in terms of frame, extended coframe and independent torsion tensor fields. As in the NC case, it is possible to consider various special cases that are obtained by truncating torsion tensor components consistently. This will be treated in section 3.3, with particular emphasis on cases that have appeared in the recent literature.

3.1. Frame fields, extended coframe fields and metric structure

In analogy to NC geometry, a local frame on a SNC manifold (with coordinates x^μ) is a collection of D frame fields $\{\tau_A{}^\mu, e_a{}^\mu\}$ that are vector fields. Here, the 'longitudinal flat' index A takes on the values 0 and 1 and can be freely raised and lowered with a two-dimensional

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[22] For earlier work on SNC geometry, see [9, 47, 48].

Minkowski metric $\eta_{AB} = \text{diag}(-1, 1)$. The 'transversal flat' index a [23] on the other hand takes on the values $2, \ldots, D-1$ and raising and lowering is done using a $(D-2)$-dimensional Euclidean metric δ_{ab}. The infinitesimal local transformation rules, according to which the structure group $(\text{SO}(1,1) \times \text{SO}(D-2)) \ltimes \mathbb{R}^{2(D-2)}$ acts on the frame fields $\tau_A{}^\mu$ and $e_a{}^\mu$, are given by:

$$\delta\tau_A{}^\mu = \lambda\epsilon_A{}^B\tau_B{}^\mu + \lambda_A{}^a e_a{}^\mu, \qquad\qquad \delta e_a{}^\mu = \lambda_a{}^b e_b{}^\mu. \tag{33}$$

Here, λ corresponds to the parameter of longitudinal $\text{SO}(1,1)$ Lorentz transformations, $\lambda^{ab} = -\lambda^{ba}$ to that of transversal $\text{SO}(D-2)$ rotations, while the λ^{Aa} are the $2(D-2)$ string Galilean boost parameters. The coframe that is dual to the frame $\{\tau_A{}^\mu, e_a{}^\mu\}$ then consists of D coframe fields $\{\tau_\mu{}^A, e_\mu{}^a\}$ that are one-forms. The duality between frame and coframe fields is in the following sense:

$$\tau_A{}^\mu\tau_\mu{}^B = \delta_A^B, \qquad\qquad \tau_A{}^\mu e_\mu{}^a = 0, \qquad\qquad e_a{}^\mu\tau_\mu{}^A = 0,$$
$$e_\mu{}^a e_b{}^\mu = \delta_b^a, \qquad\qquad \tau_\mu{}^A\tau_A{}^\nu + e_\mu{}^a e_a{}^\nu = \delta_\mu^\nu. \tag{34}$$

As in NC geometry, SNC geometry features an extra field that is now a two-form $b_{\mu\nu}$. The fields $\tau_\mu{}^A$, $e_\mu{}^a$ and $b_{\mu\nu}$ [24] transform in a reducible, indecomposable manner under the structure group, according to the following infinitesimal local transformation rules:

$$\delta\tau_\mu{}^A = \lambda\epsilon^A{}_B\tau_\mu{}^B, \qquad\qquad \delta e_\mu{}^a = \lambda^a{}_b e_\mu{}^b - \lambda_A{}^a\tau_\mu{}^A,$$
$$\delta b_{\mu\nu} = -2\epsilon_{AB}\lambda^A{}_a\tau_{[\mu}{}^B e_{\nu]}{}^a. \tag{35}$$

Note that the string Galilean boosts act in a non-linear fashion on the two-form $b_{\mu\nu}$. [25] Similar to NC geometry, we will collect the fields of the structure group representation (35) in an extended coframe $\{\tau_\mu{}^A, e_\mu{}^a, b_{\mu\nu}\}$. In what follows, the coframe fields $\tau_\mu{}^A$, $e_\mu{}^a$ will be called the 'longitudinal Vielbein' and 'transversal Vielbein' respectively. Since (34) expresses that the matrices $\begin{pmatrix} \tau_\mu{}^A & e_\mu{}^a \end{pmatrix}$ and $\begin{pmatrix} \tau_A{}^\mu \\ e_a{}^\mu \end{pmatrix}$ are each other's inverse, we will (with slight abuse of terminology) use the terms 'inverse longitudinal Vielbein' and 'inverse transversal Vielbein' for the frame fields $\tau_A{}^\mu$ and $e_a{}^\mu$ respectively.

The longitudinal and inverse transversal Vielbeine can be 'squared' to obtain two degenerate symmetric (covariant and contravariant) two-tensors that are invariant under local $\text{SO}(1,1)$, $\text{SO}(D-2)$ and string Galilean boost transformations:

$$\tau_{\mu\nu} \equiv \tau_\mu{}^A\tau_\nu{}^B\eta_{AB}, \qquad\qquad h^{\mu\nu} \equiv e_a{}^\mu e_b{}^\nu\delta^{ab}. \tag{36}$$

These two tensors constitute a degenerate metric structure on a SNC manifold. The covariant metric $\tau_{\mu\nu}$ is referred to as the 'longitudinal metric'. From (34) one sees that its kernel is spanned by the $D-2$ vectors $e_a{}^\mu$ and it thus has rank 2. The contravariant metric $h^{\mu\nu}$ is called the 'transversal metric' and has rank $D-2$, since its kernel is spanned by the two one-forms $\tau_\mu{}^A$.

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[23] Many articles on SNC-type geometries use primed capital letters $A', B', C', \cdots$ for the transversal directions instead of the lowercase $a, b, c, \cdots$ used here.

[24] As will be seen in (56), $b_{\mu\nu}$ also transforms under an extra abelian one-form gauge transformation and consequently corresponds to a connection on a U(1) gerbe. Analogously to how we dealt with the central charge (19) in NC geometry, we will take the term 'structure group' to refer only to $(\text{SO}(1,1) \times \text{SO}(D-2)) \ltimes \mathbb{R}^{2(D-2)}$, not including the extra transformation (56).

[25] In an interesting recent proposal [51], the 2-form field $b_{\mu\nu}$ is represented as a dependent expression in terms of 1-form gauge fields. The non-linear string Galilean boost transformation of $b_{\mu\nu}$ is then induced by a linear one of the vector fields.
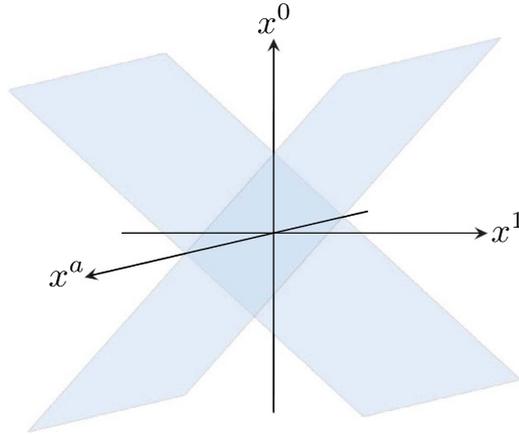
Figure 3. Local causal structure of an SNC manifold. The Lorentzian lightcone degenerates into a 'lightwedge', defined by the two hyperplanes $x^0 \pm x^1 = 0$.

Similar to NC geometry, the local causal structure of a SNC manifold can be obtained as a degenerate limit of that of a Lorentzian manifold. In this case, this limit consists of sending the velocity of light in the transverse directions in a local inertial reference frame to infinity. In local Minkowski coordinates $x^{\hat{A}} = \{x^A, x^a\}$ (with $\hat{A} = 0, \cdots, D-1$), this is achieved by rescaling the longitudinal coordinates x^A with a (dimensionless) parameter ω and taking the limit $\omega \to \infty$. The local lightcone $\omega^2 x^A x_A = -x^a x_a$ then flattens out along the transversal directions and degenerates into the two hyperplanes $x^0 = x^1$ and $x^0 = -x^1$; see figure 3. A vector can be distinguished according to whether it lies in the $(D-2)$-dimensional intersection of these two hyperplanes or not. In the former case, we will call the vector 'transversal', while in the latter case we will call it a 'worldsheet vector'. Worldsheet vectors can be further classified as time-like, space-like or null vectors, according to whether their projections onto the (x^0, x^1)-plane is time-like, space-like or null with respect to the two-dimensional Minkowski metric η_{AB}. Put covariantly, a vector X^μ is transversal whenever $\tau_\mu{}^A X^\mu = 0$ for $A = 0, 1$ and a worldsheet vector whenever $\tau_\mu{}^A X^\mu$ are not both zero. Distinguishing worldsheet vectors into time-like, space-like or null ones is done using the longitudinal metric $\tau_{\mu\nu}$. In particular, a worldsheet vector X^μ is time-like whenever $\tau_{\mu\nu} X^\mu X^\nu = \tau_\mu{}^A X^\mu \tau_{\nu A} X^\nu < 0$, space-like whenever $\tau_{\mu\nu} X^\mu X^\nu > 0$ and null whenever $\tau_{\mu\nu} X^\mu X^\nu = 0$.

Given a SNC manifold $\mathcal{M}$, the longitudinal metric $\tau_{\mu\nu}$ can be used to calculate a proper time

$$\int_0^1 d\tau \sqrt{-\tau_{\mu\nu} \dot{x}^\mu \dot{x}^\nu}, \tag{37}$$

along a curve segment $\gamma : \tau \in [0,1] \to x^\mu(\tau) \in \mathcal{M}$, for which $\dot{x}^\mu(\tau) \equiv dx^\mu(\tau)/d\tau$ is a time-like (or null) worldsheet vector for all $\tau \in (0,1)$. Similarly, if $\tilde{\gamma} : \sigma \in [0,1] \to x^\mu(\sigma) \in \mathcal{M}$ is a curve segment, for which $x'^\mu(\sigma) \equiv dx^\mu(\sigma)/d\sigma$ is a space-like (or null) worldsheet vector for all $s \in (0,1)$, one can define its proper length as

$$\int_0^1 d\sigma \sqrt{\tau_{\mu\nu} x'^\mu x'^\nu}. \tag{38}$$

Furthermore, $\tau_{\mu\nu}$ can also be used to give a notion of proper area of worldsheets, whose tangent vectors are worldsheet vectors. In particular, the proper area of a worldsheet segment Σ_φ

that is specified via an embedding map $\varphi : (\tau, \sigma) \in [0,1] \times [0,2\pi] \to x^\mu(\tau,\sigma) \in \mathcal{M}$, such that $\partial_\tau x^\mu(\tau,\sigma)$ and $\partial_\sigma x^\mu(\tau,\sigma)$ are time-like, resp. space-like worldsheet vectors, can be defined as[26]:

$$\int_0^1 d\tau \int_0^{2\pi} d\sigma \sqrt{-\det(\tau_{\alpha\beta})}, \qquad \text{with} \quad \tau_{\alpha\beta} = \tau_{\mu\nu}\partial_\alpha x^\mu \partial_\beta x^\nu, \qquad (39)$$

where the indices α, β can stand for τ or σ. The proper area is thus defined as the integral of the volume form of the induced metric $\tau_{\alpha\beta}$ that is the pull-back of the longitudinal metric $\tau_{\mu\nu}$ along the embedding φ. Assuming that $\epsilon_{AB}\tau_\mu{}^A \partial_\tau x^\mu \tau_\nu{}^B \partial_\sigma x^\nu > 0$ for all possible values of τ and σ, where ϵ_{AB} is the two-dimensional Levi–Civita epsilon symbol, normalized as $\epsilon_{01} = 1$, the integral (39) can alternatively be written as the integral of the pullback of a two-form $\ell_{\mu\nu}$ over Σ_φ:

$$2 \int_0^1 d\tau \int_0^{2\pi} d\sigma\, \ell_{\mu\nu} \partial_\tau x^\mu \partial_\sigma x^\nu, \qquad \text{with} \quad \ell_{\mu\nu} = \frac{1}{2}\epsilon_{AB}\tau_\mu{}^A \tau_\nu{}^B. \qquad (40)$$

This is analogous to how the time interval (5) in NC geometry can be given by integrating the pullback of the one-form τ_μ along a worldline. Note that this notion of proper worldsheet area does not exist in NC geometry, since there the only metric that can act on time-like vectors is of rank 1.

The rank $D - 2$ cometric $h^{\mu\nu}$ can be used to measure transversal distances to worldsheets. To do this, one proceeds similarly as in NC geometry and one views $h^{\mu\nu}$ as a well-defined and invertible map between the space of equivalence classes $[\alpha_\mu] = \{\alpha_\mu + f_A \tau_\mu{}^A | f_A \in C^\infty(\mathcal{M})\}$ of one-forms that differ by linear combinations of $\tau_\mu{}^A$ and the space of transversal vectors, where $h^{\mu\nu}$ maps $[\alpha_\nu]$ to $h^{\mu\nu}[\alpha_\nu] \equiv h^{\mu\nu}\alpha_\nu$. In analogy to the NC geometry case, one can argue that the inverse of this map is given by:

$$h_{\mu\nu} = e_\mu{}^a e_\nu{}^b \delta_{ab}, \qquad (41)$$

where one regards $h_{\mu\nu}$ as a map that assigns the equivalence class $[h_{\mu\nu}X^\nu]$ to each transversal vector X^μ. Note that $h_{\mu\nu}$ cannot be viewed as a covariant metric on the full space of vectors, since it is not invariant under local boosts: $\delta h_{\mu\nu} = -2\lambda_A{}^a \tau_{(\mu}{}^A e_{\nu)a}$.[27] It does however form a covariant metric (with Euclidean signature) on the space of transversal vectors, since $X^\mu Y^\nu h_{\mu\nu}$ is boost invariant when X^μ and Y^μ are transversal.[28] One can thus use it to define a transversal distance notion along any curve $s \in [0,1] \to x^\mu(s)$, whose tangent vectors $x'^\mu(s) \equiv dx^\mu(s)/ds$ are transversal for all $s \in (0,1)$ as follows:

$$\int_0^1 ds \sqrt{x'^\mu x'^\nu h_{\mu\nu}}. \qquad (42)$$

The frame fields $\tau_\mu{}^A$ and $e_\mu{}^a$ are a natural generalization of the time-like and spatial Vielbeine τ_μ and $e_\mu{}^a$ of NC geometry. The frame field $b_{\mu\nu}$ plays a very similar role in SNC geometry as the mass form m_μ does in NC geometry. It is not needed to specify the metric

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[26] Similar to how the time interval (5) in NC can be defined along arbitrary curves, the definitions (39) and (40) can sensibly be extended to arbitrary embedded worldsheets. Worldsheet segments, such that one or both of $\partial_\tau x^\mu(\tau,\sigma)$ or $\partial_\sigma x^\mu(\tau,\sigma)$ is transversal, then do not contribute to the integrals (39) and (40). The proper time (37) and length (38) can likewise be defined for curves with transversal tangent vectors.

[27] While $h_{\mu\nu}$ is not a boost invariant object, an equivalence class $[h_{\mu\nu}] = \{h_{\mu\nu} + 2f_{A(\mu}\tau_{\nu)}{}^A)\}$, with $f_{A\mu}$ arbitrary one-forms, is.

[28] As in the NC case, a slightly stronger statement is that the equivalence class $[h_{\mu\nu}X^\nu]$ is boost invariant, when X^μ is transversal.

structure on the NC geometry, but it becomes part of the definition of a metric compatible affine connection in terms of (extended co)frame fields and torsion tensors, as we will review in the next section.

3.2. Torsionful, metric compatible connection

To define a metric compatible affine connection in SNC geometry, we proceed analogously as in NC geometry and first introduce a structure group connection Ω_μ that takes values in the Lie algebra of (32)

$$\Omega_\mu = \omega_\mu J + \frac{1}{2}\omega_\mu{}^{ab}J_{ab} + \omega_\mu{}^{Aa}G_{Aa}, \tag{43}$$

where J, $J_{ab} = -J_{ba}$ and G_{Aa} are generators of the Lie algebras of $\mathrm{SO}(1,1)$, $\mathrm{SO}(D-2)$ and $\mathbb{R}^{2(D-2)}$. We will refer to ω_μ, $\omega_\mu{}^{ab} = -\omega_\mu{}^{ba}$ and $\omega_\mu{}^{Aa}$ as spin connections for longitudinal Lorentz transformations, transversal rotations and string Galilean boosts, respectively. They transform as follows under infinitesimal $\mathrm{SO}(1,1)$, $\mathrm{SO}(D-2)$ and string Galilean boosts:

$$\delta\omega_\mu = \partial_\mu\lambda, \qquad \delta\omega_\mu{}^{ab} = \partial_\mu\lambda^{ab} + 2\lambda^{[a|c|}\omega_{\mu c}{}^{b]},$$
$$\delta\omega_\mu{}^{Aa} = \partial_\mu\lambda^{Aa} + \lambda\epsilon^A{}_B\omega_\mu{}^{Ba} + \lambda^a{}_b\omega_\mu{}^{Ab} - \epsilon^A{}_B\lambda^{Ba}\omega_\mu + \lambda^{Ab}\omega_{\mu b}{}^a. \tag{44}$$

An affine connection $\Gamma^\rho_{\mu\nu}$ can be introduced by imposing the following 'Vielbein postulates':

$$\partial_\mu\tau_\nu{}^A - \epsilon^A{}_B\omega_\mu\tau_\nu{}^B - \Gamma^\rho_{\mu\nu}\tau_\rho{}^A = 0,$$
$$\partial_\mu e_\nu{}^a - \omega_\mu{}^{ab}e_{\nu b} + \omega_\mu{}^{Aa}\tau_{\nu A} - \Gamma^\rho_{\mu\nu}e_\rho{}^a = 0. \tag{45}$$

These postulates imply that $\Gamma^\rho_{\mu\nu}$ is compatible with the SNC metric structure (36):

$$\nabla_\mu\tau_{\nu\rho} \equiv \partial_\mu\tau_{\nu\rho} - \Gamma^\sigma_{\mu\nu}\tau_{\sigma\rho} - \Gamma^\sigma_{\mu\rho}\tau_{\nu\sigma} = 0, \qquad \nabla_\mu h^{\nu\rho} \equiv \partial_\mu h^{\nu\rho} + \Gamma^\nu_{\mu\sigma}h^{\sigma\rho} + \Gamma^\rho_{\mu\sigma}h^{\nu\sigma} = 0. \tag{46}$$

Solving $\Gamma^\rho_{\mu\nu}$ in terms of the Vielbeine $\tau_\mu{}^A$, $e_\mu{}^a$, their inverses and the spin connections ω_μ, $\omega_\mu{}^{ab}$, $\omega_\mu{}^{Aa}$, one obtains:

$$\Gamma^\rho_{\mu\nu} = \tau_A{}^\rho\partial_\mu\tau_\nu{}^A + e_a{}^\rho\partial_\mu e_\nu{}^a - \epsilon^A{}_B\omega_\mu\tau_\nu{}^B\tau_A{}^\rho - \omega_\mu{}^a{}_b e_\nu{}^b e_a{}^\rho + \omega_\mu{}^{Aa}\tau_{\nu A}e_a{}^\rho. \tag{47}$$

One can then check that (given the rules (33), (35) and (44)) this expression for the affine connection is invariant under the structure group and has the appropriate transformation rule under general coordinate transformations, providing further motivation for the form of the Vielbein postulates (45).

As in the previous section, we will view the torsion $2\Gamma^\rho_{[\mu\nu]}$ of the affine connection as an independent and *a priori* arbitrary geometric ingredient. We will split it into 'longitudinal torsion' components $T_{\mu\nu}{}^A$ along $\tau_A{}^\rho$ and 'transversal torsion' components $T_{\mu\nu}{}^a$ along $e_a{}^\rho$:

$$2\Gamma^\rho_{[\mu\nu]} = \tau_A{}^\rho T_{\mu\nu}{}^A + e_a{}^\rho T_{\mu\nu}{}^a \qquad \text{i.e.} \qquad T_{\mu\nu}{}^A \equiv 2\Gamma^\rho_{[\mu\nu]}\tau_\rho{}^A \quad \text{and} \quad T_{\mu\nu}{}^a = 2\Gamma^\rho_{[\mu\nu]}e_\rho{}^a. \tag{48}$$

Under infinitesimal local $\mathrm{SO}(1,1)$, $\mathrm{SO}(D-2)$ and string Galilean boosts, $T_{\mu\nu}{}^A$ and $T_{\mu\nu}{}^a$ then transform as follows

$$\delta T_{\mu\nu}{}^A = \lambda\epsilon^A{}_B T_{\mu\nu}{}^B, \qquad \delta T_{\mu\nu}{}^a = \lambda^a{}_b T_{\mu\nu}^b - \lambda_A{}^a T_{\mu\nu}{}^A. \tag{49}$$

By antisymmetrizing the Vielbein postulates (45), one obtains the following equations that are covariant with respect to local structure group transformations:

$$2\partial_{[\mu}\tau_{\nu]}{}^A - 2\epsilon^A{}_B\omega_{[\mu}\tau_{\nu]}{}^B = T_{\mu\nu}{}^A, \tag{50a}$$

$$2\partial_{[\mu}e_{\nu]}{}^a - 2\omega_{[\mu}{}^{ab}e_{\nu]b} + 2\omega_{[\mu}{}^{Aa}\tau_{\nu]A} = T_{\mu\nu}{}^a. \tag{50b}$$

The first of these represents a set of $D(D-1)$ equations. Of these, D equations contain the D components of ω_μ algebraically, while the remaining $D(D-2)$ ones do not contain components of ω_μ. One can thus use D of the equations (50a) to express ω_μ in terms of (co)frame fields and components of the longitudinal torsion $T_{\mu\nu}{}^A$. Doing this leads to the following expression for ω_μ:

$$\omega_\mu = \epsilon^{AB}\tau_A{}^\nu\partial_{[\mu}\tau_{\nu]B} - \frac{1}{2}\epsilon^{BC}\tau_\mu{}^A\tau_B{}^\nu\tau_C{}^\rho\partial_{[\nu}\tau_{\rho]A} + \frac{1}{2}\epsilon^{BC}\tau_{\mu A}\tau_B{}^\nu\tau_C{}^\rho T_{\nu\rho}{}^A$$
$$+ \frac{1}{2}\epsilon^A{}_B e_\mu{}^a\tau_A{}^\nu e_a{}^\rho T_{\nu\rho}{}^B. \tag{51}$$

The remaining $D(D-2)$ equations, contained in (50a), are given by:

$$2\tau_{(A|}{}^\mu e_a{}^\nu\partial_{[\mu}\tau_{\nu]|B)} = \tau_{(A|}{}^\mu e_a{}^\nu T_{\mu\nu|B)}, \qquad 2e_a{}^\mu e_b{}^\nu\partial_{[\mu}\tau_{\nu]}{}^A = e_a{}^\mu e_b{}^\nu T_{\mu\nu}{}^A. \tag{52}$$

Equation (50b) can be used to express some components of $\omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$ in terms of (co)frame fields and the transversal torsion tensor $T_{\mu\nu}{}^a$. This can however not be done for all components of these spin connections, since (50b) constitutes a set of $D(D-1)(D-2)/2$ equations, while there are $D(D+1)(D-2)/2$ components in $\omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$. One can use (50b) to express the following $D(D-1)(D-2)/2$ spin connection components

$$\tau_{[A|}{}^\mu\omega_{\mu|B]}{}^a, \qquad \tau_A{}^\mu\omega_\mu{}^{ab}, \qquad e_{(a|}{}^\mu\omega_{\mu A|b)}, \qquad e_c{}^\mu\omega_\mu{}^{ab} \tag{53}$$

in terms of (co)frame fields, $T_{\mu\nu}{}^a$ and (some of) the remaining $D(D-2)$ components of $\omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$. Since $b_{\mu\nu}$ transforms to coframe fields under string Galilean boosts, it can be used to solve some of these remaining spin connection components in terms of frame fields, extended coframe fields and torsion, similar to how m_μ is used to define the connection of NC geometry. Paralleling the discussion around (17), we thus introduce an extra independent tensor $T^{(b)}_{\mu\nu\rho}$ and set it equal to the exterior covariant derivative of the two-form $b_{\mu\nu}$ (where covariantization is with respect to string Galilean boosts):

$$3\partial_{[\mu}b_{\nu\rho]} + 6\epsilon_{AB}\omega_{[\mu}{}^{Ab}\tau_\nu{}^B e_{\rho]b} = T^{(b)}_{\mu\nu\rho}. \tag{54}$$

The left-hand side of this equation transforms to a particular combination of the left-hand sides of (50a) and (50b), while it is inert under the other structure group transformations. Requiring that the equations (50a), (50b) and (54) form an invariant set under (35), (44), (49) and (55) then leads one to conclude that $T^{(b)}_{\mu\nu\rho}$ is invariant under $SO(1,1)$ and $SO(D-2)$ transformations and transforms as follows under string Galilean boosts:

$$\delta T^{(b)}_{\mu\nu\rho} = -3\epsilon_{AB}\lambda^A{}_a T_{[\mu\nu}{}^B e_{\rho]}{}^a + 3\epsilon_{AB}\lambda^A{}_a T_{[\mu\nu}{}^a\tau_{\rho]}{}^B. \tag{55}$$

Since $T^{(b)}_{\mu\nu\rho}$ belongs to a structure group multiplet with the affine connection torsion components and corresponds to a tensor that measures the non-vanishing of the exterior covariant derivative of an extended coframe field, we view it as an extra torsion tensor in a Cartan formulation sense. Note that the left-hand side of (54) is invariant under the following one-form gauge transformation, with parameter σ_μ:

$$\delta b_{\mu\nu} = 2\partial_{[\mu}\sigma_{\nu]}. \tag{56}$$

This can be viewed as the SNC analogue of the central charge transformation (19). Like the central charge, we will not regard the gauge transformation (56) as part of the structure group of SNC geometry, since it does not act on the frame fields $\tau_A{}^\mu$ and $e_a{}^\mu$.

Of the $D(D-1)(D-2)/6$ equations (54), $(D-2)^2$ equations can be used to express the following spin connection components

$$\tau_A{}^\mu \omega_\mu{}^{Aa}, \qquad\qquad e_{[a|}{}^\mu \omega_{\mu A|b]}, \tag{57}$$

in terms of (extended co)frame fields and $T^{(b)}_{\mu\nu\rho}$. The remaining $(D-2)(D-3)(D-4)/6$ equations take the form

$$e_a{}^\mu e_b{}^\nu e_c{}^\rho T^{(b)}_{\mu\nu\rho} = 3 e_a{}^\mu e_b{}^\nu e_c{}^\rho \partial_{[\mu} b_{\nu\rho]}. \tag{58}$$

Note that, even after the introduction of the extra torsion equation (54), we cannot express all spin connection components in terms of (extended co)frame fields and torsion tensors. In particular, the following $2(D-2)$ components of $\omega_\mu{}^{Aa}$

$$W_{AB}{}^a \equiv \tau_{\{A|}{}^\mu \omega_{\mu|B\}}{}^a \equiv \tau_{(A|}{}^\mu \omega_{\mu|B)}{}^a - \frac{1}{2}\eta_{AB}\tau_C{}^\mu \omega_\mu{}^{Ca}, \tag{59}$$

where we used $\{AB\}$ to denote the symmetric traceless part of A and B, remain independent in our formalism. For the choice of extended coframe fields $\tau_\mu{}^A$, $e_\mu{}^a$ and $b_{\mu\nu}$ and torsion tensors $T_{\mu\nu}{}^A$, $T_{\mu\nu}{}^a$, $T^{(b)}_{\mu\nu\rho}$, we thus obtain a class of torsionful and metric compatible connections of SNC geometry that is parametrized by these independent components $W_{AB}{}^a$.[29] While the presence of such components is unusual from the point of view of Lorentzian geometry, it is not uncommon in non-Lorentzian geometry. It occurs for instance when discussing the Galilean and Carrollian limits of Einstein gravity in the Palatini formulation [55].

The full expressions for $\omega_\mu{}^{ab}$ and $\omega_\mu{}^{Aa}$ that can be obtained from (50*b*) and (54) are given by

$$\omega_\mu{}^{ab} = -2e^{[a|\nu|}\partial_{[\mu}e_{\nu]}{}^{b]} + e_{\mu c}e^{a\nu}e^{b\rho}\partial_{[\nu}e_{\rho]}{}^c - \frac{3}{2}\epsilon_{AB}\tau_\mu{}^A\tau^{B\nu}e^{a\rho}e^{b\sigma}\partial_{[\nu}b_{\rho\sigma]}$$
$$+ e^{[a|\nu}T_{\mu\nu}{}^{|b]} - \frac{1}{2}e_\mu{}^c e^{a\nu}e^{b\rho}T_{\nu\rho c} + \frac{1}{2}\epsilon_{AB}\tau_\mu{}^A\tau^{B\nu}e^{a\rho}e^{b\sigma}T^{(b)}_{\nu\rho\sigma},$$

$$\omega_\mu{}^{Aa} = -\tau^{A\nu}\partial_{[\mu}e_{\nu]}{}^a + e_{\mu b}\tau^{A\nu}e^{a\rho}\partial_{[\nu}e_{\rho]}{}^b + \frac{3}{2}\epsilon^{AB}\tau_B{}^\nu e^{a\rho}\partial_{[\mu}b_{\nu\rho]} + \frac{1}{2}\tau_\mu{}^B\tau_B{}^\nu\tau^{A\rho}T_{\nu\rho}{}^a$$
$$- e_{\mu b}\tau^{A\nu}e^{(a|\rho}T_{\nu\rho}{}^{|b)} - \frac{1}{4}\tau_\mu{}^A\epsilon_{CD}\tau^{C\nu}\tau^{D\rho}e^{a\sigma}T^{(b)}_{\nu\rho\sigma} - \frac{1}{2}e_{\mu b}\epsilon^A{}_B\tau^{B\nu}e^{a\rho}e^{b\sigma}T^{(b)}_{\nu\rho\sigma}$$
$$+ \tau_\mu{}^B W_B{}^{Aa}. \tag{60}$$

In order to give the explicit expression for the affine connection, we use a notation to denote certain torsion components:

$$T_{AB}{}^C = \tau_A{}^\mu\tau_B{}^\nu T_{\mu\nu}{}^C, \qquad\qquad T_{Aa}{}^B = \tau_A{}^\mu e_a{}^\nu T_{\mu\nu}{}^B, \qquad\qquad T_{ab}{}^A = e_a{}^\mu e_b{}^\nu T_{\mu\nu}{}^A \tag{61}$$

and similarly for components of $T_{\mu\nu}{}^a$ and $T^{(b)}_{\mu\nu\rho}$. This is analogous to what we have used in section 2.3. Plugging the expressions (51) and (60) for the spin connections ω_μ, $\omega_\mu{}^{Aa}$ and $\omega_\mu{}^{ab}$ into equation (47) leads to the following expression for the affine connection:

---

[29] In this paper, we do not discuss how SNC geometry, as discussed here, appears in applications. In case such applications make use of a metric compatible connection, it will typically be a particular member of the class of connections constructed here, in which the components $W_{AB}{}^a$ (as well as any possible torsion tensors) are not independent, but instead composed of other independent fields of the theory, or are absent due to, e.g. the presence of an extra Stueckelberg symmetry. For instance, the effective gravitational field theories for non-relativistic string theory [13, 50, 52–54] can be written entirely in terms of the (extended co)frame fields discussed in this paper and the $W_{AB}{}^a$ spin connection components do not appear in their equations of motion.

$$
\begin{aligned}
\Gamma^\rho_{\mu\nu} ={}& \frac{1}{2}\tau^{\rho\sigma}(\partial_\mu\tau_{\nu\sigma} + \partial_\nu\tau_{\mu\sigma} - \partial_\sigma\tau_{\mu\nu}) + \frac{1}{2}h^{\rho\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}) \\
&- \tau_\mu{}^A e_\nu{}^a \tau^{B\rho} T_{a(AB)} + \frac{1}{2}e_\mu{}^a e_\nu{}^b \tau_C{}^\rho T_{ab}{}^C \\
&+ \tau_\mu{}^A \tau_\nu{}^B e_c{}^\rho W_{AB}{}^c + \tau_\nu{}^B \tau^{A\rho}(\tau_{\mu C}T_{AB}{}^C + e_\mu{}^c T_{c[AB]}) + \frac{1}{2}e_a{}^\rho T_{\mu\nu}{}^a - e^{a\rho}e_{(\mu|b|}T_{\nu)a}{}^b \\
&+ \frac{1}{4}\tau_{\mu\nu}e^{c\rho}\epsilon^{AB}(3\tau_A{}^\eta \tau_B{}^\xi e_c{}^\sigma \partial_{[\eta}b_{\xi\sigma]} - T^{(b)}_{ABc}) \\
&- \epsilon^{AB}e^{d\rho}e_{(\mu}{}^c \tau_{\nu)A}(3\tau_B{}^\eta e_c{}^\xi e_d{}^\sigma \partial_{[\eta}b_{\xi\sigma]} - T^{(b)}_{Bcd}). \tag{62}
\end{aligned}
$$

One can explicitly check that the structure group transformations of the right-hand sides of (51) and (60) that are induced by (33), (35), (49) and (55), coincide with the rules given in (44).[30] One similarly checks that the expression for $\Gamma^\mu_{\nu\rho}$ given in (62) is invariant under the structure group, although similar to the NC case, boost invariance is not manifest. We refrain from giving an expression in similar vein as equation (22), as we cannot find a boost covariant quantity that is analogous to $\bar{h}_{\mu\nu}$. Note furthermore that the spin connections (60) and the affine connection (62) are invariant under the one-form gauge transformation (56) if all torsion tensors $T_{\mu\nu}{}^A$, $T_{\mu\nu}{}^a$ and $T^{(b)}_{\mu\nu\rho}$ are, since in that case the equations (50a), (50b) and (54) are invariant under (56). We stress however that there is no good *a priori* reason to expect that the SNC connections are invariant under (56), since we do not view the latter transformation as part of the structure group. The torsion tensors $T_{\mu\nu}{}^A$, $T_{\mu\nu}{}^a$ and $T^{(b)}_{\mu\nu\rho}$ are thus not required to be invariant under the one-form gauge transformation.

From (51) and (60), we see that the torsion tensor components $T_{ab}{}^A$, $T_{a(AB)}$ and $T^{(b)}_{abc}$ do not appear in expressions for the SNC spin connections (while all other torsion components do). We see moreover from (52) and (58) that setting these components equal to zero leads to differential constraints on extended coframe fields. They thus correspond to intrinsic torsion components in SNC geometry. Note that the term intrinsic torsion is normally used to refer to components of the affine connection torsion. As seen in section 2.2, this is how the notion of intrinsic torsion appears in NC geometry. Here however, the $e_a{}^\mu e_b{}^\nu e_c{}^\rho T^{(b)}_{\mu\nu\rho}$ components do not contribute to the torsion of the affine connection and should therefore be viewed as intrinsic torsion in a Cartan formulation sense that is more general than the one usually adopted.

In going from particles to strings, we see that effectively the central charge gauge field m_μ has been replaced by the 2-form field $b_{\mu\nu}$ which plays a very similar role as m_μ. Both are geometric fields that transform under boost transformations and both are needed to define a dependent spin connection or affine connection that transforms in the correct way. Moreover, $b_{\mu\nu}$ plays a similar physical role as m_μ. Indeed, $b_{\mu\nu}$ acts as a gauge field for the one-form transformation (56). One can thus couple it to an anti-symmetric two-tensor current that implements conservation of the string tension via an appropriate Wess–Zumino term [51], in analogy to how m_μ couples to the Noether current corresponding to particle mass conservation.

---

[30] For $\omega_\mu{}^{Aa}$ this check requires additionally that $W_{AB}{}^a$ transforms as follows under the structure group

$$
\delta W_{AB}{}^a = \tau_{\{A}{}^\mu \delta\omega_{|\mu|B\}}{}^a + \delta\tau_{\{A}{}^\mu \omega_{|\mu|B\}}{}^a, \tag{63}
$$

where $\delta\tau_A{}^\mu$ and $\delta\omega_\mu{}^{Aa}$ are given in equations (33) and (44), respectively. Note that, unlike torsion tensors, the $W_{AB}{}^a$ components transform non-covariantly, i.e. with derivatives of structure group parameters $\delta W_{AB}{}^a = \tau_{\{A}{}^\mu \partial_{|\mu|}\lambda_{B\}}{}^a + \cdots$. For this reason one has to view $W_{AB}{}^a$ as connection components and not as torsion tensor components, even though in our discussion both $W_{AB}{}^a$ and torsion tensors are left as independent fields.
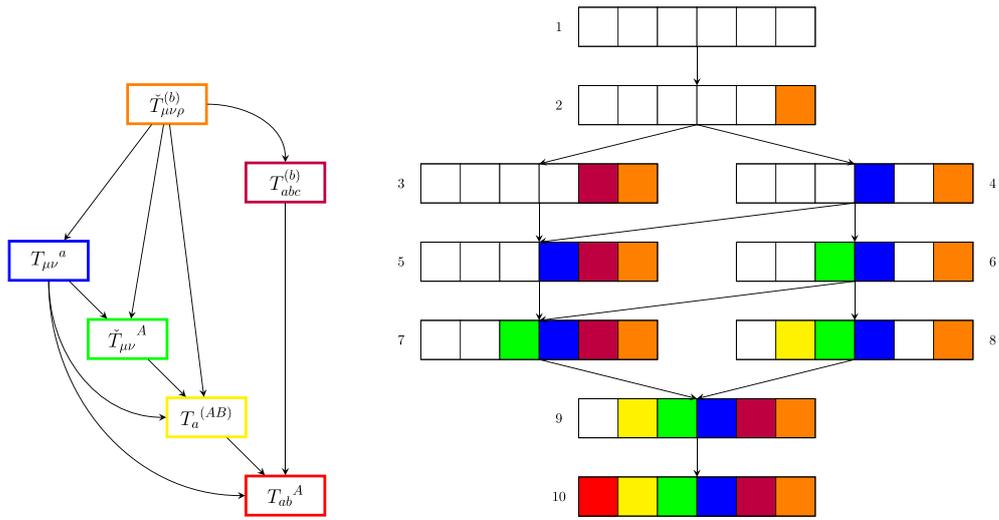
(a) This diagram gives a schematic representation of the boost transformations of the different torsion components. Arrows display relations between the different torsion components under boost transformations. If the boost transformation of a torsion component contains another torsion component, an arrow points from the former component towards the latter, e.g. the arrow from the purple to the red box represents the boost transformation $\delta T_{abc}^{(b)} = -3\epsilon_{AB}\lambda^A{}_{[c}T_{ab]}{}^B$.

(b) The colours in every rectangle of this diagram represent the parts of the corresponding torsion tensors that are unconstrained. In particular, if a colour is missing, it implies that this part of the torsion tensor is zero. In Figure 4a, we have indicated what colour corresponds to what part of the torsion tensors. Furthermore, arrows point from more special cases towards more general cases.

Figure 4. Classification of constraints on the torsion tensors (b) that are consistent with the local structure group transformations (a).

3.3. Special cases and examples

Similarly as in the particle case in section 2.3, a generic torsionful affine connection that is compatible with SNC geometry includes the torsion tensors $T_{\mu\nu}{}^A$, $T_{\mu\nu}{}^a$ and $T_{\mu\nu\rho}^{(b)}$. Those torsion tensors transform under Lorentz transformations, spatial rotations and string Galilean boosts. Some components of the torsion tensors transform to other components, and hence, those torsion tensors cannot be set to zero independently from other torsion components. All possible scenarios in which components of the torsion tensors can be set to zero consistently are displayed in figure 4(b). The structure of this figure is similar to figure 1(b).

In the following, we find it useful to define the following notation which separates the intrinsic torsion components by defining:

$$\check{T}_{\mu\nu}^A = T_{\mu\nu}{}^A - 2e_{[\mu}{}^a\tau_{\nu]}^B\eta_{BC}T_a{}^{(AC)} - e_\mu{}^a e_\nu{}^b T_{ab}{}^A, \tag{64a}$$

$$\check{T}_{\mu\nu\rho}^{(b)} = T_{\mu\nu\rho}^{(b)} - e_\mu{}^a e_\nu{}^b e_\rho{}^c T_{abc}^{(b)}. \tag{64b}$$

The torsion components $\check{T}_{\mu\nu}^A$ and $\check{T}_{\mu\nu\rho}^{(b)}$, are the torsion components $T_{\mu\nu}{}^A$ and $T_{\mu\nu\rho}^{(b)}$ but with the intrinsic torsion projected out. These intrinsic torsion components are given by (52) and (58).
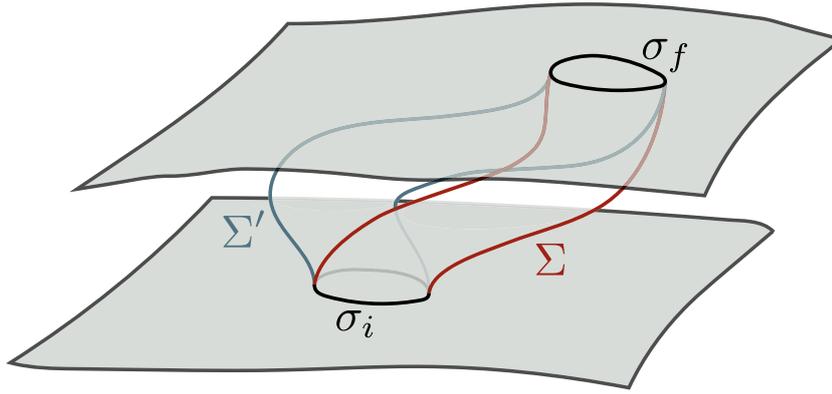
Figure 5. Two worldsheets Σ and Σ' stretching between two leaves of the manifold with $\partial\Sigma = \partial\Sigma'$. For geometries with zero time-like torsion $T_{\mu\nu}^A = 0$, the volume two-form $\ell = 1/2\,\ell_{\mu\nu}dx^\mu \wedge dx^\nu$ is closed. We can thus conclude that the proper area (40) traced out by Σ and Σ' is the same, i.e. $\int_\Sigma \ell = \int_{\Sigma'} \ell$.

Analogously to the particle in section 2.3, it is convenient to subdivide those cases. We will do this in the following list.

- In cases 1, 2 and 3, we have that the anti-symmetrization of the affine connection, i.e. its torsion, is zero. By (48), this is equivalent to setting $T_{\mu\nu}^A = 0$ and $T_{\mu\nu}^a = 0$. Those cases are commonly referred to as 'torsionless String Newton-Cartan geometry'.
- Cases 1, 2, 3, 4 and 5 all have zero longitudinal torsion $T_{\mu\nu}^A = 0$. In case 4 and 5, we let $T_{\mu\nu}^a$ unconstrained. In all those cases, the two-form $\ell_{\mu\nu}$ as defined in (40) is closed, that is $3\partial_{[\mu}\ell_{\nu\rho]} = 0$. By Stokes' theorem, we obtain that the proper area (40) is independent of the chosen worldsheet segment Σ_φ and only depends on the initial and final position of the string, that is, on the curves $\varphi(0,\cdot) = \sigma_i$ and $\varphi(1,\cdot) = \sigma_f$. This implies that the same amount of proper area has been swept out by two strings starting and ending at the same position, irrespective of the worldsheet segment they have traced out throughout space-time. See figure 5. This can be rephrased as the statement that SNC manifolds with zero longitudinal torsion admit an absolute area function.
- Cases 6 and 7 correspond to setting $T_a^{(AB)} = 0$ and $T_{ab}^A = 0$ and letting $T_{\mu\nu}^a$ unconstrained.
- Cases 8 and 9 correspond to setting $T_{ab}^A = 0$ and letting $T_{\mu\nu}^a$ and $T_a^{(AB)}$ unconstrained. This situation has for instance been encountered in [56], where it occurs naturally in the string $1/c$ expansion of the Einstein equations (at leading order) in vacuum. In the cases 6, 7, 8 and 9, there does not exist an absolute area function anymore. This means that the area of the worldsheet between two events does not only depend on the initial and final positions of the string, but also on the worldsheet segment a string traces out. As $T_{ab}^A = 0$, though, there is still a notion of absolute transversal simultaneity. The condition $T_{ab}^A = 0$ is, by Frobenius' theorem, equivalent to stating that there is a foliation of $(D-2)$-dimensional transversal submanifolds, i.e. submanifolds of the space-time manifold $\mathcal{M}$ such that the tangent vectors to all curves on those submanifolds are transversal, as defined in section 3.1. A notion of string causality that distinguishes between past and future can be defined as the following statement: a string defined by the embedding $\sigma_f : [0,1] \to \mathcal{M}$ is in the future with respect to a string defined by an embedding $\sigma_i : [0,1] \to \mathcal{M}$ if there exists a worldsheet segment Σ_φ with $\varphi(0,\cdot) = \sigma_i$ and $\varphi(1,\cdot) = \sigma_f$ such that the integral in (40) is positive.

- Case 10 is generic torsion. As we let $T_{ab}{}^A$ unconstrained, consistency with boost transformations requires that all other torsion components are also unconstrained. There is no notion of absolute area or absolute transversal simultaneity anymore.

The above classification needs a further refinement if we include the intrinsic torsion constraints that describe the DSNC$^-$ geometry underlying non-relativistic string theory with $N = 1$ supersymmetry [13], since these constraints set part of the torsion tensors

$$T_{ab}{}^A \qquad \text{and} \qquad T_{aA}{}^B \tag{65}$$

equal to zero without changing the basic structure of the classification. This proceeds in two steps. First one picks out those intrinsic torsion components that are invariant under local (anisotropic) dilatations $\delta \tau_\mu{}^A = \lambda_D \tau_\mu{}^A$, since they are an emerging symmetry in non-relativistic string theory. One thus ends up with the components

$$T_{ab}{}^A \qquad \text{and} \qquad T_{a\{AB\}}. \tag{66}$$

Formally, these tensors can be obtained by discarding $T_{aA}{}^A$. Since it transforms as a (dependent) dilatation gauge field, it should not be seen as part of the intrinsic torsion of the geometry. In a second step, using light-cone notation $A = (+, -)$, we set half of the intrinsic torsion components given in (66) equal to zero:

$$T_{ab}{}^- = T_{a+}{}^- = 0. \tag{67}$$

One then obtains the DSNC$^-$ case by setting these torsion components together with $T_{aA}{}^A$ to zero in the above classification.

Let us now give an example of dependent torsion tensors, similar to what we considered at the end of section 2.3. We assume that we are in a dimension where there exists a vector-spinor ψ_μ that satisfies the Majorana condition. A concrete example in ten dimensions has been worked out in detail in [13]. The vector-spinor forms a representation of the local $SO(1, 1) \times SO(D - 2)$ transformations

$$\delta \psi_\mu = -\frac{1}{2} \lambda \gamma_{01} \psi_\mu + \frac{1}{4} \lambda^{ab} \gamma_{ab} \psi_\mu, \tag{68}$$

where the gamma matrices (γ_A, γ_a) form a Clifford algebra with signature $(- + + \cdots +)$. In order to specify the boost representation, it is useful to split the spinor as $\psi_\mu = \psi_{\mu+} + \psi_{\mu-}$, where the components are eigenspinors under $\gamma_{01} = \gamma_0 \gamma_1$ as follows: $\gamma_{01} \psi_{\mu\pm} = \pm \psi_{\mu\pm}$. Equivalently, one can define $\psi_{\mu\pm} = 1/2(\mathbb{1} \pm \gamma_{01}) \psi_\mu$. The transformation under String Galilean boosts with parameters λ^{Aa} is then given as

$$\delta \psi_{\mu+} = 0, \qquad \delta \psi_{\mu-} = \frac{1}{2} \lambda^{Aa} \gamma_A \gamma_a \psi_{\mu+}. \tag{69}$$

The projected Majorana spinors are also the natural building blocks for constructing independent spinor bilinears as follows:

$$T_{\mu\nu}{}^A = \frac{1}{2} \bar{\psi}_{[\mu+} \gamma^A \psi_{\nu]+}, \qquad T_{\mu\nu}{}^a = \bar{\psi}_{[\mu+} \gamma^a \psi_{\nu]-}. \tag{70}$$

Using the transformation rules (68) and (69), one can show that the two-forms $T_{\mu\nu}{}^A$ and $T_{\mu\nu}{}^a$ transform as given in equation (49). Due to the identity $\epsilon^{AB} \gamma_B = -\gamma^A \gamma_{01}$ and the properties of the projected spinors, we find that $\epsilon^A{}_B T_{\mu\nu}{}^B = -T_{\mu\nu}{}^A$. This is equivalent to the statement that $T_{\mu\nu}{}^- = 2^{-1/2}(T_{\mu\nu}{}^0 - T_{\mu\nu}{}^1) = 0$ identically. The three-form torsion can analogously be defined as

$$T^{(b)}_{\mu\nu\rho} = 3 \bar{\psi}_{[\mu-} \gamma^A \psi_{\nu-} \tau_{\rho]A} - 3 T_{[\mu\nu}{}^a e_{\rho]a}. \tag{71}$$

It is straightforward to check that this three-form transforms under the local structure group as in (55). The tensors given in equations (70) and (71) provide explicit examples of the dependent torsion tensors. Consequently, they gives rise to an affine connection that is invariant under string Galilean boosts.

4. Conclusions and outlook

In this work we gave an in-depth description of generalized NC geometries for particles and strings using a frame formulation. In the case of particles, such a frame formulation stresses the relation with the underlying structure group and makes it possible to derive several results in an elegant way. An important feature of our discussion was the introduction of independent torsion tensors which makes it possible to define spin connections and affine connections that transform in the right way under the symmetries of the structure group. We gave a rather extensive set of solutions of different constraints that one can impose on the intrinsic torsion tensors leading to different constrained geometries. Furthermore, we gave a physical interpretation of the geometric fields at several places, thereby extending the notion of absolute time to the string case.

One might wonder whether there is a natural interpretation of the 2-form field $b_{\mu\nu}$ similar to the interpretation of the gauge field m_μ as the one associated with the central extension of the Galilei algebra. One interesting proposal, inspired by earlier work in supergravity, was recently given in [51] where the 2-form field $b_{\mu\nu}$ was represented as a dependent expression in terms of two 1-form gauge fields, at the expense of introducing extra Stueckelberg symmetries. An alternative option is to go to loop space geometry, thereby replacing coordinates x^μ by $x^\mu(\sigma)$, where the coordinate σ parametrizes a circle, and replacing fields $\phi(x)$ defined over ordinary geometry by fields $\phi(x(\sigma))$ defined over loop space geometry. Within such a geometry, it is natural to define a loop space covariant derivative involving the 2-form $b_{\mu\nu}$ as follows [57]:

$$\mathcal{D}_\mu(\sigma) = \frac{\delta}{\delta x^\mu(\sigma)} - b_{\mu\nu} x'^\nu. \tag{72}$$

This naturally corresponds to a loop algebra with generators $T(\sigma)$. Although promising, it is not yet clear how useful these approaches are. At the moment, perhaps a more practical approach is to work immediately in terms of fields and ignore a possible relation with an underlying algebra which is not needed at least for the purpose of this work.

In [17], the intrinsic torsion of non-Lorentzian geometric structures was systematically studied and classified using cohomological techniques. The classification derived there agrees with the one given in section 2. It would be interesting to see whether the analysis based on Spencer cohomology can be extended to the study of SNC-type geometries as presented in section 3. Furthermore, it would be natural to generalize that to G-structures with G = $(SO(1,p) \times SO(D-p-1)) \ltimes \mathbb{R}^{(p+1)(D-p-1)}$—that is, so-called p-brane geometries [22].

It is natural to consider the extension of our work to non-relativistic string theory with $N = 2$ supersymmetry and to M-theory or membranes. In the case of $N = 2$ string theory, one expects more constraints than the ones characterizing the DSNC⁻ geometry given in equation (67). These will also include fermionic intrinsic torsion tensors. We expect the same to happen for M-theory with the understanding that in that case one uses a membrane foliation [58, 59] with $A = 0, 1, 2$ and $a = 3, \ldots, 10$. This suggests the existence of a degenerated supergeometry whose proper formulation might require the use of superfields and superspace. The non-relativistic torsion constraints we find are reminiscent of the superspace torsion constraints

that one imposes in the relativistic case to define a relativistic supergravity theory. Once constructed, by consistency the non-relativistic M-theory geometry one finds should reduce to the DSNC$^-$ geometry considered in this work by performing a double dimensional reduction over a spatial membrane direction followed by a truncation. We hope to come back to these issues in a forthcoming work.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

ORCID iDs

E A Bergshoeff ⬤ https://orcid.org/0000-0003-1937-6537
K van Helden ⬤ https://orcid.org/0000-0002-0830-6835
J Lahnsteiner ⬤ https://orcid.org/0000-0002-6326-3401
L Romano ⬤ https://orcid.org/0000-0001-9033-1345
J Rosseel ⬤ https://orcid.org/0000-0003-3569-4893

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