# Supersymmetric Poisson and Poisson-supersymmetric sigma models

Thomas Basile ⓘ,[a] Athanasios Chatzistavrakidis ⓘ[b,c] and Sylvain Lavau ⓘ[d]

[a]*Service de Physique de l'Univers, Champs et Gravitation, Université de Mons,*
*20 place du Parc, 7000 Mons, Belgium*

[b]*Division of Theoretical Physics, Rudjer Bošković Institute,*
*Bijenička 54, 10000 Zagreb, Croatia*

[c]*Institute for Theoretical Physics, University of Wroclaw,*
*pl. Maxa Borna 9, 50-204 Wroclaw, Poland*

[d]*Department of Mathematics, Galatasaray Üniversitesi,*
*Çırağan Caddesi No:36, 34349 İstanbul, Turkey*

*E-mail:* thomas.basile@umons.ac.be, athanasios.chatzistavrakidis@irb.hr,
lavau@gsu.edu.tr

Abstract: We revisit and construct new examples of supersymmetric 2D topological sigma models whose target space is a Poisson supermanifold. Inspired by the AKSZ construction of topological field theories, we follow a graded-geometric approach and identify two commuting homological vector fields compatible with the graded symplectic structure, which control the gauge symmetries and the supersymmetries of the sigma models. Exemplifying the general structure, we first revisit the $\mathcal{N} = 1$ dilaton supergravity in 2 dimensions and then we show that two distinguished cases exhibiting global supersymmetry exist, one being the differential Poisson sigma model constructed before by Arias, Boulanger, Sundell and Torres-Gomez and the other a contravariant differential Poisson sigma model. The new model features nonlinear global supersymmetry transformations that are generated by the Poisson structure on the body of the target supermanifold, giving rise to a Poisson supersymmetry. Further examples are characterised by supersymmetry transformations controlled by the anchor map of a Lie algebroid, when this map is invertible, in which case we determine the geometric conditions for invariance under supersymmetry and closure of the supersymmetry algebra. Moreover, we show that the common thread through this type of models is that their supersymmetry-generating vector field is the coadjoint representation up to homotopy of a Lie algebroid.

# Contents

# 1 Introduction

The Poisson sigma model is directly related to two important results of 90s mathematical physics: the solution to the problem of deformation quantization for Poisson manifolds by Kontsevich [1] and the geometrization of the classical master equation for topological field theory by Alexandrov, Kontsevich, Schwarz and Zaboronsky (AKSZ) [2]. In the former case, the diagrammatic expression for the star product of functions given in [1] was found to correspond to a correlation function between two observables of the Poisson sigma model on a disc [3], in other words the diagrams in the star product expansion can actually be interpreted as Feynman diagrams. In the second case, the Poisson sigma model was embedded within the general construction of [2] (inspired by previous work of A. Schwarz [4]) in [5]. The underlying mathematical structure behind $(n + 1)$-dimensional AKSZ sigma models, namely graded symplectic supermanifolds with symplectic structure of degree $n$ (sometimes

called $\mathrm{QP}_n$ manifolds),[1] corresponds exactly to Poisson geometry for $n = 1$ [7], making the Poisson sigma model *the* 2D AKSZ sigma model.

The above statements refer to the bosonic Poisson sigma model. When sigma models with fermions and supersymmetry are considered, one may ask whether the above statements carry over in one way or another. However, on the one hand, the literature on the supersymmetric version of the AKSZ construction is limited, see for instance [8] and [9]. On the other hand, deformation quantization on superspace in a spirit similar to [3] was studied in [10], see also [11–13] for alternative approaches. More recently, a different supersymmetric Poisson sigma model was constructed in [14, 15], within the context of differential Poisson algebras. The quantization of such differential Poisson sigma models, which is expected to yield a star product on the algebra of differential forms, has not been performed yet, let alone compared to previous proposals.

These suggest that much less is known about supersymmetric extensions of Poisson sigma models and that the topic deserves a fresh look from a modern and systematic perspective. Nonetheless, the general form of a supersymmetric Poisson sigma model was already described as a nonlinear super-gauge theory in the original publication of Ikeda [16].[2] Indeed, the model of [14] is an interesting special case of Ikeda's model, essentially an example of it, different from 2D supergravity, which was the example studied in [16] (and further analyzed in, e.g. [18–22]). More specifically, it is an example of supersymmetric Poisson sigma model wherein the target space is the shifted cotangent bundle of the parity-reversed tangent bundle $\Pi TM$ of a smooth manifold $M$, endowed with a structure of *Poisson supermanifold*. In particular, this means that $M$ is itself an ordinary Poisson manifold, and that its Poisson bracket extends to the algebra of differential forms (isomorphic to the algebra of functions on the parity-reversed tangent bundle).

A closer look to this model reveals an interesting interplay between two different homological vector fields: the one associated with the structure of Poisson supermanifold on $\Pi TM$ controls the gauge symmetry of the model, including local i.e. spacetime dependent supersymmetry transformations, whereas the one associated with the de Rham differential on $M$ generates its (global) supersymmetry. We may ask whether this is an isolated model or there exist other examples of such differential Poisson sigma models featuring other types of global supersymmetry. We answer this question affirmatively in this paper, showing that there exists a contravariant model with target space the shifted cotangent bundle of the parity-reversed *cotangent* bundle of a smooth manifold and that the two models are selected based on general consistency conditions. This also reveals their structural kinship through the relation of their construction to the concept of representations up to homotopy for Lie algebroids, a notion that extends usual representations from vector bundles to graded vector bundles (chain complexes) [23], in particular to the coadjoint representation.

In more detail, we set up our analysis by first studying non-negatively graded (NQ) supermanifolds in section 2. A supermanifold carries by construction a $\mathbb{Z}_2$-grading, and it can be identified with a parity-reversed vector bundle $\Pi E$ over a smooth manifold $M$, which is the body of the supermanifold. Endowing it with an additional $\mathbb{Z}$-grading, we can

---

[1]Referred as $\Sigma_n$-manifolds in [6], aka symplectic Lie $n$-algebroids.

[2]The bosonic model was also described around the same time in [17].

consider super-vector bundles over the supermanifold. In this very general setting, where we have a $(\mathbb{Z} \times \mathbb{Z}_2)$-bidegree, two different degree 1 vector fields can be considered, namely one with $\mathbb{Z}$-degree 1 (denoted $\mathcal{Q}$) and one with $\mathbb{Z}_2$-degree 1 (denoted $\mathcal{Q}_S$). Asking that their graded commutator vanishes[3],

$$[\mathcal{Q}, \mathcal{Q}] = 0 \quad \text{and} \quad [\mathcal{Q}_S, \mathcal{Q}_S] = 0 \, ,$$

imposes consistency conditions that correspond to the notions of a Lie superalgebroid [24, 25] and of a differential super-vector bundle [24, 26]. The latter is closely related to the concept of representation of a Lie algebroid, notably to the richer notion of representation up to homotopy, where the module is a chain complex (a graded vector bundle). This notion allows for a generalization of the adjoint and coadjoint representations for Lie algebras to the Lie algebroid realm [23]. A central element in these constructions, which we describe in section 2.2 following [23] and adjusting accordingly for supermanifolds, is the so-called basic connection together with the accompanying notion of basic curvature. We then discuss how these elements are elegantly encoded in the homological vector field $\mathcal{Q}_S$.

In section 3, we specialize to the case of Poisson supermanifolds, where the graded super-vector bundle is a shifted cotangent bundle over the Poisson supermanifold. The existence of a graded symplectic structure $\omega$ on the cotangent bundle prompts us to impose the compatibility conditions

$$\mathcal{L}_{\mathcal{Q}} \omega = 0 \quad \text{and} \quad \mathcal{L}_{\mathcal{Q}_S} \omega = 0 \, .$$

These conditions allow us to define Hamiltonian functions $\mathcal{H}$ and $\mathcal{H}_S$ that can be used to construct a general form of supersymmetric Poisson sigma models. In particular, the first condition together with the fact that $\mathcal{Q}$ is homological result in the Jacobi identity for the super-Poisson bracket on the algebra of functions on the supermanifold. Working toward finding explicit solutions, in section 3.2 we expand the generic super-Poisson bracket up to second order in the fermionic coordinate of the Poisson supermanifold and report the conditions stemming from the Jacobi identity up to order 5 in the fermionic coordinate in manifestly covariant form. Notably, this requires that the body of the supermanifold is an ordinary Poisson manifold $M$, that there exists a $T^*M$-connection on the dual bundle $E^*$, and an inner metric on $E$, not necessarily nondegenerate, which is covariantly constant with respect to this connection.

In section 4 we take the final step toward constructing explicit examples of supersymmetric Poisson sigma models in this fashion, which requires to impose that the graded commutator of the two homological vector fields vanishes too,

$$[\mathcal{Q}, \mathcal{Q}_S] = 0 \, ,$$

alternatively that the corresponding Hamiltonian functions have vanishing Poisson bracket,

$$\{\mathcal{H}, \mathcal{H}_S\} = 0 \, .$$

Then the general form of the action functional for a supersymmetric Poisson sigma model becomes

$$S[\Phi] = \int \omega_{AB} \, \Phi^A \wedge \mathrm{d}\Phi^B + \Phi^*(\mathcal{H}) \, ,$$

---

[3]The graded commutator of vector fields in degree 1 is the anticommutator, yet we denote it with ordinary brackets to avoid confusion with the Poisson bracket of functions.

where $\Phi = (\Phi^A)$ is a map from the parity-reversed tangent bundle of a 2D world sheet to the parity-shifted cotangent bundle of the Poisson supermanifold, and $\Phi^*$ the pull-back by this map. By virtue of the consistency conditions, the model has gauge symmetries generated by $\mathcal{Q}$ and global supersymmetries generated by $\mathcal{Q}_S$, it is a Cartan integrable Hamiltonian system, and it corresponds to the nonlinear super-gauge theory of [16]. In section 4.2 we revisit the well-known case of $\mathcal{N} = 1$ dilaton supergravity in 2 dimensions in our approach as a first example of a physically motivated model.

The two distinguished examples of the general construction exhibiting global sypersymmetry follow directly for the choices $E = TM$ and $E = T^*M$ for a Poisson manifold $M$, and they are presented in sections 4.3 and 4.4. The first is identical to the one constructed in ref. [14] using differential Poisson algebras and the second is a nontrivial "dual" model. In both cases, the Hamiltonian features a quadratic part controlled by the Poisson structure on the body of the supermanifold and a quartic part whose coefficient is associated with the basic curvature of a connection on the Poisson supermanifold. This results in a supersymmetry-generating homological vector field that can be identified with the coadjoint representation of the tangent and cotangent Lie algebroids, providing a common ground for the two models. This is justified in section 4.5, where we analyze further the general conditions for invariance under supersymmetry transformations and closure of their algebra for arbitrary Lie algebroids $E$. We show that the only other option includes models based on Lie algebroids whose anchor is invertible, for instance arising from a Lie algebra action or from endomorphisms of the tangent bundle with vanishing Nijenhuis tensor.

**A note on notation.** We will routinely work with geometrical objects in a basis independent fashion and also present local coordinate expressions in a chosen basis. To keep the notation as transparent and light as possible, we use the following rules. For "$E$-on-$V$" connections, which are $\mathbb{R}$-linear maps $\Gamma(E) \times \Gamma(V) \to \Gamma(V)$, we use the symbol $\nabla^E$. For their torsion and curvature tensors, we use $T^{\nabla^E}$ and $R^{\nabla^E}$. When $E = TM$ (ordinary connections), we simplify to the symbol $\nabla$, and for dual ordinary connections ($TM$-on-$V^*$) we use $\nabla^*$. When entries or indices are included, we strip the symbols from additional superscripts and let it be understood from the entries/indices which objects they are. For example, for a $T^*M$-on-$E$ connection we write any of the following: $\nabla^{T^*M}$, $\nabla_\eta$, $\nabla^\mu$, where $\eta = \eta_\mu \mathrm{d}x^\mu$ is an 1-form. For its curvature, we write $R^{\nabla^{T^*M}}$ or $R^{\mu\nu a}{}_b$ in a basis $\mathfrak{e}_a$ of $E$, etc.

## 2 NQ-supermanifolds

### 2.1 Homological vector fields

Any $\mathbb{Z}_2$-graded manifold, aka supermanifold, $\mathcal{M}$ is (non-canonically) diffeomorphic to a parity-shifted vector bundle $\Pi E$ where $E \twoheadrightarrow M$ is an ordinary (meaning non-graded, smooth) vector bundle over a smooth manifold $M$ which is the body of the supermanifold $\mathcal{M}$ [27, 28]. We are interested in supermanifolds equipped with an additional $\mathbb{Z}$-grading, i.e. we will consider $(\mathbb{Z} \times \mathbb{Z}_2)$-graded manifolds in this work. We will denote the bidegree of various objects on such a $\mathbb{Z}$-graded supermanifold, and the corresponding shift functor, as $(\cdot, \cdot)$ and $[\cdot, \cdot]$ respectively, where the first placeholder corresponds to the $\mathbb{Z}$ degree and the second one

to the $\mathbb{Z}_2$ degree. In this context, the aforementioned diffeomorphism $\mathcal{M} \cong \Pi E$ should be rewritten as $\mathcal{M} \cong E[0,1]$, thereby highlighting the fact that the coordinates of $E$ along its fibres are parity-shifted and given $\mathbb{Z}_2$-degree 1, and the whole supermanifold is thought of as being concentrated in $\mathbb{Z}$-degree 0. Put differently, local coordinates $(x^\alpha) = (x^\mu, \theta^a)$ on $E[0,1]$ are of bidegrees $(0,0)$ and $(0,1)$ respectively. The indices run through $\mu = 1, \ldots, \dim M$ and $a = 1, \ldots, \operatorname{rk} E$. We shall use the sign convention of summing the $\mathbb{Z}$ and $\mathbb{Z}_2$ degrees to a total degree $|\cdot|$ and using

$$\varphi^\alpha \varphi^\beta = (-1)^{|\varphi^\alpha| \cdot |\varphi^\beta|} \varphi^\beta \varphi^\alpha, \tag{2.1}$$

for any elements $\varphi$ (coordinates, fields, etc.)

Applying the shift functor $[1,0]$ to supermanifolds, we therefore obtain $\mathbb{Z}$-graded supermanifolds, concentrated in $\mathbb{Z}$-degree 0 and 1. Consider for instance a super-vector bundle $\pi : \mathcal{V} \twoheadrightarrow \mathcal{M}$ over a supermanifold $\mathcal{M}$. Its suspension $\mathcal{V}[1,0]$ in the $\mathbb{Z}$ degree makes it a $(\mathbb{Z} \times \mathbb{Z}_2)$-graded manifold, with local coordinates $x^\alpha = (x^\mu, \theta^a)$ on $\mathcal{M}$, of the same bidegrees as before, and coordinates $a^A = (a^m, \chi^I)$ along the fibres of $\mathcal{V}[1,0]$, of bidegrees $(1,0)$ and $(1,1)$ respectively. Indices $m$ and $I$ run from 1 to each of the ranks of $\mathcal{V}$.[4] In this coordinate system, the most general vector field of degree $(1,0)$ takes the form

$$\mathcal{Q} = a^A \rho_A{}^\alpha(x) \frac{\partial}{\partial x^\alpha} - \tfrac{1}{2} a^B a^C f_{BC}{}^A(x) \frac{\partial}{\partial a^A}, \tag{2.2}$$

where both $\rho_A{}^\alpha(x)$ and $f_{AB}{}^C(x)$ are functions of $x^\alpha$, and hence of $\mathbb{Z}$-degree 0, and of the same parity as $a^A \frac{\partial}{\partial x^\alpha}$ and $a^B a^C \frac{\partial}{\partial a^C}$ respectively. More explicitly, the components of such a vector field take the form

$$\mathcal{Q}^\mu = a^m \rho_m{}^\mu(x, \theta^2) + \chi^I \theta^a \rho_{I\,a}{}^\mu(x, \theta^2), \tag{2.3a}$$

$$\mathcal{Q}^a = a^m \theta^b \rho_{m\,b}{}^a(x, \theta^2) + \chi^I \rho_I{}^a(x, \theta^2) \tag{2.3b}$$

$$\mathcal{Q}^p = -\tfrac{1}{2} a^m a^n f_{mn}{}^p(x, \theta^2) - a^m \chi^I \theta^a f_{m\,I\,a}{}^p(x, \theta^2) - \tfrac{1}{2} \chi^I \chi^J f_{IJ}{}^p(x, \theta^2), \tag{2.3c}$$

$$\mathcal{Q}^I = -\tfrac{1}{2} a^m a^n \theta^a f_{mn\,a}{}^I(x, \theta^2) - a^m \chi^J f_{m\,J}{}^I(x, \theta^2) - \tfrac{1}{2} \chi^J \chi^K \theta^a f_{JK\,a}{}^I(x, \theta^2), \tag{2.3d}$$

where now *all* structure functions that appear depend smoothly on the degree $(0,0)$ coordinate $x^\mu$ and are *even* polynomials in the degree $(0,1)$ coordinates $\theta^a$ — as suggested by the notation $(x, \theta^2)$. The equations resulting from the requirement that $\mathcal{Q}$ is homological,

$$\tfrac{1}{2}[\mathcal{Q}, \mathcal{Q}] = \mathcal{Q}^2 = 0, \tag{2.4}$$

i.e. that it 'squares' to zero, simply read

$$0 = (-1)^{|B|} \rho_A{}^\beta \partial_\beta \rho_B{}^\alpha + (-1)^{|A||B|+|A|} \rho_B{}^\beta \partial_\beta \rho_A{}^\alpha - (-1)^{|A||B|} f_{AB}{}^C \rho_C{}^\alpha, \tag{2.5a}$$

$$0 = (-1)^{|B|(|C|+1)} \left( \rho_B{}^\alpha \partial_\alpha f_{CD}{}^A - (-1)^{|D|} f_{DB}{}^E f_{CE}{}^A \right)$$
$$+ (-1)^{|C|(|D|+1)} \left( \rho_C{}^\alpha \partial_\alpha f_{DB}{}^A - (-1)^{|B|} f_{BC}{}^E f_{DE}{}^A \right) \tag{2.5b}$$
$$+ (-1)^{|D|(|B|+1)} \left( \rho_D{}^\alpha \partial_\alpha f_{BC}{}^A - (-1)^{|C|} f_{CD}{}^E f_{BE}{}^A \right),$$

---

[4]This is a double vector bundle structure [29]. For instance, if $\mathcal{V}$ is the tangent bundle over $E$, then $TE$ is a vector bundle over $E$ and also a vector bundle over $TM$, each having its own rank.

where $|A|$ denotes the *total degree* of the fibre coordinate $a^A$. These are the defining conditions of a Lie superalgebroid, that is to say, a Lie algebroid structure on a super-vector bundle. In other words, the standard result of Vaintrob establishing a bijective correspondence between NQ-manifolds (graded manifolds equipped with a homological vector field and concentrated in non-negative degrees) of degree 1 and Lie algebroids [30] carries over to the category of supermanifolds, as shown by Mehta in [24, section 2.4] and [25, section 4]. An expanded form of these conditions is presented in appendix A.1.

Apart from a $\mathbb{Z}$-degree 1 homological vector field, it is also possible to consider one with $\mathbb{Z}_2$-degree 1. The most general vector field of degree $(0,1)$ reads

$$\mathcal{Q}_S = V^\alpha(x)\,\frac{\partial}{\partial x^\alpha} + a^A\,U_A{}^B(x)\,\frac{\partial}{\partial a^B}\,, \tag{2.6}$$

where the components $V^\alpha(x)$ and $U_A{}^B(x)$ are functions of the $\mathbb{Z}$-degree 0 coordinates $x^\alpha$, and have parity *opposite* to $\frac{\partial}{\partial x^\alpha}$ and $a^A\frac{\partial}{\partial a^B}$ respectively. The subscript $S$ refers to "supersymmetry", since this vector field will generate supersymmetry transformations for the sigma models we will consider in later sections. More explicitly, its components can be parametrized as

$$\mathcal{Q}_S^\mu = \theta^a\,t_a{}^\mu(x,\theta^2)\,, \tag{2.7a}$$

$$\mathcal{Q}_S^a = V^a(x,\theta^2)\,, \tag{2.7b}$$

$$\mathcal{Q}_S^m = a^n\theta^a\,U_{n\,a}{}^m(x,\theta^2) + \chi^I\,W_I{}^m(x,\theta^2)\,, \tag{2.7c}$$

$$\mathcal{Q}_S^I = a^m\,Y_m{}^I(x,\theta^2) + \chi^J\theta^a\,Z_{J\,a}{}^I(x,\theta^2)\,, \tag{2.7d}$$

where, as previously, all structure functions depend in a smooth manner on the degree $(0,0)$ coordinates and are even polynomially in the degree $(0,1)$ ones. Since this vector field is also of total degree 1 on $\mathcal{V}$, it also makes sense to require it to be homological, which yields the conditions

$$0 = V^\beta\,\frac{\partial}{\partial x^\beta}V^\alpha\,, \tag{2.8a}$$

$$0 = U_B{}^C\,U_C{}^A + (-1)^{|B|}\,V^\alpha\,\frac{\partial}{\partial x^\alpha}U_B{}^A\,. \tag{2.8b}$$

These are analyzed in more detail in appendix A.2. A first thing one can notice is that if $\mathcal{Q}_S$ is homological on $\mathcal{V}$, it induces a homological vector field on $\mathcal{M}$ via $\mathcal{Q}_\mathcal{M} := \pi_*\mathcal{Q}_S$ where $\pi$ is the projection of $\mathcal{V}$ onto $\mathcal{M}$. In the above coordinate system, it is simply given by $V^\alpha\frac{\partial}{\partial x^\alpha}$. In other words, $(\mathcal{V},\mathcal{Q}_S)$ is a *differential* super-vector bundle, meaning a super-vector bundle whose total space and base are both equipped with a homological vector field, and such that the projection preserves the latter. One can think of this as the counterpart of a *differential graded* (dg for short) vector bundle (also known as $\mathcal{Q}$-bundle [24, 26], see e.g. [31–34] for recent developments in the context of gauge theories).

## 2.2 Representations up to homotopy

$\mathcal{Q}$-bundles are closely related to the notion of representation up to homotopy [23]. In the simplest case, the data $(\mathcal{V},\mathcal{Q}_S)$ can be considered as a *representation* of the $\mathcal{Q}$-supermanifold $(\mathcal{M},\mathcal{Q}_\mathcal{M})$, in the sense of [35, section 2], and based on Vaintrob's insight. Indeed, recall that

the paradigmatic example of a $\mathcal{Q}$-manifold is the suspension $E[1]$ of a Lie algebroid $E \twoheadrightarrow M$, with its Chevalley-Eilenberg differential as homological vector field. We denote by $\rho$ and $[\cdot, \cdot]$ the anchor and the Lie bracket of the Lie algebroid, and by $\rho_a{}^\mu$ and $f_{ab}{}^c$ their components in a local basis. A representation of this Lie algebroid is another vector bundle $V \twoheadrightarrow M$ equipped with a *flat $E$-on-$V$ connection*[5] $\nabla^E$. Correspondingly, the $\mathcal{Q}$-manifold $E[1] \oplus V$ with the Chevalley-Eilenberg differential associated with the representation $V$ as $\mathcal{Q}$-vector defines a $\mathcal{Q}$-bundle over $E[1]$. In this case, we consider the homological vector field with components

$$\mathcal{Q}_S^\mu = \theta^a\, \rho_a{}^\mu(x)\,, \qquad \mathcal{Q}_S^a = -\tfrac{1}{2}\,\theta^b\theta^c\, f_{bc}{}^a(x)\,, \qquad \mathcal{Q}_S^m = 0\,, \qquad \mathcal{Q}_S^I = -\chi^J \theta^a\, \Gamma_{a\,J}^I(x)\,, \qquad (2.9)$$

where $\Gamma_{a\,J}^I$ are the components of the flat $E$-on-$V$ connection. A direct computation yields

$$(\mathcal{Q}_S)^2 = 0 \qquad \Longleftrightarrow \qquad \begin{cases} 0 = \rho_{[a}{}^\nu\, \partial_\nu \rho_{b]}{}^\mu - \tfrac{1}{2}\, f_{ab}{}^c\, \rho_c{}^\mu\,, \\ 0 = \rho_{[a}{}^\mu\, \partial_\mu f_{bc]}{}^d + f_{[ab}{}^e f_{c]e}{}^d\,, \\ 0 = \rho_{[a}{}^\mu\, \partial_\mu \Gamma_{b]J}^I - \Gamma_{[a\,J}^K\, \Gamma_{b]K}^I - \tfrac{1}{2}\, f_{ab}{}^c\, \Gamma_{c\,J}^I =: \tfrac{1}{2} R_{ab}{}^I{}_J\,, \end{cases} \qquad (2.10)$$

the first two equations being the condition that the anchor defines a morphism of Lie algebras and the Jacobi identity for $E$ (which define a Lie algebroid), while the third is the vanishing of the curvature $R^{\nabla^E}$ of the connection $\nabla^E$ with components $R_{ab}{}^I{}_J$ (which defines a representation).

More generally, one can encode representations *up to homotopy* in $\mathcal{Q}_S$. We first recall the original definition from [23] in our notation (for more details, see e.g. [36, section 2.1]). Given the Lie algebroid $E$, a representation up to homotopy is a graded vector bundle $\mathcal{V}$ together with

(i) An operator $\partial : \mathcal{V} \to \mathcal{V}$ of degree 1 that turns $(\mathcal{V}, \partial)$ into a complex.

(ii) An $E$-connection $\nabla^E$ on the complex $(\mathcal{V}, \partial)$.

(iii) A total degree 1, endomorphism-valued 2-form of $E$, $\omega_2 \in \Omega^2(E; \mathrm{End}^{-1}(\mathcal{V}))$ such that the curvature of $\nabla^E$ is counterbalanced by $[\partial, \omega_2]$ as in

$$[\partial, \omega_2] + R^{\nabla^E} = 0\,, \qquad (2.11)$$

where the brackets $[\cdot, \cdot]$ above denote the graded-commutator.

(iv) A collection of total degree 1, endomorphism-valued $n$-forms on $E$ with $n > 2$, denoted $\omega_n \in \Omega^n(E; \mathrm{End}^{1-n}(\mathcal{V}))$ which satisfy for every $n > 2$ the recursive relations

$$[\partial, \omega_n] + \mathrm{d}_{\nabla^E}\, \omega_{n-1} + \omega_2 \circ \omega_{n-2} + \cdots + \omega_{n-2} \circ \omega_2 = 0\,, \qquad (2.12)$$

---

[5]We recall from the introduction that the statement "$E$-on-$V$ connection" refers to a connection $\nabla^E :$ $\Gamma(E) \times \Gamma(V) \to \Gamma(V)$ with the usual properties of linearity, homogeneity and Leibniz rule for functions. We will simply use the symbol $\nabla$ in case $E = TM$ and refer to this as an "ordinary" connection on $V$. Similarly, we will use $\nabla^*$ in case $E = T^*M$ and refer to this as a "contravariant" connection (not to be confused with the dual connection, which instead refers to connections on $V^*$).

where $\mathrm{d}_{\nabla^E}$ is the differential on the space $\Omega(E, \mathcal{V}) = \Gamma(\wedge^\bullet E^* \otimes \mathcal{V})$ of $\mathcal{V}$-valued differential forms on $E$ defined via the Koszul formula using the $E$-connection $\nabla^E$:

$$
\begin{aligned}
(\mathrm{d}_{\nabla^E}\, \eta)(e^{(1)}, \ldots, e^{(n+1)}) = {}& \sum_{i<j}(-1)^{i+j}\eta([e^{(i)}, e^{(j)}], e^{(1)}, \ldots, e^{(n+1)}) \\
& + \sum_{i=1}^{n+1}(-1)^{i+1}\nabla^E_{e^{(i)}}\big(\eta(e^{(1)}, \ldots, e^{(n+1)})\big).
\end{aligned}
\tag{2.13}
$$

It was shown in [23] that this definition is equivalent to the pair $(\mathcal{V}, \mathrm{D})$ with the degree 1 structure operator $\mathrm{D} : \Omega(E, \mathcal{V}) \to \Omega(E, \mathcal{V})$ being a differential, namely $\mathrm{D}^2 = 0$, and a graded derivation, namely for $\omega \in \Omega^n(E)$ and $\eta \in \Omega(E; \mathcal{V})$ it satisfies

$$
\mathrm{D}(\omega\, \eta) = \mathrm{d}_E\, \omega\, \eta + (-1)^n \omega\, \mathrm{D}\, \eta,
\tag{2.14}
$$

where $\mathrm{d}_E$ is the differential on the Lie algebroid given through the Koszul formula

$$
\begin{aligned}
(\mathrm{d}_E\, \omega)(e^{(1)}, \ldots, e^{(n+1)}) = {}& \sum_{i<j}(-1)^{i+j}\omega([e^{(i)}, e^{(j)}], e^{(1)}, \ldots, e^{(n+1)}) \\
& + \sum_{i=1}^{n+1}(-1)^{i+1}\mathcal{L}_{\rho(e^{(i)})}\big(\omega(e^{(1)}, \ldots, e^{(n+1)})\big).
\end{aligned}
\tag{2.15}
$$

The relation of the two definitions is given through

$$
\mathrm{D}\, \eta = \partial\eta + \mathrm{d}_{\nabla^E}\, \eta + \sum_{n \geq 2}\omega_n \wedge \eta.
\tag{2.16}
$$

The usual notion of representation of a Lie algebroid that we recalled earlier is included in this definition in degree 0.

The adjoint and coadjoint representations of a Lie algebroid $E$ are the paradigmatic examples of this notion. Although it will turn out that the coadjoint one will be related to the field theories we will study below, we begin with the adjoint representation, which is defined as follows: given any $TM$-on-$E$ connection $\nabla$, one can construct an $E$-on-$(E[0] \oplus TM[-1])$ connection $\overline{\nabla}^E$, via

$$
\overline{\nabla}^E_e \begin{pmatrix} e' \\ X \end{pmatrix} = \begin{pmatrix} \nabla_{\rho(e')}e + [e, e'] \\ \rho(\nabla_X e) + [\rho(e), X] \end{pmatrix},
\tag{2.17}
$$

for any vector field $X \in \Gamma(TM)$ and any pair of sections $e, e' \in \Gamma(E)$, which is called the *basic connection*. Note that, as suggested by the notation, the direct sum $E[0] \oplus TM[-1]$ is considered as a *complex* of vector bundles over $M$, with $E$ in degree 0 and $TM$ in degree 1 and with degree 1 differential given by the anchor $\rho : E \longrightarrow TM$. Denoting by

$$
\Gamma^b_{\mu a}\, \mathfrak{e}_b = \nabla_{\partial_\mu}\mathfrak{e}_a,
\tag{2.18}
$$

the components of $\nabla$ in a basis $\{\mathfrak{e}_a\}$ of the fibres of $E$, the components of the basic connections read

$$
\overline{\Gamma}^c_{ab} = \rho_b{}^\mu\, \Gamma^c_{\mu a} + f_{ab}{}^c, \qquad\qquad \overline{\Gamma}^\nu_{a\mu} = \Gamma^b_{\mu a}\, \rho_b{}^\nu - \partial_\mu \rho_a{}^\nu.
\tag{2.19}
$$

These suggest that we should choose $E[0] \oplus TM[-1]$ as the graded vector bundle $\mathcal{V}$ in the definition of a representation up to homotopy, together with $\rho = \partial$ and the basic connection $\overline{\nabla}^E$ as the $E$-connection on the complex. That this choice is compatible with the complex is guaranteed by the fact that the basic connection commutes with the anchor:

$$[\overline{\nabla}^E, \rho] := \overline{\nabla}^E \circ \rho - \rho \circ \overline{\nabla}^E = 0. \tag{2.20}$$

This is easily proven using the definition of the basic connection and the fact that $\rho$ is a (Lie algebra) homomorphism.

The basic connection is not flat in general, so that it does not allow one to define a representation of $E$ on $E[0] \oplus TM[-1]$ in the usual sense, however it satisfies

$$R^{\overline{\nabla}^E} + [\rho, S^\nabla] = 0, \tag{2.21}$$

where $R^{\overline{\nabla}^E}$ is the curvature of $\overline{\nabla}^E$ and

$$S^\nabla : \Gamma(E \wedge E) \otimes \Gamma(TM) \longrightarrow \Gamma(E), \tag{2.22}$$

is called the basic curvature, and it is defined as

$$S^\nabla(e, e')X = \nabla_{\overline{\nabla}^E_e X} e' - \nabla_{\overline{\nabla}^E_{e'} X} e + \nabla_X [e, e'] - [\nabla_X e, e'] - [e, \nabla_X e'], \tag{2.23}$$

for any $e, e' \in \Gamma(E)$ and $X \in \Gamma(TM)$. The concept was implicitly introduced in ref. [37] in the study of Cartan connections for Lie algebroids (see also [38]) and later defined in ref. [23] in the present context. Eq. (2.21) is a short version of the following two equations:

$$R^{\overline{\nabla}^E}(e, e')X = -\rho(S^\nabla(e, e')X), \tag{2.24a}$$

$$R^{\overline{\nabla}^E}(e, e')e'' = -S^\nabla(e, e')\rho(e''), \tag{2.24b}$$

for $e, e', e'' \in \Gamma(E)$ and $X \in \mathfrak{X}(M)$. In the above basis, the components of the basic curvature

$$S_{ab\,\mu}{}^c \mathfrak{e}_c := S^\nabla(\mathfrak{e}_a, \mathfrak{e}_b)\partial_\mu, \tag{2.25}$$

are given by

$$S_{ab\,\mu}{}^c = \partial_\mu f_{ab}{}^c + f_{ab}{}^d \Gamma^c_{\mu d} + 2\Gamma^d_{\mu[a} f_{b]d}{}^c - 2\rho_{[a}{}^\nu \partial_\nu \Gamma^c_{\mu b]} - 2\left(\partial_\mu \rho_{[a}{}^\nu - \rho_d{}^\nu \Gamma^d_{\mu[a}\right) \Gamma^c_{\nu b]}. \tag{2.26}$$

These suggest that we should make the choice that the 2-form $\omega_2$ is equal to the basic curvature of the ordinary connection $\nabla$ on $E$, $\omega_2 = S^\nabla$, since this satisfies the requirement (2.11) in item (iii) of the definition. Finally, the basic curvature is $d_{\overline{\nabla}^E}$-closed [23],

$$d_{\overline{\nabla}^E} S^\nabla = 0, \tag{2.27}$$

and the requirement (2.12) is satisfied with all $\omega_{n>2} = 0$. This representation up to homotopy, alternatively presented via the structure operator as

$$D = \rho + \overline{\nabla}^E + S^\nabla, \tag{2.28}$$

is called the adjoint representation of the Lie algebroid $E$.

Returning to the homological vector field $\mathcal{Q}_S$, we can essentially identify it with the structure operator, a *differential* on the space of '$E$-forms' valued in $E \oplus TM[-1]$, that is the space of sections $\Gamma(\wedge E^* \otimes (E \oplus TM[-1]))$. The latter can be identified with the space of sections of the super-vector bundle $E[0,1] \times (E[0,0] \oplus T[0,1]M) \twoheadrightarrow E[0,1]$, where as before the second entry of the suspension functor is reserved for the $\mathbb{Z}_2$-grading. Upon shifting the fibres of this super-vector bundle by one in the $\mathbb{Z}$-degree, as outlined before, we end up with

$$\pi : E[0,1] \times (E[1,0] \oplus T[1,1]M) \twoheadrightarrow E[0,1], \tag{2.29}$$

which becomes a differential super-vector bundle, with homological vector field given by

$$\begin{aligned} \mathcal{Q}_S = {} & \theta^a \, \rho_a{}^\mu(x) \, \tfrac{\partial}{\partial x^\mu} - \tfrac{1}{2} \, \theta^b \theta^c \, f_{bc}{}^a(x) \, \tfrac{\partial}{\partial \theta^a} \\ & - \left( \theta^b a^c \, \overline{\Gamma}_{bc}^a(x) + \tfrac{1}{2} \, \theta^b \theta^c \chi^\mu S_{bc\,\mu}{}^a \right) \tfrac{\partial}{\partial a^a} + \left( a^a \, \rho_a{}^\mu - \theta^a \chi^\nu \, \overline{\Gamma}{}^\mu_{a\,\nu} \right) \tfrac{\partial}{\partial \chi^\mu} \, , \end{aligned} \tag{2.30}$$

and where $\theta^a$ are degree $(0,1)$ coordinates on $E[0,1]$, while $a^a$ are degree $(1,0)$ coordinates on $E[1,0]$ and $\chi^\mu$ are degree $(1,1)$ coordinates along the fibres of $T[1,1]M$. The projection of this vector field on the base, which is $E[0,1]$ here, concretely amounts to setting the fibre coordinates $a^a$ and $\chi^\mu$ of the super-vector bundle to zero, which leaves only the first line of the above equation. Explicitly,

$$\pi_* \mathcal{Q}_S = \mathcal{Q}_E \in \Gamma(TE[0,1]) \qquad \text{with} \qquad \mathcal{Q}_E = \theta^a \, \rho_a{}^\mu(x) \, \tfrac{\partial}{\partial x^\mu} - \tfrac{1}{2} \, \theta^b \theta^c \, f_{bc}{}^a(x) \, \tfrac{\partial}{\partial \theta^a} \, . \tag{2.31}$$

Finally, to confirm that $\mathcal{Q}_S$, as presented in local coordinates in (2.30), is the structure operator for the representation up to homotopy, we should compute its square. It is useful to restructure it as follows

$$\mathcal{Q}_S = \mathcal{Q}_S^{(0)} + \mathcal{Q}_S^{(1)} + \mathcal{Q}_S^{(2)} \, , \tag{2.32}$$

where we decomposed the vector field in three pieces according to the number of graded generators, 0, 1 and 2 respectively, (called the *arity* [39]):

$$\mathcal{Q}_S^{(0)} = a^a \rho_a{}^\mu \tfrac{\partial}{\partial \chi^\mu} \, , \tag{2.33a}$$

$$\mathcal{Q}_S^{(1)} = \theta^a \, \rho_a{}^\mu(x) \, \tfrac{\partial}{\partial x^\mu} - \tfrac{1}{2} \, \theta^b \theta^c \, f_{bc}{}^a(x) \, \tfrac{\partial}{\partial \theta^a} - \theta^b a^c \, \overline{\Gamma}_{bc}^a(x) \tfrac{\partial}{\partial a^a} - \theta^a \chi^\nu \, \overline{\Gamma}{}^\mu_{a\,\nu} \tfrac{\partial}{\partial \chi^\mu} \, , \tag{2.33b}$$

$$\mathcal{Q}_S^{(2)} = -\tfrac{1}{2} \, \theta^b \theta^c \chi^\mu S_{bc\,\mu}{}^a \tfrac{\partial}{\partial a^a} \, . \tag{2.33c}$$

Note immediately that $(\mathcal{Q}_S^{(0)})^2 = 0 = (\mathcal{Q}_S^{(2)})^2$. Further calculation shows that:

$$[\mathcal{Q}_S^{(0)}, \mathcal{Q}_S^{(1)}] = 0 \qquad \Longleftrightarrow \qquad \overline{\Gamma}^\mu_{a\nu} \rho_b{}^\nu - \overline{\Gamma}^c_{ab} \rho_c{}^\mu + \rho_a{}^\nu \partial_\nu \rho_b{}^\mu = 0 \, . \tag{2.34}$$

This condition is the local coordinate expression of eq. (2.20), i.e. the basic connection commuting with the anchor. Crucially, we also find that

$$[\mathcal{Q}_S^{(1)}, \mathcal{Q}_S^{(1)}] + [\mathcal{Q}_S^{(0)}, \mathcal{Q}_S^{(2)}] = 0 \Longleftrightarrow R_{ab}{}^\nu{}_\mu + \rho_c{}^\nu S_{ab\,\mu}{}^c = 0, \quad R_{ab}{}^d{}_c + \rho_c{}^\mu S_{ab\,\mu}{}^d = 0, \tag{2.35}$$

which are the coordinate expressions that correspond to eq. (2.21) with the curvature components being those of the basic connection. Finally, the closure condition (2.27) is found in:

$$[\mathcal{Q}_S^{(1)}, \mathcal{Q}_S^{(2)}] = 0 \Longleftrightarrow \rho_{[a}{}^\nu \partial_{|\nu|} S_{bc]\mu}{}^d - f_{[ab}{}^e S_{|e|c]\,\mu}{}^d - \overline{\Gamma}^\nu_{[a|\mu|} S_{bc]\nu}{}^d + S_{[ab|\mu|}{}^e \overline{\Gamma}^d_{c]e} = 0, \tag{2.36}$$

with the last equation being its component form.

Similarly to Lie algebras, there exists a dual representation to the adjoint, the coadjoint representation. In this case, we consider the dual complex

$$T^*M \xrightarrow{\rho^*} E^*, \qquad (2.37)$$

with the cotangent bundle $T^*M$ in degree $-1$ and the dual bundle $E^*$ in degree 0. The degree 1 operator is now the dual anchor, with components $\rho^\mu{}_a$, which are equal to the components $\rho_a{}^\mu$ of the anchor. The connection in the present case is the dual one to the basic connection, which is an $E$-on-$(T^*M[1] \oplus E^*[0])$ connection. It is defined using the canonical duality of the bundles, collectively denoted as $\langle \cdot, \cdot \rangle$ for any vector bundle, through

$$\langle \overline{\nabla}^{E^*}_{e'} e^*, e \rangle + \langle e^*, \overline{\nabla}^E_{e'} e \rangle = \langle \mathrm{d}_E \langle e, e^* \rangle, e' \rangle, \qquad (2.38)$$

for any $e, e' \in \Gamma(E)$ and $e^* \in \Gamma(E^*)$. The components of this dual basic connection are opposite to the ones of the straight basic connection, namely $-\overline{\Gamma}^c_{ab}$ and $-\overline{\Gamma}^\nu_{a\mu}$ as they appear in (2.19). Finally, the dual of the basic curvature is the tensor operator

$$(S^\nabla)^* : \Gamma(E \wedge E) \otimes \Gamma(E^*) \longrightarrow \Gamma(T^*M), \qquad (2.39)$$

defined via

$$(S^\nabla)^*(e, e')e^* = -e^* \circ S^\nabla(e, e'). \qquad (2.40)$$

In components, this tells us that $S_{ab}{}^c{}_\mu = -S_{ab\mu}{}^c$, where the left hand side refers to the dual basic curvature and the right hand side to the straight one. The statement that these data collect into a representation up to homotopy is equivalent to the dual properties to the ones of the adjoint representation, namely

$$[\overline{\nabla}^{E^*}, \rho^*] = 0, \qquad (2.41\text{a})$$

$$R^{\overline{\nabla}^{E^*}} + [\rho^*, (S^\nabla)^*] = 0, \qquad (2.41\text{b})$$

$$\mathrm{d}_{\overline{\nabla}^{E^*}}(S^\nabla)^* = 0. \qquad (2.41\text{c})$$

We recall that the curvature of the dual connection is given as

$$\langle R^{\overline{\nabla}^{E^*}}(e, e')e^*, e'' \rangle = -\langle e^*, R^{\overline{\nabla}^E}(e, e')e'' \rangle, \qquad (2.42)$$

and similarly for the other leg of the connection. To encode the coadjoint representation in a homological vector field, we follow the same logic as before, replacing the differential super-vector bundle (2.29) with

$$\pi : E[0,1] \times (T^*[1,0]M \oplus E^*[1,1]) \twoheadrightarrow E[0,1]. \qquad (2.43)$$

The corresponding coordinates are $x^\mu, \theta^a$ of bidegrees $(0,0)$ and $(0,1)$ and $a_\mu, \chi_a$ of bidegrees $(1,0)$ and $(1,1)$ respectively and we can write the homological vector field

$$\begin{aligned} \mathcal{Q}_S = {} & \theta^a \, \rho_a{}^\mu(x) \, \tfrac{\partial}{\partial x^\mu} - \tfrac{1}{2} \, \theta^b \theta^c \, f_{bc}{}^a(x) \, \tfrac{\partial}{\partial \theta^a} \\ & + \left( \theta^b a_\nu \, \overline{\Gamma}^\nu_{b\mu}(x) + \tfrac{1}{2} \theta^b \theta^c \chi_a S_{bc}{}^a{}_\mu \right) \tfrac{\partial}{\partial a_\mu} - \left( a_\mu \, \rho^\mu{}_a - \theta^c \chi_b \, \overline{\Gamma}^b_{ca} \right) \tfrac{\partial}{\partial \chi_a}, \end{aligned} \qquad (2.44)$$

where we recognize the components of the dual anchor, the dual basic curvature and the dual basic connection (the latter written in terms of the straight components, but with the opposite sign, as they should be). That this vector field is homological is proven using the same local coordinate formulas as before and the relation between straight and dual objects.

## 3 Poisson supermanifolds

### 3.1 General structure

For every ordinary Poisson manifold $(M, \Pi)$, $\Pi \in \Gamma(\wedge^2 TM)$ denoting the Poisson bivector, its cotangent bundle $T^*M$ has a canonical Lie algebroid structure. The anchor is given by the induced map $\Pi^\sharp : T^*M \to TM$ with $\Pi^\sharp(\eta) = \Pi(\eta, \cdot)$ for $\eta \in \Gamma(T^*M)$, and the Lie bracket is given by the Koszul-Schouten bracket on 1-forms

$$[\eta, \eta']_{\mathrm{KS}} = \mathcal{L}_{\Pi^\sharp(\eta)} \eta' - \mathcal{L}_{\Pi^\sharp(\eta')} \eta - \mathrm{d}\, \Pi(\eta, \eta') \,. \tag{3.1}$$

In this case we may describe the Lie algebroid in terms of the graded manifold $T^*[1]M$ with a suitable homological vector field. This is a special case since apart from the $\mathcal{Q}$-structure there is also a symplectic structure on the graded manifold, since it is a cotangent bundle. Requiring that the graded symplectic form is invariant under the flow of the homological vector field fixes the latter in terms of the Poisson structure on $M$. In fact, any $\mathbb{Z}$-graded manifold concentrated in degrees 0 and 1, and equipped with a symplectic 2-form of degree 1 invariant under a homological vector field $\mathcal{Q}$ arises in this manner, i.e. is obtained as the shifted cotangent bundle of an ordinary Poisson manifold [7]. We do not provide additional details yet, since we will now address all these in a broader context.

It turns out that these statements naturally carry over in the category of supermanifolds. Adding a bidegree $(1, 0)$ symplectic structure $\omega$ on $\mathcal{V}[1, 0]$ makes it diffeomorphic to the shifted cotangent bundle of $\mathcal{M} = E[0, 1]$ (e.g. [4]), i.e. $\mathcal{V}[1, 0] \cong T^*[1, 0]\mathcal{M}$. Denoting by $p_\alpha$ the fibre coordinate of $\mathbb{Z}$-degree 1, and by $\alpha \equiv |x^\alpha|$ the *total* degree of the coordinate $x^\alpha$ (so that $|p_\alpha| = 1 + \alpha$), the symplectic structure $\omega$ and homological field $\mathcal{Q}$ locally take the form

$$\omega = \mathrm{d}p_\alpha \, \mathrm{d}x^\alpha \,, \qquad \mathcal{Q} = p_\alpha \, \rho^{\alpha\beta} \, \frac{\partial}{\partial x^\beta} - \tfrac{1}{2} \, p_\alpha p_\beta \, f_\gamma{}^{\alpha\beta} \, \frac{\partial}{\partial p_\gamma} \,, \tag{3.2}$$

in Darboux coordinates. The compatibility condition of $\mathcal{Q}$ with $\omega$ yields[6]

$$\mathcal{L}_{\mathcal{Q}}\, \omega = 0 \qquad \Longleftrightarrow \qquad \begin{cases} 0 = \rho^{\alpha\beta} - (-1)^{(\alpha+1)(\beta+1)} \, \rho^{\beta\alpha} \,, \\ f_\gamma{}^{\alpha\beta} = -(-1)^{(\alpha+\beta)\gamma + (\alpha+1)(\beta+1)} \partial_\gamma \rho^{\alpha\beta} \,, \end{cases} \tag{3.3}$$

which tell us that the homological vector field $\mathcal{Q}$ is completely determined by the graded-symmetric part of the 'anchor', and squares to zero if and only if

$$(-1)^{\gamma(\alpha+1)} \, \rho^{\alpha\delta} \, \partial_\delta \rho^{\beta\gamma} + (-1)^{\alpha(\beta+1)} \, \rho^{\beta\delta} \, \partial_\delta \rho^{\gamma\alpha} + (-1)^{\beta(\gamma+1)} \, \rho^{\gamma\delta} \, \partial_\delta \rho^{\alpha\beta} = 0 \,, \tag{3.4}$$

as can be verified upon plugging the components of this $\omega$-compatible $\mathcal{Q}$-vector in the conditions (2.5). The above identity is nothing but the graded version of the Jacobi identity for a super-Poisson bivector on $\mathcal{M}$. Indeed, defining

$$\mathcal{P}^{\alpha\beta} := (-1)^\alpha \rho^{\alpha\beta} \qquad \Longrightarrow \qquad \mathcal{P}^{\beta\alpha} = -(-1)^{\alpha\beta} \, \mathcal{P}^{\alpha\beta} \,, \tag{3.5}$$

the homological $\mathcal{Q}$ reads

$$\mathcal{Q} = (-1)^\alpha \big( p_\alpha \, \mathcal{P}^{\alpha\beta} \, \frac{\partial}{\partial x^\beta} - \tfrac{1}{2} \, (-1)^{\alpha\beta} \, \partial_\gamma \mathcal{P}^{\alpha\beta} \, p_\alpha p_\beta \, \frac{\partial}{\partial p_\gamma} \big) \,, \tag{3.6}$$

---

[6]We suppress a third condition which is trivially satisfied as a consequence of the two equations in (3.3). The expanded form of these conditions appears in appendix A.4.

and the above identity becomes

$$\mathcal{Q}^2 = 0 \iff (-1)^{\gamma\alpha}\,\mathcal{P}^{\alpha\delta}\,\partial_\delta\mathcal{P}^{\beta\gamma} + (-1)^{\alpha\beta}\,\mathcal{P}^{\beta\delta}\,\partial_\delta\mathcal{P}^{\gamma\alpha} + (-1)^{\beta\gamma}\,\mathcal{P}^{\gamma\delta}\,\partial_\delta\mathcal{P}^{\alpha\beta} = 0 \,, \qquad (3.7)$$

which is precisely the Jacobi identity satisfied by the components of a super-Poisson bivector on $\mathcal{M}$. In other words, requiring that $\mathcal{Q}$ is compatible with the canonical symplectic structure $\omega$ on the shifted cotangent bundle $T^*[1,0]\mathcal{M}$ implies that $\mathcal{M}$ is a Poisson supermanifold, i.e. a supermanifold whose algebra of functions is endowed with a structure of Poisson algebra, with Poisson bracket $\{\cdot,\cdot\}$ determined by $\{x^\alpha, x^\beta\} = \mathcal{P}^{\alpha\beta}$. This type of supermanifolds were discussed in e.g. [40], and were also identified in [16] as the target space of the supersymmetric version of the Poisson sigma model.[7]

We can repeat the exercise of requiring the compatibility between the symplectic structure on $T^*[1,0]\mathcal{M}$ and the parity odd vector field $\mathcal{Q}_S$, to find[8]

$$\mathcal{L}_{\mathcal{Q}_S}\omega = 0 \qquad \iff \qquad U^\alpha{}_\beta = (-1)^{\beta(\alpha+1)}\,\frac{\partial}{\partial x^\beta}V^\alpha \,. \qquad (3.8)$$

These conditions, given in more detail in appendix A.5, lead to the following local expression,

$$\mathcal{Q}_S = V^\alpha\,\frac{\partial}{\partial x^\alpha} + (-1)^{\alpha(\beta+1)}\,p_\beta\,\frac{\partial}{\partial x^\alpha}V^\beta\,\frac{\partial}{\partial p_\alpha} \,, \qquad (3.9)$$

which we can recognize as that of the Lie derivative along $\mathcal{Q}_\mathcal{M}$ on *polyvector fields on* $\mathcal{M}$, seen as a homological vector field on $T^*[1,0]\mathcal{M}$. This result is consistent with the idea that a $\mathbb{Z}$-graded supermanifold concentrated in degrees 0 and 1, and equipped with a homological vector field of bidegree $(0,1)$, is a $\mathcal{Q}$ super-vector bundle.

In the rest of this section, we will focus on Poisson supermanifolds. Before presenting examples, let us rewrite the components of the super-Poisson bivector, its associated homological vector field and the corresponding Jacobi identity, in terms of the even and odd coordinates of the supermanifold $\mathcal{M}$, respectively $x^\mu$ and $\theta^a$, as well as their momenta, respectively $a_\mu$ and $\chi_a$. Accordingly, one distinguishes between three types of components of the Poisson bivector, namely

$$\mathcal{P}^{\mu\nu} = -\mathcal{P}^{\nu\mu}\,, \qquad \mathcal{P}^{\mu a} = -\mathcal{P}^{a\mu}\,, \qquad \mathcal{P}^{ab} = \mathcal{P}^{ba}\,, \qquad (3.10)$$

whose dependency on the odd coordinates $\theta^a$ reads

$$\mathcal{P}^{\mu\nu}(x,\theta) = \mathcal{P}^{\mu\nu}(x,\theta^2)\,, \qquad \mathcal{P}^{\mu a}(x,\theta) = \theta^b\,\mathcal{P}^{\mu a}_b(x,\theta^2)\,, \qquad \mathcal{P}^{ab}(x,\theta) = \mathcal{P}^{ab}(x,\theta^2)\,, \quad (3.11)$$

i.e. $\mathcal{P}^{\mu\nu}$ and $\mathcal{P}^{ab}$ are *even* polynomials in $\theta$ whereas $\mathcal{P}^{\mu a}$ are *odd* ones. The associated homological vector field on $T^*[1,0]\mathcal{M}$ is, following (3.6), given by

$$\begin{aligned}
\mathcal{Q} = {}&\left(a_\mu\,\mathcal{P}^{\mu\nu} + \chi_a\,\mathcal{P}^{\nu a}\right)\frac{\partial}{\partial x^\nu} + \left(a_\mu\,\mathcal{P}^{\mu a} - \chi_b\,\mathcal{P}^{ba}\right)\frac{\partial}{\partial\theta^a} \\
&- \tfrac{1}{2}\left(a_\mu a_\nu\,\partial_\lambda\mathcal{P}^{\mu\nu} - 2\,a_\mu\chi_a\,\partial_\lambda\mathcal{P}^{\mu a} + \chi_a\chi_b\,\partial_\lambda\mathcal{P}^{ab}\right)\frac{\partial}{\partial a_\lambda} \\
&- \tfrac{1}{2}\left(a_\mu a_\nu\,\partial_a\mathcal{P}^{\mu\nu} - 2\,a_\mu\chi_b\,\partial_a\mathcal{P}^{\mu b} + \chi_b\chi_c\,\partial_a\mathcal{P}^{bc}\right)\frac{\partial}{\partial\chi_a}\,.
\end{aligned} \qquad (3.12)$$

---

[7]It is worth mentioning that another recent appearance of graded Poisson structures in physics and of their relation to generalised geometry is found in ref. [41].

[8]Here again, we suppressed another condition that comes out of this computation, as it is redundant — trivially satisfied as a consequence of (3.8).

The Jacobi identity (3.7) splits into four identities,

$$0 = \mathcal{P}^{[\mu|\kappa} \partial_\kappa \mathcal{P}^{\nu\lambda]} + \mathcal{P}^{[\mu|a} \partial_a \mathcal{P}^{\nu\lambda]} \,, \tag{3.13a}$$

$$0 = -\mathcal{P}^{\kappa a} \partial_\kappa \mathcal{P}^{\mu\nu} + \mathcal{P}^{ab} \partial_b \mathcal{P}^{\mu\nu} + 2\, \mathcal{P}^{[\mu|\kappa} \partial_\kappa \mathcal{P}^{\nu]a} + 2\, \mathcal{P}^{[\mu|b} \partial_b \mathcal{P}^{\nu]a} \,, \tag{3.13b}$$

$$0 = \mathcal{P}^{\mu\nu} \partial_\nu \mathcal{P}^{ab} + \mathcal{P}^{\mu c} \partial_c \mathcal{P}^{ab} + 2\, \mathcal{P}^{\nu(a} \partial_\nu \mathcal{P}^{\mu|b)} - 2\, \mathcal{P}^{c(a} \partial_c \mathcal{P}^{\mu|b)} \,, \tag{3.13c}$$

$$0 = -\mathcal{P}^{\mu(a} \partial_\mu \mathcal{P}^{bc)} + \mathcal{P}^{d(a} \partial_d \mathcal{P}^{bc)} \,, \tag{3.13d}$$

relating the three different types of components of $\mathcal{P}^{\alpha\beta}$. Note that each one of these four relations can in fact generate more than one identities since they have to be evaluated order by order in the odd coordinates $\theta^a$, as we will see in examples spelled out below. A quick counting shows that eqs. (3.13a) and (3.13c) solely contain even powers of $\theta$, while eqs. (3.13b) and (3.13d) only odd powers.

The homological vector fields $\mathcal{Q}$ and $\mathcal{Q}_S$ being compatible with the symplectic form on $T^*[1,0]\mathcal{M}$, they admit each a Hamiltonian function, that is a degree 2 and parity-even function $\mathcal{H} \in \mathcal{C}_2(T^*[1,0]\mathcal{M})$ and $\mathcal{H}_S \in \mathcal{C}_2(T^*[1,0]\mathcal{M})$ such that

$$\mathcal{Q} = \{\mathcal{H}, -\} \,, \qquad \mathcal{Q}_S = \{\mathcal{H}_S, -\} \tag{3.14}$$

where $\{-,-\}$ denotes the degree $-1$ Poisson bracket on $T^*[1,0]\mathcal{M}$ induced by the canonical symplectic form. More explicitly, this bracket reads

$$\{f, g\} = (-1)^{\alpha |f|+1} \left( \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial p_\alpha} + (-1)^{|f|} \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial x^\alpha} \right) \,, \qquad f, g \in \mathcal{C}(T^*[1,0]\mathcal{M}) \,, \tag{3.15}$$

which leads to

$$\mathcal{H} = \tfrac{1}{2} (-1)^{\alpha(\beta+1)} \mathcal{P}^{\alpha\beta} p_\alpha p_\beta = \tfrac{1}{2} \mathcal{P}^{\mu\nu} a_\mu a_\nu + \mathcal{P}^{\mu a} a_\mu \chi_a + \tfrac{1}{2} \mathcal{P}^{ab} \chi_a \chi_b \,, \tag{3.16a}$$

$$\mathcal{H}_S = (-1)^\alpha V^\alpha p_\alpha = V^\mu a_\mu - V^a \chi_a \,, \tag{3.16b}$$

for the Hamiltonian functions associated with $\mathcal{Q}$ and $\mathcal{Q}_S$ respectively.

### 3.2 Expansion up to second order in the fermionic coordinate

Let us unpack some of the structures encoded in a generic super-Poisson bracket, focusing on the case where all of its components are at most quadratic in the odd coordinates $\theta^a$, that we write in suggestive notation as

$$\mathcal{P}^{\mu\nu}(x,\theta) = \Pi^{\mu\nu}(x) + \tfrac{1}{2} \theta^a \theta^b \mathcal{P}_{ab}{}^{\mu\nu}(x) \,, \tag{3.17a}$$

$$\mathcal{P}^{\mu a}(x,\theta) = \theta^b \Gamma_b^{\mu a}(x) \,, \tag{3.17b}$$

$$\mathcal{P}^{ab}(x,\theta) = g^{ab}(x) + \tfrac{1}{2} \theta^c \theta^d \mathcal{P}_{cd}{}^{ab}(x) \,. \tag{3.17c}$$

Without loss of generality, we will consider the supermanifold to be a shifted vector bundle, i.e. $\mathcal{M} \cong E[0,1]$, with $E \twoheadrightarrow M$ and $M$ being an ordinary (non-graded, smooth) manifold, so that $\theta^a$ can be thought of as a basis of the fibers of $E^*$ and the algebra of functions on $\mathcal{M}$ is isomorphic to the exterior algebra of sections of $E^* \twoheadrightarrow M$. Next, we will study the Jacobi identity for $\mathcal{P}^{\alpha\beta}$, order by order in $\theta^a$.

**Order 0.** We expect two conditions at this order, one from (3.13a) and one from (3.13c). The order 0 component in $\theta$ of (3.13a) reads

$$\Pi^{\kappa[\mu}\,\partial_\kappa \Pi^{\nu\lambda]} = 0\,, \tag{3.18}$$

and therefore simply tells us that $\Pi$ is a Poisson bivector on the base manifold $M$.

To understand the second condition, we need to clarify the geometrical meaning of the mixed component $\mathcal{P}^{\mu a}$. As the notation suggests, the coefficients $\Gamma^{\mu a}_b$ are those of a *contravariant* connection. In other words, the Poisson bracket between a function $f \in \mathscr{C}^\infty(M)$ and a section $e^* \in \Gamma(E^*)$ defines a $T^*M$-on-$E^*$ connection, i.e. a map

$$\nabla^{T^*M} : \Gamma(T^*M) \times \Gamma(E^*) \longrightarrow \Gamma(E^*)\,, \tag{3.19}$$

satisfying all standard properties of homogeneity, linearity and Leibniz rule for a vector bundle connection, via[9]

$$\nabla_{\mathrm{d}f} = \{f, -\}\,, \qquad \forall f \in \mathscr{C}^\infty(M)\,. \tag{3.20}$$

This identification follows from the fact that *any* derivation $\mathcal{D} : \mathscr{C}^\infty(\mathcal{M}) \longrightarrow -(\mathcal{E}^*)$ *factors through* the $\mathscr{C}^\infty(M)$-bimodule of 1-forms $\Omega^1(M)$ via the de Rham differential $\mathrm{d}$. To be more precise, given the derivation $\mathcal{D}$, there exists a (unique, up to an isomorphism) $\mathscr{C}^\infty(M)$-linear map $\phi : \Omega^1(M) \longrightarrow \Gamma(E^*)$ such that the diagram

$$\begin{array}{ccc}
\mathscr{C}^\infty(M) & \xrightarrow{\ \mathrm{d}\ } & \Omega^1(M) \\
& \searrow{\scriptstyle \mathcal{D}} & \ \downarrow{\scriptstyle \phi} \\
& & \Gamma(E^*)
\end{array} \tag{3.21}$$

commutes (for more details about this property, in more general contexts, see e.g. [42, 43] and references therein).

Now let us come back to the Poisson bracket between a function and a section $e^* \in \Gamma(E^*)$, and consider the map defined by $\mathcal{D}_{e^*} := \{-, e^*\} : \mathscr{C}^\infty(M) \to \Gamma(E^*)$. Since the Poisson bracket obeys the Leibniz rule in both arguments, this map is a derivation of the $\mathscr{C}^\infty(M)$-bimodule $\Gamma(E^*)$, and hence by the previous universal property, there exists a $\mathscr{C}^\infty(M)$-linear endomorphism $\phi_{e^*} : \Omega^1(M) \to \Gamma(E^*)$ such that $\mathcal{D}_{e^*}(f) = \{f, e^*\} = \phi_{e^*}(\mathrm{d}f)$, for any $f \in \mathscr{C}^\infty(M)$ and $e^* \in \Gamma(E^*)$. The Leibniz rule of the Poisson bracket also yields $\mathcal{D}_{f\,e^*}(g) = f\,\mathcal{D}_{e^*}(g) + \{g, f\}\,e^*$ when multiplying $e^*$ by a function $f$. The second term on the right hand side can be written as $\{g, f\} = \Pi^\sharp(\mathrm{d}g)(f)$, i.e. recognized as the action on $f$ of the vector field given by the anchor of the cotangent Lie algebroid applied to $\mathrm{d}g$. In summary, $\phi_{e^*}(\mathrm{d}f)$ is $\mathscr{C}^\infty(M)$-linear in $\mathrm{d}f$ and obeys the Leibniz rule in $e^*$ with respect to the cotangent Lie algebroid anchor, which are the defining properties of a $T^*M$-on-$E^*$ connection, hence the identification (3.19).[10]

---

[9]In accordance with our general practice in this paper, we simplify the notation for the connection when it is clear from its entries. Therefore $\nabla_{\mathrm{d}f}$ refers to $\nabla^{T^*M}$, as it is clear from its 1-form entry $\mathrm{d}f$.

[10]Note that this type of connection is known to arise naturally as the semiclassical data of the formal deformation (meaning first order) of a bimodule over an algebra, which is itself deformed (and hence naturally gives rise to a Poisson structure), see [44–48].

Given a basis $\{\mathfrak{e}^a\}$ of the fiber of $E^*$, the components of $\nabla^{T^*M}$ are defined as

$$\nabla_{\mathrm{d}x^\mu}\mathfrak{e}^a = \Gamma_b^{\mu a}\,\mathfrak{e}^b\,, \tag{3.22}$$

and its action on a section $e^* = e_a^*\,\mathfrak{e}^a \in \Gamma(E^*)$ for $\eta \in \Omega^1(M)$ reads

$$\nabla_\eta e^* = \eta_\mu\left(\Pi^{\mu\nu}\partial_\nu e_a^* + \Gamma_a^{\mu b}\,e_b^*\right)\mathfrak{e}^a \equiv \eta_\mu\,\nabla^\mu e_a^*\,\mathfrak{e}^a\,, \tag{3.23}$$

which is exactly what one finds when computing $\{f, e^*\}$ upon identifying the section $e^*$ with the odd function $e_a^*(x)\,\theta^a$. The curvature of such a connection is defined as

$$R^{\nabla^{T^*M}}(\eta, \eta')e^* := [\nabla_\eta, \nabla_{\eta'}]\,e^* - \nabla_{[\eta,\eta']_{\mathrm{KS}}}e^*\,, \qquad \eta, \eta' \in \Omega^1(M)\,, \quad e^* \in \Gamma(E^*)\,, \tag{3.24}$$

whose components are given by

$$R^{\mu\nu\,a}{}_b := \langle R^{\nabla^{T^*M}}(\mathrm{d}x^\mu, \mathrm{d}x^\nu)\mathfrak{e}^a, \mathfrak{e}_b\rangle = 2\left(\Pi^{[\mu|\lambda}\partial_\lambda\Gamma_b^{\nu]a} + \Gamma_b^{[\mu|c}\Gamma_c^{\nu]|a} - \tfrac{1}{2}\partial_\lambda\Pi^{\mu\nu}\Gamma_b^{\lambda a}\right)\,, \tag{3.25}$$

with $\mathfrak{e}_a$ a basis of the fibers of $E$, and $\langle\cdot,\cdot\rangle$ the canonical pairing between $E$ and $E^*$. Note that any ordinary connection $\nabla$ on $E$, meaning a $TM$-on-$E$ linear connection, gives rise to a $T^*M$-on-$E^*$ connection by taking

$$\overset{\bullet}{\nabla}{}^{T^*M}_\eta = \nabla^*_{\Pi^\sharp(\eta)}\,, \qquad \eta \in \Omega^1(M)\,, \tag{3.26}$$

with $\nabla^*$ the dual connection of $\nabla$. We denoted this connection with a bullet to highlight the fact that it is canonically induced by an ordinary connection, since not all such connections arise this way. Concretely, this means taking

$$\Gamma_b^{\mu a} = \overset{\bullet}{\Gamma}{}_b^{\mu a} := -\Pi^{\mu\nu}\,\Gamma_{\nu b}^a\,, \tag{3.27}$$

in which case,

$$R^{\overset{\bullet}{\nabla}} = R^\nabla(\Pi^\sharp, \Pi^\sharp) \qquad \Longleftrightarrow \qquad R^{\mu\nu\,a}{}_b \equiv \Pi^{\mu\kappa}\Pi^{\nu\lambda}\,R_{\kappa\lambda}{}^a{}_b\,, \tag{3.28}$$

with $R^\nabla$ being the curvature of $\nabla$, as noted in e.g. [14, 49].

We are now ready to turn our attention to the degree 0 component of eq. (3.13c), which we can write as

$$\nabla^\mu g^{ab} \equiv \Pi^{\mu\nu}\,\partial_\nu g^{ab} - 2\,\Gamma_c^{\mu(a}\,g^{b)c} = 0\,, \tag{3.29}$$

i.e. we recognize it as the condition that the symmetric tensor $g \in \Gamma(S^2 E)$ be covariantly constant with respect to the dual of the $T^*M$-on-$E^*$ connection identified before. Note that we have not made any assumption on the nondegeneracy of this symmetric tensor. Quite on the contrary, we will see later that in most of the examples we will assume that it vanishes completely. For the induced connection $\nabla^{T^*M} = \overset{\bullet}{\nabla}$, this condition boils down to

$$\overset{\bullet}{\nabla}{}^\mu g^{ab} := \Pi^{\mu\nu}\nabla_\nu g^{ab} = 0\,, \tag{3.30}$$

i.e. $g$ need not be preserved 'on the nose' by $\nabla$, but its covariant derivative should sit in the kernel of $\Pi^\sharp$.

**Splitting the supermanifold.** At this point, a couple of remarks are in order. First, note that given a $TM$-on-$E$ connection $\nabla$, any contravariant connection on $E$ can be written as

$$\nabla^{T^*M} = \overset{\bullet}{\nabla}{}^{T^*M} + \phi\,, \qquad \text{with} \qquad \phi : \Omega^1(M) \longrightarrow \Gamma\big(\text{End}(E^*)\big)\,, \tag{3.31}$$

since the difference between any two linear connections is a tensor. In other words, we can always choose the induced connection $\overset{\bullet}{\nabla}$ as reference point (on the affine space of contravariant connections), and parametrize any other contravariant connection $\nabla^{T^*M}$ 'in reference to the latter', i.e. by specifying the $\text{End}(E^*)$-valued vector field $\phi \in \Gamma(TM \otimes \text{End}\, E^*)$. This suggests that we may write

$$\mathcal{P}^{\mu a} = \big( -\Pi^{\mu\nu}\,\Gamma^a_{\nu b} + \phi^\mu{}_b{}^a \big)\,\theta^b + \mathcal{O}(\theta^3)\,, \tag{3.32}$$

thereby extracting a tensorial piece, $\phi$, out of the lowest order $\theta$-component of $\mathcal{P}^{\mu a}$, and express the non-tensorial piece in terms of the lowest order component of $\mathcal{P}^{\mu\nu}$ and a $TM$-on-$E$ connection $\nabla$. Note that when $M$ is a symplectic manifold, such a connection is directly obtained from the bracket of a differential graded Poisson algebra, as shown in [50, 51]. Then it is part of the geometrical data on the target space, since it determines the contravariant connection found above. This can also happen for Poisson manifolds $M$, as will be the case in the examples that we will study below. However, a $TM$-on-$E$ connection can also serve a different purpose. Note that for target spaces that are second (or higher) order bundles, there is no canonical vector bundle structure over the base manifold. Since in physics we are used to work with vector bundles and their sections (e.g. differential forms), it is technically desirable to give the target space the structure of a vector bundle over the base, which is possible using an auxiliary linear connection such as $\nabla$, which splits the supermanifold. This construction is non-canonical and it is the price to pay in order to have manifest target space covariance. In fact, such auxiliary connections can be useful even when there is no higher bundle structure and in a purely bosonic setting, see for example [52, 53] in the context of the bosonic Poisson sigma model. In such cases, one can also proceed without introducing this auxiliary connection, since the AKSZ construction only requires using the target space geometry.

Then it is useful to write the various identities that are extracted from (3.13a)–(3.13d) in terms of tensors for $E$ and $TM$ (and their duals). A choice of $TM$-on-$E$ connection defines an isomorphism

$$\Gamma(TE[0,1]) \overset{\nabla}{\cong} \Gamma(\wedge E^*) \underset{\mathscr{C}^\infty(M)}{\otimes} \Big( \Gamma(TM) \oplus \Gamma(E) \Big)\,, \tag{3.33}$$

as modules over $\Gamma(\wedge E^*)$. Under this isomorphism the components of $\mathcal{P}$, that we shall denote with a subscript $\nabla$, are related to the initial ones via

$$\mathcal{P}^{\mu a}_\nabla = \mathcal{P}^{\mu a} + \mathcal{P}^{\mu\nu}\,\Gamma^a_{\nu b}\,\theta^b\,, \tag{3.34a}$$

$$\mathcal{P}^{ab}_\nabla = \mathcal{P}^{ab} - 2\,\mathcal{P}^{\mu(a}\,\Gamma^{b)}_{\mu c}\,\theta^c + \mathcal{P}^{\mu\nu}\,\Gamma^a_{\mu c}\Gamma^b_{\nu d}\,\theta^c\theta^d\,, \tag{3.34b}$$

and their expansion in powers of $\theta$ consist of tensors on the body of $\mathcal{M}$, which is also the base $M$ of the model vector bundle $E$. In terms of these new components, the Jacobi identity becomes

$$0 = \mathcal{P}^{[\mu|\kappa} \nabla_\kappa \mathcal{P}^{\nu\lambda]} + \mathcal{P}_{\dot\nabla}^{[\mu|a} \partial_a \mathcal{P}^{\nu\lambda]}, \tag{3.35a}$$

$$0 = -\mathcal{P}_{\dot\nabla}^{\kappa a} \nabla_\kappa \mathcal{P}^{\mu\nu} + \mathcal{P}_{\dot\nabla}^{ab} \partial_b \mathcal{P}^{\mu\nu} + 2\mathcal{P}^{[\mu|\kappa} \nabla_\kappa \mathcal{P}_{\dot\nabla}^{\nu]a} + 2\mathcal{P}_{\dot\nabla}^{[\mu|b} \partial_b \mathcal{P}_{\dot\nabla}^{\nu]a} - \mathcal{P}^{\mu\kappa} \mathcal{P}^{\nu\lambda} R_{\kappa\lambda}{}^a{}_b \theta^b, \tag{3.35b}$$

$$0 = \mathcal{P}^{\mu\nu} \nabla_\nu \mathcal{P}_{\dot\nabla}^{ab} + \mathcal{P}_{\dot\nabla}^{\mu c} \partial_c \mathcal{P}_{\dot\nabla}^{ab} + 2\mathcal{P}_{\dot\nabla}^{\nu(a} \nabla_\nu \mathcal{P}_{\dot\nabla}^{\mu|b)} - 2\mathcal{P}_{\dot\nabla}^{c(a} \partial_c \mathcal{P}_{\dot\nabla}^{\mu|b)} + 2\mathcal{P}^{\mu\kappa} \mathcal{P}_{\dot\nabla}^{\lambda(a} R_{\kappa\lambda}{}^{b)}{}_c \theta^c, \tag{3.35c}$$

$$0 = -\mathcal{P}_{\dot\nabla}^{\mu(a} \nabla_\mu \mathcal{P}_{\dot\nabla}^{bc)} + \mathcal{P}_{\dot\nabla}^{d(a} \partial_d \mathcal{P}_{\dot\nabla}^{bc)} - \mathcal{P}_{\dot\nabla}^{\mu(a|} \mathcal{P}_{\dot\nabla}^{\nu|b} R_{\mu\nu}{}^{c)}{}_d \theta^d, \tag{3.35d}$$

where $\nabla$ acts on sections of tensors products of $E$ and $\wedge E^*$, the latter part being considered as polynomials in $\theta^a$, as

$$\nabla_\mu T^{a_1 \cdots a_k}(x, \theta) = \left( \partial_\mu - \Gamma_{\mu b}^c \, \theta^b \, \tfrac{\partial}{\partial \theta^c} \right) T^{a_1 \cdots a_k} + \sum_{l=1}^{k} \Gamma_{\mu b}^{a_l} \, T^{\cdots a_{l-1} \, b \, a_{l+1} \cdots}. \tag{3.36}$$

Under this isomorphism, the 'covariant' components of $\mathcal{P}$ read

$$\mathcal{P}_{\dot\nabla}^{\mu a} = \phi^\mu{}_b{}^a \, \theta^b, \qquad \mathcal{P}_{\dot\nabla}^{ab} = g^{ab} + \tfrac{1}{2} \theta^c \theta^d \, R_{cd}{}^{ab}, \tag{3.37}$$

up to second order in $\theta$, where $\phi^\mu{}_b{}^a$ are the components of the $\mathrm{End}(E)$-valued vector field relating an arbitrary contravariant connection to the induced one $\overset{\bullet}{\nabla}$ as discussed in the previous paragraphs, and where $R_{ab}{}^{cd}$ are the components of a section of $\wedge^2 E^* \otimes S^2 E$.

**Higher orders.** Now that we have re-expressed the content of $\mathcal{P}^{\alpha\beta}$ in terms of tensors over $M$, let us go back to the higher orders in $\theta^a$ of its Jacobi identity. We collect the conditions order by order in $\theta$ in table 1.

In the remainder of the paper, we will mostly focus on the case where $\phi = 0$, with the notable exception of dilaton supergravity in section 4.2, i.e. we will assume that the contravariant connection induced by the super-Poisson bivector is induced from an ordinary $TM$-on-$E$ connection, and where the quadratic component (and in fact, all higher order components) in $\mathcal{P}^{\mu\nu}$ vanishes, i.e. $\mathcal{P}_{ab}{}^{\mu\nu} = 0$. In this situation, the above identities boil down to the flatness of the induced contravariant connection,

$$\Pi^{\mu\kappa} \Pi^{\nu\lambda} R_{\kappa\lambda}{}^a{}_b = 0 \qquad \Longleftrightarrow \qquad R^\nabla(\Pi^\sharp, \Pi^\sharp) = 0, \tag{3.38}$$

and the conditions

$$0 = g^{e(a} R_{de}{}^{bc)}, \tag{3.39a}$$

$$0 = \Pi^{\mu\nu} \nabla_\nu R_{cd}{}^{ab}, \tag{3.39b}$$

$$0 = R_{[de}{}^{g(a} R_{f]g}{}^{bc)}, \tag{3.39c}$$

on the tensor $R_{cd}{}^{ab}$, which are reminiscent of the algebraic and differential Bianchi identities. We will see in section 4.5 that these constraints allow one to define a supersymmetric Poisson sigma model, with supersymmetry transformations being encoded in a Lie algebroid structure on $E$.

| Order | Conditions |
|---|---|
| $0$ | $$\Pi^{\kappa[\mu}\,\partial_\kappa\Pi^{\nu\lambda]} = 0$$ $$\nabla^\mu g^{ab} = 0$$ |
| $\theta$ | $$\Pi^{\mu\kappa}\,\Pi^{\nu\lambda}\,R_{\kappa\lambda}{}^a{}_b + \phi^\kappa{}_b{}^a\,\nabla_\kappa\Pi^{\mu\nu} + 2\,\Pi^{\kappa[\mu}\,\nabla_\kappa\phi^{\nu]}{}_b{}^a - 2\,\phi^{[\mu}{}_b{}^c\,\phi^{\nu]}{}_c{}^a + g^{ac}\,\mathcal{P}_{bc}{}^{\mu\nu} = 0$$ $$\phi^\mu{}_d{}^{(a}\,\nabla_\mu g^{bc)} + g^{e(a}\,R_{de}{}^{bc)} = 0$$ |
| $\theta^2$ | $$\Pi^{\kappa[\mu}\,\nabla_\kappa\mathcal{P}_{ab}{}^{\nu\lambda]} + \mathcal{P}_{ab}{}^{\kappa[\mu}\,\nabla_\kappa\Pi^{\nu\lambda]} + 2\,\phi^{[\mu}{}_{[a}{}^c\,\mathcal{P}_{b]c}{}^{\nu\lambda]} = 0$$ $$\tfrac{1}{2}\,\Pi^{\mu\nu}\,\nabla_\nu R_{cd}{}^{ab} + \tfrac{1}{2}\,\mathcal{P}_{cd}{}^{\mu\nu}\,\nabla_\nu g^{ab} - \phi^\mu{}_{[c}{}^e\,R_{d]e}{}^{ab} +$$ $$+ 2\,\phi^\nu{}_{[c}{}^{(a}\,\nabla_\nu\phi^\mu{}_{d]}{}^{b)} - R_{cd}{}^{e(a}\,\phi^\mu{}_e{}^{b)} + 2\,\Pi^{\mu\kappa}\,\phi^\lambda{}_{[c}{}^{(a}\,R_{\kappa\lambda}{}^{b)}{}_{d]} = 0$$ |
| $\theta^3$ | $$\phi^{\kappa}{}_{[b}{}^a\,\nabla_\kappa\mathcal{P}_{cd]}{}^{\mu\nu} - 2\,\mathcal{P}_{[bc}{}^{[\mu|\kappa}\,\nabla_\kappa\phi^{|\nu]}{}_{d]}{}^a + 2\,\Pi^{[\mu|\kappa}\,\mathcal{P}_{[bc}{}^{|\nu]\lambda}\,R_{\kappa\lambda}{}^a{}_{d]} = 0$$ $$\phi^\mu{}_{[d}{}^{(a}\,\nabla_\mu R_{ef]}{}^{bc)} + R_{[de}{}^{g(a}\,R_{f]g}{}^{bc)} + 2\,\phi^\mu{}_{[d}{}^{(a}\,\phi^\nu{}_e{}^b\,R_{\mu\nu}{}^{c)}{}_{f]} = 0$$ |
| $\theta^4$ | $$\mathcal{P}_{[ab}{}^{[\mu|\kappa}\,\nabla_\kappa\mathcal{P}_{cd]}{}^{|\nu\lambda]} = 0$$ $$\mathcal{P}_{[cd}{}^{\mu\nu}\,\nabla_\nu R_{ef]}{}^{ab} + 2\,\mathcal{P}_{[cd}{}^{\mu\kappa}\,\phi^\lambda{}_e{}^{(a}\,R_{\kappa\lambda}{}^{b)}{}_{f]} = 0$$ |
| $\theta^5$ | $$\mathcal{P}_{[bc}{}^{\mu\kappa}\,\mathcal{P}_{de}{}^{\nu\lambda}\,R_{\kappa\lambda}{}^a{}_{f]} = 0$$ |

**Table 1.** Covariant form of the general conditions obtained from the Jacobi identity of the super-Poisson structure with components at most quadratic in the odd coordinates $\theta^a$.

## 4 Supersymmetric Poisson sigma models

### 4.1 The general Ikeda supermodel

A nonlinear supergauge theory was defined constructively in ref. [16]. Here we revisit this theory within the context presented in the previous sections. We are interested in 2D supersymmetric topological sigma models, where the target space is an NQ-supermanifold with a degree 1 compatible symplectic structure. In general, the non-supersymmetric version of such models requires the existence of a QP structure on the target space $\mathcal{M}$, which

induces a corresponding QP structure on the mapping space $\mathsf{Maps}(\mathcal{X}, \mathcal{M})$, where $\mathcal{X}$ is the source dg manifold; this is the backbone of the AKSZ construction [2]. We would like to reconstruct the classical action of the model from the geometrical data, therefore we focus on degree-preserving maps in this work. In the supersymmetric case, the geometrical data are the source data $(\mathcal{X}, \mathrm{d})$ and the target data $(\mathcal{M}, \omega, \mathcal{Q}, \mathcal{Q}_S)$. There are two homological vector fields, $\mathcal{Q}$ of degree $(1, 0)$ that controls the gauge symmetries of the model, which include bosonic and fermionic ones, and $\mathcal{Q}_S$ of degree $(0, 1)$ that controls the global supersymmetries. To construct an action that is both gauge invariant and supersymmetric, we require that both homological vector fields are compatible with the graded symplectic structure $\omega$ and that they are mutually commuting,

$$[\mathcal{Q}, \mathcal{Q}] = 0 = [\mathcal{Q}_S, \mathcal{Q}_S], \qquad [\mathcal{Q}, \mathcal{Q}_S] = 0, \qquad \mathcal{L}_{\mathcal{Q}}\omega = 0 = \mathcal{L}_{\mathcal{Q}_S}\omega. \tag{4.1}$$

The expanded form of all these conditions appears in appendix A.

The general supersymmetric sigma model is constructed with target space $T^*[1, 0]E[0, 1]$. We recall that we have assigned to it the $\mathbb{Z}$-degree 0 coordinates $x^\alpha = (x^\mu, \theta^a)$ and the $\mathbb{Z}$-degree 1 coordinates $p_\alpha = (a_\mu, \chi_a)$. We will denote the corresponding fields in the mapping space as $X^\alpha = (X^\mu, \theta^a)$ and $P_\alpha = (A_\mu, \chi_a)$ respectively, and when we want to refer collectively to all of them as $\Phi \equiv (\Phi^A) = (X^\alpha, P_\alpha)$, where $\Phi$ can be considered as a sigma model map from the graded source manifold to the graded target space, $\Phi : \mathcal{X} \to \mathcal{M}$ (note that there is some abuse of notation for the $\mathbb{Z}_2$-odd fields in that we have refrained from capitalizing them). The action functional of the supersymmetric sigma model in our conventions is

$$S[X^\alpha, P_\alpha] = \int \left( P_\alpha \wedge \mathrm{d}X^\alpha + \tfrac{1}{2}(-1)^{\alpha(\beta+1)} \mathcal{P}^{\alpha\beta}(X) P_\alpha \wedge P_\beta \right) \tag{4.2}$$

$$= \int \left( A_\mu \wedge \mathrm{d}X^\mu + \chi_a \wedge \mathrm{d}\theta^a + \tfrac{1}{2}\mathcal{P}^{\mu\nu} A_\mu \wedge A_\nu + \mathcal{P}^{\mu a} A_\mu \wedge \chi_a + \tfrac{1}{2}\mathcal{P}^{ab} \chi_a \wedge \chi_b \right),$$

where the interaction term corresponds to the Hamiltonian for $\mathcal{Q}$ given in (3.16a) above. This action appeared already in ref. [16, section 5]. The gauge symmetries of this general model can be found in a straightforward manner from the homological vector field $\mathcal{Q}$. They are (see e.g. [31, 49, 54] for detailed explanations)

$$\delta_\varepsilon \Phi = [\mathrm{d} + \mathcal{Q}, \varepsilon], \tag{4.3}$$

where the gauge symmetry parameters are $\varepsilon = (0, \varepsilon_\alpha)$ with $|\varepsilon| = 1$ and $\varepsilon_\alpha = (\varepsilon_\mu, \varepsilon_a)$ the even and odd scalar parameters for the fields $A_\mu$ and $\chi_a$ respectively. The pull-back to the space of fields on the right hand side of (4.3) is implicit. We can calculate the graded commutator for the homological vector field given in the chosen basis of section 2 and obtain the gauge symmetries for each component:

$$\delta_\varepsilon X^\mu = \varepsilon_\nu \mathcal{P}^{\nu\mu} - \varepsilon_a \theta^b \mathcal{P}_b^{a\mu}, \tag{4.4a}$$

$$\delta_\varepsilon A_\mu = \mathrm{d}\varepsilon_\mu - \varepsilon_\nu A_\rho \partial_\mu \mathcal{P}^{\nu\rho} - \varepsilon_\nu \chi_a \theta^b \partial_\mu \mathcal{P}_b^{a\nu} - \varepsilon_a A_\nu \theta^b \partial_\mu \mathcal{P}_b^{a\nu} - \varepsilon_a \chi_b \partial_\mu \mathcal{P}^{ba}, \tag{4.4b}$$

$$\delta_\varepsilon \theta^a = \varepsilon_\mu \theta^b \mathcal{P}_b^{\mu a} - \varepsilon_b \mathcal{P}^{ba}, \tag{4.4c}$$

$$\delta_\varepsilon \chi_a = \mathrm{d}\varepsilon_a - \varepsilon_\mu A_\nu \partial_a \mathcal{P}^{\mu\nu} - \varepsilon_\mu \chi_b \partial_a(\theta^c \mathcal{P}_c^{\mu b}) - \varepsilon_b A_\mu \partial_a(\theta^c \mathcal{P}_c^{\mu b}) - \varepsilon_b \chi_c \partial_a \mathcal{P}^{cb}, \tag{4.4d}$$

where we expressed all transformations in terms of the components of $\mathcal{P}^{\alpha\beta}$ and we denoted $\partial_a \equiv \partial/\partial\theta^a$. They comprise ordinary bosonic gauge symmetries with parameter $\varepsilon_\mu$ and local supersymmetries with parameter $\varepsilon_a$. This form of the gauge symmetries agrees with what was described in ref. [16].

The global supersymmetry transformations of the fields that leave the action invariant are controlled by the homological vector field $\mathcal{Q}_S$ and they are given as

$$\delta_S \Phi = [\mathcal{Q}_S, \Phi]. \tag{4.5}$$

We can rewrite them in more explicit terms as

$$\delta_S X^\mu = \theta^a t_a{}^\mu, \tag{4.6a}$$

$$\delta_S A_\mu = A_\nu \theta^a U^\nu{}_{a\mu} + \chi_b W^b{}_\mu, \tag{4.6b}$$

$$\delta_S \theta^a = V^a, \tag{4.6c}$$

$$\delta_S \chi_a = A_\mu Y^\mu{}_a + \chi_b \theta^c Z^b{}_{ca}. \tag{4.6d}$$

This is in general a nonlinear supersymmetry transformation and we also recall that all coefficients may also depend on even powers of $\theta^a$, in addition to their $X^\mu$ dependence. Moreover, for a given model, there might exist more than one supersymmetry [15], giving rise to extended supersymmetric models. The kinetic sector of the general model is invariant under these supersymmetry transformations provided that the symplectic form in Darboux coordinates is invariant under the supersymmetry-generating homological vector field $\mathcal{Q}_S$. The analysis of this condition is simple, see appendix A.5, and it fixes all coefficients in terms of only $t_a{}^\mu$ and $V^a$. We repeat the result here:

$$Y^\mu{}_a = -t_a{}^\mu + \theta^b \partial_a t_b{}^\mu, \qquad U^\mu{}_{a\nu} = \partial_\nu t_a{}^\mu, \qquad W^a{}_\mu = \partial_\mu V^a, \qquad \theta^c Z^a{}_{cb} = \partial_b V^a. \tag{4.7}$$

The invariance of the interaction sector under supersymmetry corresponds to the condition $[\mathcal{Q}, \mathcal{Q}_S] = 0$, which is more complicated in its general form, see appendix A.3. We will describe specific solutions later in this section.

As for any gauge theory, the field strengths of the various fields are given in terms of the components of the homological vector field $\mathcal{Q}$. Define the operator

$$F := \mathrm{d} \circ \Phi^* - \Phi^* \circ \mathcal{Q}, \tag{4.8}$$

where $\Phi^*$ is the pull-back of the sigma model map between graded manifolds $\Phi : \mathcal{X} \to \mathcal{M}$ and $\circ$ denotes composition. Note that if $F = 0$ then the map $\Phi$ is a morphism of dg-manifolds. Applying this to the local coordinates of $\mathcal{M}$, say $\varphi^A$, we find that $F(\varphi^A) \equiv F^A$ is given by the formula

$$F^A = \mathrm{d}\Phi^A - \mathcal{Q}^A(\Phi). \tag{4.9}$$

In more detail, this may be expanded as follows,

$$F^\mu = \mathrm{d}X^\mu - A_\nu \mathcal{P}^{\nu\mu} + \chi_a \theta^b \mathcal{P}_b^{a\mu}, \tag{4.10a}$$

$$F_\mu = \mathrm{d}A_\mu + \tfrac{1}{2} A_\nu \wedge A_\rho \, \partial_\mu \mathcal{P}^{\nu\rho} - A_\nu \wedge \chi_b \theta^a \, \partial_\mu \mathcal{P}_a^{\nu b} + \tfrac{1}{2} \chi_a \wedge \chi_b \, \partial_\mu \mathcal{P}^{ab}, \tag{4.10b}$$

$$F^a = \mathrm{d}\theta^a - A_\mu \theta^b \, \mathcal{P}_b^{\mu a} + \chi_b \, \mathcal{P}^{ba} \,, \tag{4.10c}$$

$$F_a = \mathrm{d}\chi_a + \tfrac{1}{2} A_\mu \wedge A_\nu \, \partial_a \mathcal{P}^{\mu\nu} + A_\mu \wedge \chi_b \, \partial_a \mathcal{P}^{\mu b} + \tfrac{1}{2} \chi_b \wedge \chi_c \, \partial_a \mathcal{P}^{bc} \,. \tag{4.10d}$$

The equations of motion for the topological sigma model are obtained by setting all these field strengths to zero. These field strengths are often said to be 'Cartan integrable', meaning that setting them to zero is consistent with the fact that the de Rham differential is nilpotent. More concretely, this means that they verify

$$\mathrm{d}F^\alpha + F^\beta \, \partial_\beta \mathcal{Q}^\alpha = \mathcal{Q}^\beta \, \partial_\beta \mathcal{Q}^\alpha \equiv 0 \,, \tag{4.11}$$

which can be read as a kind of generalized covariant constancy condition for the curvature defined by $\mathcal{Q}$ and hence is often referred to as a Bianchi identity (see, e.g., [55, section 3.3]).

The formulation presented so far, which is due to Ikeda, is direct and simple, but it does not exhibit manifest target space covariance. This is particularly important in the supersymmetric case, since there is a $T^*M$-on-$E^*$ connection hidden in the components $\mathcal{P}^{\mu a}$, as we discussed in section 3, which is not the case for the bosonic model. To account for this, we make the assumption that this $T^*M$-on-$E^*$ connection is induced by a $TM$-on-$E$ connection $\nabla$ with coefficients $\Gamma_{\mu a}^b$. Since it determines the contravariant connection, it is part of the target space geometry. Moreover, it allows us to work with vector bundles, sections and manifestly tensorial objects in the target space. For future reference, we mention that the AKSZ construction depends only on the target space geometry and there is no need to introduce an additional auxiliary connection to split the supermanifold. We emphasize again that the contravariant connection from the components $\mathcal{P}^{\mu a}$ is part of the target space data and it is not auxiliary, whether or not it arises from an ordinary connection. Next, we perform the field redefinition.

$$A_\mu^\nabla = A_\mu + \Gamma_{\mu a}^b \theta^a \chi_b \,. \tag{4.12}$$

Writing the action functional in terms of the redefined 1-form, we obtain

$$S[X, A^\nabla, \theta, \chi] = \int \left( A_\mu^\nabla \wedge \mathrm{d}X^\mu + \chi_a \wedge \nabla\theta^a + \tfrac{1}{2}\mathcal{P}^{\mu\nu} A_\mu^\nabla \wedge A_\nu^\nabla + (\mathcal{P}^{\mu a} + \mathcal{P}^{\mu\nu}\Gamma_{\nu b}^a \theta^b) A_\mu^\nabla \wedge \chi_a \right.$$

$$\left. + \tfrac{1}{2}(\mathcal{P}^{ab} - 2\mathcal{P}^{\mu a}\Gamma_{\mu c}^b \theta^c + \mathcal{P}^{\mu\nu}\Gamma_{\mu c}^a \Gamma_{\nu d}^b \theta^c \theta^d)\chi_a \wedge \chi_b \right), \tag{4.13}$$

where the covariant exterior derivative is given as

$$\nabla\theta^a = \mathrm{d}\theta^a + \Gamma_{\mu b}^a \mathrm{d}X^\mu \theta^b \,. \tag{4.14}$$

This aligns with the definitions we presented in section 3, in particular we may write the action in the compact form

$$S[X^\alpha, P_\alpha^\nabla] = \int \left( P_\alpha^\nabla \wedge \nabla X^\alpha + \tfrac{1}{2}(-1)^{\alpha(\beta+1)} \mathcal{P}_\nabla^{\alpha\beta}(X) P_\alpha^\nabla \wedge P_\beta^\nabla \right), \tag{4.15}$$

with $P_\alpha^\nabla = (A_\mu^\nabla, \chi_a)$ and the covariant components $\mathcal{P}_\nabla^{\alpha\beta}$ defined in (3.34). This equivalent formulation will be useful in understanding some specific cases below. Each of the two

equivalent formulations, related through the simple field redefinition (4.12), highlights different features of the model. For example, although the formulation in terms of the redefined 1-form highlights target space covariance, the original one is much simpler in the specification of the geometrical data. Indeed, the degree $(1,0)$ symplectic 2-form of the model reads

$$\omega = \mathrm{d}A_\mu \wedge \mathrm{d}X^\mu + \mathrm{d}\chi_a \wedge \mathrm{d}\theta^a \qquad (4.16\mathrm{a})$$

$$= \mathrm{d}A_\mu^\nabla \wedge \mathrm{d}X^\mu + \mathrm{d}\chi_a \wedge (\mathrm{d}\theta^a + \mathrm{d}X^\mu \Gamma^a_{\mu b}\theta^b)$$

$$- \chi_a \Gamma^a_{\mu b}\mathrm{d}X^\mu \wedge \mathrm{d}\theta^b - \chi_a \partial_\kappa \Gamma^a_{\nu b}\theta^b \mathrm{d}X^\nu \wedge \mathrm{d}X^\kappa , \qquad (4.16\mathrm{b})$$

and it is canonical in the original formulation, instead containing various off diagonal terms in terms of the redefined field content, where we do not use Darboux coordinates. It will be useful to express the supersymmetry transformations in terms of this new field. For the fields $X^\mu$ and $\theta^a$, they do not get modified, whereas for the remaining two fields the new supersymmetry transformations are

$$\delta_S A_\mu^\nabla = A_\nu^\nabla \theta^a \left(U^\nu{}_{a\mu} + \Gamma^b_{\mu a} Y^\nu{}_b\right) + \chi_b \left(W^b{}_\mu + \Gamma^b_{\mu a}V^a + (t_c{}^\nu \partial_\nu \Gamma^b_{\mu d} - \Gamma^e_{\mu c} Z^b{}_{de})\theta^c \theta^d\right), \qquad (4.17\mathrm{a})$$

$$\delta_S \chi_a = A_\mu^\nabla Y^\mu{}_a + \chi_b \theta^c \left(Z^b{}_{ca} - Y^\mu{}_a \Gamma^b_{\mu c}\right), \qquad (4.17\mathrm{b})$$

with coefficients as in (4.7).

## 4.2 Dilaton supergravity in 2 dimensions

A characteristic example of such a supersymmetric Poisson sigma model is $\mathcal{N} = 1$ dilaton supergravity in 2 dimensions, based on the super-Poincaré algebra [16], or the orthosymplectic algebra $\mathfrak{osp}(1|2,\mathbb{R})$ in the presence of a cosmological constant, e.g. [56] or [57] for a recent review. The super Poisson manifold $E[0,1]$ in that case is the dual of $\mathfrak{osp}(1|2,\mathbb{R})$, with body the bosonic part and fiber the fermionic one. The dictionary between our notations and those of [57] is:

$$X^\mu \longmapsto (X^a, X), \qquad A_\mu \longmapsto (e_a, \omega), \qquad \theta^a \longmapsto \lambda^\alpha, \qquad \chi_a \longmapsto -\bar{\psi}_\alpha .$$

This correspondence allows to recast action (2.1) of [57] into the form of action (4.2). Local supersymmetry transformations (2.3) and (2.4) of [57] correspond to the odd part of the gauge symmetries generated by the homological vector field $\mathcal{Q}$ as given in equations (4.4). The contravariant connection (3.19) associated to this model is the representation of the bosonic part of $\mathfrak{osp}(1|2,\mathbb{R})$ onto the fermionic part.

In fact, any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where the subscripts 0 and 1 respectively denote the even/bosonic and odd/fermionic subspaces, gives rise to a Poisson supermanifold: its linear dual $\mathfrak{g}^*$ is equipped with a super-Poisson bivector whose components are linear in the coordinates defined by a choice of basis on $\mathfrak{g}$, and given by the structure constants of the Lie superalgebra in this basis [40]. In other words, the situation is identical to the bosonic case, where the dual of any Lie algebra is a Poisson manifold, and the corresponding sigma model is — up to a boundary term obtained from integrating by parts the kinetic term — a two-dimensional BF-theory.

Let us also remark that the dual of a Lie superalgebra can be given the structure of a super-vector bundle,

$$E[0,1] := \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}_0^*, \tag{4.18}$$

with the base being the bosonic subalgebra. According to the previous discussion, both the base and the total space are Poisson supermanifolds. On top of that, taking into account the isomorphisms

$$\Gamma(T^*\mathfrak{g}_0^*) \cong \mathscr{C}^\infty(\mathfrak{g}_0^*) \otimes \mathfrak{g}_0 \,, \qquad \Gamma(E^*) \cong \mathscr{C}^\infty(\mathfrak{g}_0^*) \otimes \mathfrak{g}_1 \,, \tag{4.19}$$

where the second term denotes sections of $E^* = \mathfrak{g}_1 \oplus \mathfrak{g}_0^* \twoheadrightarrow \mathfrak{g}_0^*$, the vector bundle dual to $\mathfrak{g}^* \twoheadrightarrow \mathfrak{g}_0^*$, one can check that the flat contravariant connection appearing as the linear odd piece of any super-Poisson bivector is nothing but the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$, as explained before in the case of the orthosymplectic algebra.

Let us be more specific, and start by introducing a basis $\mathfrak{e}^\mu$ of the bosonic subalgebra $\mathfrak{g}_0$, and $\mathfrak{e}^a$ for the fermionic subspace $\mathfrak{g}_1$. The structure constants of the Lie superalgebra $\mathfrak{g}$ are given by

$$[\mathfrak{e}^\mu, \mathfrak{e}^\nu]_\mathfrak{g} = C^{\mu\nu}{}_\rho\, \mathfrak{e}^\rho \,, \qquad [\mathfrak{e}^\mu, \mathfrak{e}^a]_\mathfrak{g} = \Gamma^{\mu\,a}{}_b\, \mathfrak{e}^b \,, \qquad [\mathfrak{e}^a, \mathfrak{e}^b]_\mathfrak{g} = U^{ab}{}_\mu\, \mathfrak{e}^\mu \,, \tag{4.20}$$

where we avoided using the anticommutator notation for the super-Lie bracket evaluated on two fermionic generators, so as not to cause any confusion with the super-Poisson bracket on $\mathfrak{g}^*$. The basis of $\mathfrak{g}$ introduced here also defines a coordinate system on the supermanifold $\mathfrak{g}^*$, with even coordinates denoted $x_\mu$, and odd ones denoted $\theta_a$. The super-Poisson bracket reads

$$\{x^\mu, x^\nu\}_{\mathfrak{g}^*} = C^{\mu\nu}{}_\rho\, x^\rho \,, \qquad \{x^\mu, \theta^a\}_{\mathfrak{g}^*} = \Gamma^{\mu\,a}{}_b\, \theta^b \,, \qquad \{\theta^a, \theta^b\}_{\mathfrak{g}^*} = U^{ab}{}_\mu\, x^\mu \,, \tag{4.21}$$

and is extended to arbitrary functions on $\mathfrak{g}^*$ as an even biderivation. Put differently, the super-Poisson bivector $\mathcal{P}_{\mathfrak{g}^*}$ on the supermanifold $\mathfrak{g}^*$ has components

$$\mathcal{P}_{\mathfrak{g}^*}^{\mu\nu} := C^{\mu\nu}{}_\rho\, x^\rho \,, \qquad \mathcal{P}_{\mathfrak{g}^*}^{\mu a} := \Gamma^{\mu a}{}_b\, \theta^b \,, \qquad \mathcal{P}_{\mathfrak{g}^*}^{ab} := U^{ab}{}_\mu\, x^\mu \,, \tag{4.22}$$

which are all (exactly) linear in the coordinates of the supermanifold (a defining feature of super-Poisson structures obtained from super-Lie brackets). In terms of the power expansion in $\theta$, this super-Poisson bivector only contains the components of lowest possible orders. Note that since we are dealing with a *flat* supermanifold here, we can choose a trivial connection to split it, which we will assume from now on. These different components encode three geometrical structures on $\mathfrak{g}^*$, seen as a super-vector bundle over $\mathfrak{g}_0^*$, namely

- A Poisson structure on the base $\mathfrak{g}_0^*$ with components $\Pi^{\mu\nu}(x) \equiv C^{\mu\nu}{}_\rho\, x^\rho$;

- A contravariant connection with constant components $\Gamma^{\mu a}{}_b$ stemming from the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$;

- A fiberwise symmetric bilinear form with components $g^{ab}(x) \equiv U^{ab}{}_\mu\, x^\mu$.

The Jacobi identity for $\mathcal{P}_{\mathfrak{g}^*}$ is equivalent to that of the super-Lie bracket $[-,-]_{\mathfrak{g}}$, and in terms of the structure constants, amounts to the identities

$$0 = C^{[\mu\nu}{}_\sigma C^{\rho]\sigma}{}_\lambda \, , \tag{4.23a}$$

$$0 = \Gamma^{[\mu|c}{}_b \Gamma^{|\nu]a}{}_c - \tfrac{1}{2} C^{\mu\nu}{}_\rho \Gamma^{\rho a}{}_b \, , \tag{4.23b}$$

$$0 = \Gamma^{\mu\,(a}{}_c U^{b)c}{}_\nu - \tfrac{1}{2} U^{ab}{}_\rho C^{\mu\rho}{}_\nu \, , \tag{4.23c}$$

$$0 = \Gamma^{\mu(a}{}_d U^{bc)}{}_\mu \, . \tag{4.23d}$$

The first of the above equations is the Jacobi identity for $\mathfrak{g}_0$, while the second one is the condition that $\mathfrak{g}_1$ carries a representation of $\mathfrak{g}_0$, obtained as the restriction of the adjoint action. Geometrically, these conditions tell us that $\mathfrak{g}_0^*$ is a Poisson manifold, and $\mathfrak{g}^* \twoheadrightarrow \mathfrak{g}_0^*$ is equipped with a contravariant connection, as mentioned above. The third equation imposes that the restriction of the super-Lie bracket to the fermionic subspace, i.e. $S^2(\mathfrak{g}_1) \longrightarrow \mathfrak{g}_0$ is a $\mathfrak{g}_0$-equivariant map.[11] Geometrically, this corresponds to the requirement that the fiberwise metric $g$ be covariantly constant with respect to the contravariant derivative, as can be checked by plugging the definitions for $\Pi$, $\Gamma$ and $g$ in the case of $\mathfrak{g}^*$,

$$\nabla^\mu g^{ab} = \Pi^{\nu\mu}\, \partial_\nu g^{ab} + 2\,\Gamma^{\mu(a}{}_c g^{b)c} = 0 \qquad \overset{\mathfrak{g}^*}{\Longleftrightarrow} \qquad \Gamma^{\mu\,(a}{}_c U^{b)c}{}_\nu - \tfrac{1}{2} U^{ab}{}_\rho C^{\mu\rho}{}_\nu = 0 \, . \tag{4.24}$$

The fourth and last identity can be re-written as

$$0 = \Gamma^{\mu(a}{}_d U^{bc)}{}_\mu = \Gamma^{\mu(a}{}_d \partial_\mu g^{bc)} = \phi^{\mu}{}_d{}^{(a} \partial_\mu g^{bc)} \, , \tag{4.25}$$

where we used the fact that, having chosen the trivial connection to split $\mathfrak{g}^*$, the tensor $\phi^\mu{}_b{}^a$ expressing the difference between an arbitrary contravariant connection and the one induced by an 'ordinary one', identifies with the contravariant connection in question. This reproduces the second equation listed at order $\theta$ in table 1 (as in this case, $R_{ab}{}^{cd} = 0 = \mathcal{P}_{ab}{}^{\mu\nu}$, i.e. there are no quadratic terms in $\theta$ in the expression of $\mathcal{P}_{\mathfrak{g}^*}$). The first equation at the same order in $\theta$ (i.e. linear) in this table reduces to the requirement that $\Gamma^{\mu\,a}{}_b$ defines a *flat* contravariant connection — equivalently a $\mathfrak{g}_0$-representation on $\mathfrak{g}_1$. Since $\mathcal{P}_{\mathfrak{g}^*}$ is at most linear in $\theta$, there are no non-trivial conditions at higher orders in $\theta$, so that we have recovered all the conditions encoded in the Jacobi identity of the super-Poisson bracket on $\mathfrak{g}^*$ induced by the super-Lie bracket on $\mathfrak{g}$.

The action (4.2) of the super-Poisson sigma model, for the target space $\mathcal{M} = \mathfrak{g}^*$ reads

$$S[X, A, \theta, \chi] = \int_\Sigma A_\mu \,\mathrm{d}X^\mu + \chi_a\, \mathrm{d}\theta^a + \tfrac{1}{2} C^{\mu\nu}{}_\lambda X^\lambda A_\mu A_\nu + \Gamma^{\mu a}{}_b \theta^b A_\mu \chi_a + \tfrac{1}{2} U^{ab}{}_\mu X^\mu \chi_a \chi_b$$

$$= \int_\Sigma X^\mu F_\mu[A, \chi] + \theta^a \nabla \chi_a - \int_{\partial\Sigma} A_\mu X^\mu + \chi_a \theta^a \, , \tag{4.26}$$

with

$$F_\mu[A, \chi] = \mathrm{d}A_\mu + \tfrac{1}{2} C^{\kappa\lambda}{}_\mu A_\kappa A_\lambda + \tfrac{1}{2} U^{ab}{}_\mu \chi_a \chi_b \, , \qquad \nabla\chi_a = \mathrm{d}\chi_a + \Gamma^{\mu b}{}_a A_\mu \chi_b \tag{4.27}$$

---

[11]In terms of the super-Lie bracket, this identity reads $\{\mathrm{ad}_X \xi_1, \xi_2\} + \{\xi_1, \mathrm{ad}_X \xi_2\} = \mathrm{ad}_X\{\xi_1, \xi_2\}$ for any $X \in \mathfrak{g}_0$ and $\xi_1, \xi_2 \in \mathfrak{g}_1$.

the bosonic and fermionic parts of the field strength of the pair $(A, \chi)$, seen as a gauge field for the superalgebra $\mathfrak{g}$. In other words, we recover here the 'super-version' of the standard result that the Poisson sigma model associated with the dual of a Lie algebra coincides with the non-Abelian BF theory for this algebra, up to a boundary term.

For the sake of concreteness, let us have a closer look at the superalgebra $\mathfrak{osp}(1|2, \mathbb{R})$, whose bosonic and fermionic parts are given by

$$\mathfrak{osp}(1|2, \mathbb{R})_0 = \mathfrak{sp}(2, \mathbb{R}) = \mathrm{span}\{\mathfrak{L}, \mathfrak{P}^\mu \mid \mu = 0, 1\}, \tag{4.28}$$

$$\mathfrak{osp}(1|2, \mathbb{R})_1 = \mathcal{S} = \mathrm{span}\{\mathfrak{Q}^a \mid a = \pm\}, \tag{4.29}$$

where $\mathcal{S}$ denotes the spinor representation of the Lorentz group in two dimensions. In this basis, the structure constants of the bosonic subalgebra read

$$[\mathfrak{L}, \mathfrak{P}^\mu]_{\mathfrak{osp}(1|2, \mathbb{R})} = \epsilon^\mu{}_\nu \mathfrak{P}^\nu, \qquad [\mathfrak{P}^\mu, \mathfrak{P}^\nu]_{\mathfrak{osp}(1|2, \mathbb{R})} = \Lambda \epsilon^{\mu\nu} \mathfrak{L}, \tag{4.30}$$

where $\Lambda$ is the cosmological constant, $\epsilon^{\mu\nu}$ is the Levi-Civita symbol in two dimensions and indices $\mu, \nu, \ldots$ are raised and lowered with the two-dimensional Lorentzian flat metric. These relations show that $\mathfrak{L}$ can be identified as the generator of the Lorentz subalgebra $\mathfrak{so}(1, 1) \cong \mathbb{R} \subset \mathfrak{sp}(2, \mathbb{R})$ and $\mathfrak{P}^\mu$ denote the translation generators, which are Abelian in the flat limit $\Lambda \to 0$. The brackets of these bosonic generators with the fermionic ones are given by[12]

$$[\mathfrak{L}, \mathfrak{Q}^a]_{\mathfrak{osp}(1|2, \mathbb{R})} = \tfrac{1}{2} \Gamma^a{}_b \mathfrak{Q}^b, \qquad [\mathfrak{P}^\mu, \mathfrak{Q}^a]_{\mathfrak{osp}(1|2, \mathbb{R})} = \tfrac{\sqrt{\Lambda}}{2} (\Gamma^\mu)^a{}_b \mathfrak{Q}^b, \tag{4.31}$$

where $\Gamma^a{}_b$ and $(\Gamma^\mu)^a{}_b$ are the components of the following $2 \times 2$ matrices,

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.32}$$

Note that they obey the relations

$$\{\Gamma^\mu, \Gamma^\nu\} = 2 \eta^{\mu\nu} \mathbf{1}_2, \qquad \Gamma = \Gamma^0 \Gamma^1, \tag{4.33}$$

i.e. $\Gamma^\mu$ define gamma matrices in two dimensions while $\Gamma$ is the chirality operator, and as a consequence one finds,

$$\{\Gamma, \Gamma^\mu\} = 0, \qquad \Gamma^2 = \mathbf{1}_2, \tag{4.34}$$

thereby showing that $\Gamma$ and $\Gamma^\mu$ for a set of gamma matrices in three dimensions (and Lorentzian signature). The matrix $\Gamma$ defines the action of the Lorentz subalgebra $\mathfrak{so}(1, 1)$ on the spinor module $\mathcal{S}$, which is extended to an action of $\mathfrak{sp}(2, \mathbb{R})$ by the matrices $\Gamma^\mu$. Finally, the bracket of two fermionic generators reads

$$[\mathfrak{Q}^a, \mathfrak{Q}^b]_{\mathfrak{osp}(1|2, \mathbb{R})} = -\sqrt{\Lambda} (C\Gamma)^{ab} \mathfrak{L} - (C\Gamma^\mu)^{ab} \mathfrak{P}_\mu, \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.35}$$

with $C$ the charge conjugation matrix, which can be used to raise and lower the spinor indices. With this in place, we can read off the structure constants of $\mathfrak{osp}(1|2, \mathbb{R})$, and hence

---

[12]For the sake of simplicity, we will assume that the cosmological constant is positive so that its square root makes sense. The case $\Lambda < 0$ can also be treated, modulo a few signs in the Lie brackets spelled out here.

the components of the super-Poisson bivector on its dual: the structure constants of the bosonic subalgebra $\mathfrak{sp}(2,\mathbb{R})$ generically denoted by $C^{ab}_{\ c}$ above are given by the antisymmetric tensor $\epsilon^{\mu\nu}$ (and $\Lambda$), the components of the contravariant connection generically denoted by $\Gamma^{\mu a}_{\ \ b}$ previously are given by the components of the gamma matrices $\Gamma$ and $\Gamma^\mu$, and the structure constants generically denoted by $U^{ab}_{\ \ \mu}$ are the components of these same matrices multiplied from the left by the conjugation matrix $C$. In particular, the degree $-2$ (in $\theta$) piece of the super-Poisson bivector reads

$$g^{ab}(x^\bullet, x^\mu) = -\sqrt{\Lambda}\,(C\Gamma)^{ab}\,x^\bullet - (C\Gamma^\mu)^{ab}\,x_\mu\,, \tag{4.36}$$

where $x^\bullet$ is the coordinates in the direction $\mathfrak{L}$, and $x^\mu$ in the direction $\mathfrak{P}^\mu$.

As a final comment, let us note that there does not seem to be an obvious Lie algebroid structure on $\mathfrak{g}^*$, seen as super-vector bundle over $\mathfrak{g}_0^*$, which could provide a natural candidate for the odd homological vector field $\mathcal{Q}_S$ generating additional supersymmetry transformations. In the rest of this section, we will construct two examples based on Lie super-algebroids and show that they are essentially singled out through an analysis of the general case.

### 4.3 The ABST-G model and de Rham supersymmetry

The *differential Poisson sigma model*, introduced in [14, 15], is an example of super-Poisson sigma model, wherein the target space is obtained from a dg-Poisson manifold concentrated in degrees 0 and 1. As discussed previously, such a target space is equivalent to a shifted vector bundle $E[1]$, for which the space of sections of the exterior algebra of its dual, $\Gamma(\wedge E^*)$, is equipped with a (degree 0) Poisson bracket. In fact, the differential Poisson sigma model is obtained from the general model of section 4.1 for the choice $E = TM$, in other words, the target space is $T^*[1,0]T[0,1]M$.

To be specific, the fields of this differential Poisson sigma model are as in section 4.1 for $E = TM$, namely $(X^\mu, \theta^\mu, A^\nabla_\mu, \chi_\mu)$, and its action functional is

$$S_{TM} = \int \left( A^\nabla_\mu \wedge \mathrm{d}X^\mu + \chi_\mu \wedge \nabla\theta^\mu + \tfrac{1}{2}\Pi^{\mu\nu} A^\nabla_\mu \wedge A^\nabla_\nu + \tfrac{1}{4}R_{\kappa\lambda}^{\ \ \mu\nu}\chi_\mu \wedge \chi_\nu\, \theta^\kappa\theta^\lambda \right), \tag{4.37}$$

with the coefficient $R_{\kappa\lambda}^{\ \ \mu\nu}$ to be specified. This action is obtained by the following choice of components for the structure on the Poisson supermanifold:

$$\mathcal{P}^{\mu\nu} = \Pi^{\mu\nu}\,, \tag{4.38a}$$

$$\mathcal{P}^{\mu a} = -\Pi^{\mu\kappa}\Gamma^a_{\kappa\lambda}\theta^\lambda\,, \tag{4.38b}$$

$$\mathcal{P}^{ab} = \tfrac{1}{2}\big(R_{\rho\sigma}^{\ \ ab} + 2\Pi^{\kappa\lambda}\Gamma^{(a}_{\kappa\rho}\Gamma^{b)}_{\lambda\sigma}\big)\theta^\rho\theta^\sigma\,, \tag{4.38c}$$

where we kept the Latin indices to distinguish between different components without introducing an unnecessary layer of notation, even though in this case all indices are of the same type and in the position they appear. We observe that with respect to the expansion (3.17) of the super-Poisson structure, in this case the (inverse) metric $g^{\mu\nu}$ is completely degenerate and there is no quadratic piece in $\mathcal{P}^{\mu\nu}$. Moreover, the $T^*M$-connection is chosen to be the canonically induced one from an ordinary connection and thus the endomorphism $\phi$ vanishes.

With these identifications, we can now gradually unveil the particular graded geometrical structure of the target space. Before doing so, it is useful to mention that in the original formulation in terms of the field $A_\mu$, the action may be equivalently written as

$$S_{TM} = \int \left( A_\mu \wedge \mathrm{d}X^\mu + \chi_\mu \wedge \mathrm{d}\theta^\mu + \tfrac{1}{2}\Pi^{\mu\nu} A_\mu \wedge A_\nu \right.$$
$$\left. - \Pi^{\mu\kappa}\Gamma^\nu_{\kappa\lambda}\theta^\lambda A_\mu \wedge \chi_\nu + \tfrac{1}{4}(R_{\rho\sigma}{}^{\mu\nu} + 2\Pi^{\kappa\lambda}\Gamma^\mu_{\kappa\rho}\Gamma^\nu_{\lambda\sigma})\chi_\mu \wedge \chi_\nu \theta^\rho\theta^\sigma \right). \tag{4.39}$$

Turning to the homological vector fields that dictate the gauge symmetries and the supersymmetries respectively, once again it is much simpler to reveal the structure in the original formulation. Starting from the $(1,0)$ vector field $\mathcal{Q}$, it is completely specified by the components (4.38a)–(4.38c), together with the requirement that it is compatible with the graded symplectic structure. The result in this case is

$$\mathcal{Q} = \left(\Pi^{\mu\nu}a_\mu - \Pi^{\nu\kappa}\Gamma^\mu_{\kappa\rho}\chi_\mu\theta^\rho\right)\frac{\partial}{\partial x^\nu}$$
$$+ \left(-\Pi^{\mu\kappa}\Gamma^\nu_{\kappa\lambda}a_\mu\theta^\lambda + \tfrac{1}{2}\chi_\mu(R_{\rho\sigma}{}^{\mu\nu} + 2\Pi^{\kappa\lambda}\Gamma^{(\mu}_{\kappa\rho}\Gamma^{\nu)}_{\lambda\sigma})\theta^\rho\theta^\sigma\right)\frac{\partial}{\partial\theta^\nu}$$
$$+ \left(-\tfrac{1}{2}\partial_\lambda\Pi^{\mu\nu}a_\mu a_\nu - \Pi^{\mu\kappa}\Gamma^\nu_{\kappa\lambda}a_\mu\chi_\nu\theta^\lambda + \tfrac{1}{4}\partial_\lambda(R_{\rho\sigma}{}^{\mu\nu} + 2\Pi^{\kappa\lambda}\Gamma^{(\mu}_{\kappa\rho}\Gamma^{\nu)}_{\lambda\sigma})\chi_\mu\chi_\nu\theta^\rho\theta^\sigma\right)\frac{\partial}{\partial a_\lambda}$$
$$+ \left(\Pi^{\mu\kappa}\Gamma^\nu_{\kappa\sigma}a_\mu\chi_\nu + \tfrac{1}{2}(R_{\sigma\rho}{}^{\mu\nu} + 2\Pi^{\kappa\lambda}\Gamma^{(\mu}_{\kappa\sigma}\Gamma^{\nu)}_{\lambda\rho})\chi_\mu\chi_\nu\theta^\rho\right)\frac{\partial}{\partial\chi_\sigma}. \tag{4.40}$$

The supersymmetry generating homological vector field of degree $(0,1)$ in this case is

$$\mathcal{Q}_S = \theta^\mu\frac{\partial}{\partial x^\mu} - a_\mu\frac{\partial}{\partial\chi_\mu}. \tag{4.41}$$

It corresponds to the simple and linear supersymmetry transformations

$$\delta_S X^\mu = \theta^\mu, \qquad \delta_S\chi_\mu = -A_\mu, \qquad \delta_S\theta^\mu = 0 = \delta_S A_\mu, \tag{4.42}$$

which is easily checked to be a symmetry of the action $S_{TM}$. As noticed in [15], this supersymmetry is associated to the de Rham differential and we call it de Rham supersymmetry. That $\mathcal{Q}_S$ is homological is obvious, while its compatibility with the graded symplectic structure follows in a simple way from the fact that the conditions (A.18) hold. Regarding its commutator with the vector field $\mathcal{Q}$, according to the detailed analysis in appendix A.3, we get three independent conditions,[13] which we write directly in terms of the covariant component of $\mathcal{P}^{ab}_\nabla$, denoted as $R_{\kappa\lambda}{}^{\mu\nu}$ in (3.37):

$$\overline{\nabla}_\kappa\Pi^{\mu\nu} = 0, \tag{4.43a}$$

$$R_{\kappa\lambda}{}^{\mu\nu} = -\Pi^{(\mu|\rho}R_{\kappa\lambda}^{\overline{\nabla}}{}^{|\nu)}{}_\rho, \tag{4.43b}$$

$$\overline{\nabla}_{[\rho}R_{\kappa\lambda]}{}^{\mu\nu} - T^{\overline{\nabla}}_{[\rho\kappa}{}^\sigma R_{\lambda]\sigma}{}^{\mu\nu} = 0, \tag{4.43c}$$

where we introduced the opposite (or conjugated) connection $\overline{\nabla}$ defined by

$$\overline{\nabla}_X Y := \nabla_Y X + [X,Y] \qquad \Longleftrightarrow \qquad \overline{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}. \tag{4.44}$$

---

[13]The other two conditions are trivial because both $g^{\mu\nu}$ and $V^\mu$ vanish in this example.

The opposite connection is such that the "average" connection $\frac{1}{2}(\nabla + \overline{\nabla})$ is torsion-free. Note that the same result is obtained from (2.19), meaning that the opposite connection is the same as the basic connection introduced to facilitate the concept of representations up to homotopy, which explains why we use the same notation. We observe that in this example we have obtained the components of $\mathcal{P}_\nabla^{ab}$, which are fixed in terms of the Poisson structure on the base and the curvature of the basic connection. It is also worth mentioning that the curvature of the basic connection in the case of the tangent bundle is opposite to the basic curvature of the ordinary connection.

In the covariant formulation, we can write the vector field in terms of $a_\mu^\nabla = a_\mu + \Gamma_{\mu\nu}^\rho \theta^\nu \chi_\rho$ as

$$\mathcal{Q}_S = \theta^\mu \frac{\partial}{\partial x^\mu} - \left(a_\nu^\nabla \theta^\rho \overline{\Gamma}_{\rho\mu}^\nu + \frac{1}{2}\chi_\nu R_{\kappa\lambda}^{\overline{\nabla}}{}^\nu{}_\mu \theta^\kappa \theta^\lambda\right)\frac{\partial}{\partial a_\mu^\nabla} - \left(a_\mu^\nabla - \overline{\Gamma}_{\mu\rho}^\nu \theta^\rho \chi_\nu\right)\frac{\partial}{\partial \chi_\mu}\,, \qquad (4.45)$$

which generates the same supersymmetry transformations for the redefined fields and it is a symmetry of the classical action. Comparing this with the structure operator (2.44) in the representation up to homotopy perspective, we observe that the supersymmetry-generating homological vector field corresponds to the coadjoint representation of the tangent Lie algebroid with unit anchor map and the usual Lie bracket of vector fields.

### 4.4 The contravariant model and Poisson supersymmetry

We now introduce a different supersymmetric Poisson sigma model, this time with choice of vector bundle $E = T^*M$. Its target space is $T^*[1,0]T^*[0,1]M$ and the fields of the sigma model are $(X^\mu, \theta_\mu, A_\mu^\nabla, \chi^\mu)$.[14] The main difference is that the fermionic fields take values in the dual bundle compared to the previous case. For this reason, we will refer to this case as the contravariant supersymmetric Poisson sigma model. Its action functional is

$$S_{T^*M} = \int \left(A_\mu^\nabla \wedge \mathrm{d}X^\mu + \chi^\mu \wedge \nabla\theta_\mu + \frac{1}{2}\Pi^{\mu\nu}A_\mu^\nabla \wedge A_\nu^\nabla + \frac{1}{4}R^{\kappa\lambda}{}_{\mu\nu}\chi^\mu \wedge \chi^\nu \theta_\kappa \theta_\lambda\right), \qquad (4.46)$$

where the coefficient of the quartic term $R^{\kappa\lambda}{}_{\mu\nu}$ must be determined anew (and it is different than in the previous model). To understand the structure of this dual model, we should specify the geometric meaning of the coefficients that appear in it and the new type of supersymmetry it features. In accordance with our analysis in section 3, the bivector $\Pi^{\mu\nu}$ is again a Poisson structure on the body of the target supermanifold. From a non-manifestly covariant perspective, the choice of coefficients in the sigma model is

$$\mathcal{P}^{\mu\nu} = \Pi^{\mu\nu}\,, \qquad (4.47\text{a})$$

$$\mathcal{P}^\mu{}_\nu = \Pi^{\mu\kappa}\Gamma_{\kappa\nu}^\lambda \theta_\lambda\,, \qquad (4.47\text{b})$$

$$\mathcal{P}_{\mu\nu} = \frac{1}{2}\left(R^{\kappa\lambda}{}_{\mu\nu} - 2\Pi^{\rho\sigma}\Gamma_{\rho(\mu}^\kappa \Gamma_{\sigma\nu)}^\lambda\right)\theta_\kappa\theta_\lambda\,, \qquad (4.47\text{c})$$

corresponding to a degree 0 Poisson structure. Note the sign difference in the mixed component $\mathcal{P}^\mu{}_\nu$, since this refers to the dual connection coefficients. This fixes completely the homological

---

[14]Note that models with a double cotangent bundle target space where constructed in [58–61], dubbed (twisted) R-Poisson sigma models due to their particular geometric structure. These are different models, since they are purely bosonic and they were mainly considered in higher than 2 dimensions.

vector field $\mathcal{Q}$ and determines the full set of gauge symmetries of the model. We do not write their explicit form here, since they are similar to the ones of the previous model.

With regard to supersymmetry, the contravariant model exhibits a new feature; the supersymmetry transformations are controlled by the Poisson structure and its derivatives, and they are nonlinear. To track them correctly, let us first separate the action into kinetic and interaction terms. The kinetic sector is the one that contains derivatives, namely the first two terms in the action $S_{T^*M}$. The invariance of the kinetic sector under supersymmetry transformations is equivalent to the compatibility of the homological vector field $\mathcal{Q}_S$ with the symplectic structure. Referring to the data of $\mathcal{Q}_S$ as they appear in eq. (2.7) and to the compatibility conditions eq. (A.18), we observe that in the present case it suffices to determine the coefficients $t^{\mu\nu}$ and $V_\mu$. We fix these coefficients to be

$$t^{\mu\nu} = \Pi^{\mu\nu}\,, \tag{4.48a}$$

$$V_\mu = -\tfrac{1}{2}\,\partial_\mu \Pi^{\kappa\lambda}\theta_\kappa\theta_\lambda\,. \tag{4.48b}$$

Thus $t^{\mu\nu}$ is identified with the Poisson structure on the body of the supermanifold, which now plays a dual role, inducing both the gauge symmetries and the supersymmetry of the model. The kinetic sector of the sigma model respects the supersymmetry transformations generated by this $\mathcal{Q}_S$ provided that the rest of the coefficients are

$$Y^{\mu\nu} = \Pi^{\mu\nu}\,, \quad U^{\nu\rho}{}_\mu = \partial_\mu\Pi^{\rho\nu}\,, \quad Z_\mu{}^{\nu\rho} = \partial_\mu\Pi^{\nu\rho}\,, \quad W_{\mu\nu} = -\tfrac{1}{2}\partial_\mu\partial_\nu\Pi^{\rho\sigma}\theta_\rho\theta_\sigma\,. \tag{4.49}$$

We observe that the supersymmetry is completely determined by the Poisson structure already at kinematical level.

In addition, we must address the supersymmetry of the interaction sector. This amounts to the compatibility between the two homological vector fields $\mathcal{Q}$ and $\mathcal{Q}_S$, which is also studied in full generality in appendix A. There are three conditions that should be met, as in the previous case. The first condition reads

$$\Pi^{\mu\kappa}\Pi^{\nu\lambda}(\Gamma^\rho_{\kappa\lambda} - \Gamma^\rho_{\lambda\kappa}) = 0 \quad \Longleftrightarrow \quad T^\nabla\big(\Pi^\sharp(\eta_1), \Pi^\sharp(\eta_2)\big) = 0\,, \quad \forall\,\eta_1, \eta_2 \in \Gamma(T^*M)\,, \tag{4.50}$$

and it has an interesting geometric interpretation that ties it together with the previous case. Considering the basic connection in the present case, its coefficients are generally given by eq. (2.19), and focusing on its $T^*M$-on-$T^*M$ component, say $\overline{\nabla}^*$, we obtain

$$\overline{\Gamma}^{\mu\nu}{}_\rho = -\Pi^{\nu\kappa}\Gamma^\mu_{\kappa\rho} + \partial_\rho\Pi^{\mu\nu}\,. \tag{4.51}$$

A straightforward calculation leads to

$$T^\nabla\big(\Pi^\sharp(\eta_1), \Pi^\sharp(\eta_2)\big) = 0 \qquad \Longleftrightarrow \qquad \overline{\nabla}^*\Pi(\eta_1, \eta_2) = 0\,. \tag{4.52}$$

We observe that we obtain a condition analogous to the case of the tangent bundle; the Poisson structure is covariantly constant with respect to the basic connection in each case. Note that the other component of the basic connection is a $T^*M$-on-$TM$ one with components

$$\overline{\Gamma}^\mu{}_\rho{}^\nu = \Pi^{\nu\kappa}\Gamma^\mu_{\rho\kappa} - \partial_\rho\Pi^{\mu\nu}\,, \tag{4.53}$$

and that (4.51) and (4.53) are not a pair of dual connections, which is the reason we carefully placed their indices. The second condition reads

$$\Pi^{\nu\sigma}\big(R^{\kappa\lambda}{}_{\sigma\rho} + S^{\kappa\lambda}{}_{\sigma\rho}\big) = 0\,, \tag{4.54}$$

where $S$ is the basic curvature in this case, whose precise local coordinate expression is

$$S^{\kappa\lambda}{}_{\mu\nu} = \partial_\mu\partial_\nu\Pi^{\kappa\lambda} - 2\Gamma^{[\kappa}_{\mu\rho}\partial_\nu\Pi^{\lambda]\rho} - \Gamma^\rho_{\mu\nu}\partial_\rho\Pi^{\kappa\lambda} + 2\Gamma^{[\lambda}_{\sigma\nu}\partial_\mu\Pi^{\kappa]\sigma} - 2\Pi^{\sigma[\kappa}\partial_\sigma\Gamma^{\lambda]}_{\mu\nu} + 2\Pi^{\rho\sigma}\Gamma^{[\kappa}_{\mu\rho}\Gamma^{\lambda]}_{\sigma\nu}\,. \tag{4.55}$$

This means that we can take the coefficient $R^{\kappa\lambda}{}_{\mu\nu}$ to be opposite to the basic curvature up to terms in the kernel of the map $\Pi^\sharp$. To complete the analysis and fully determine the coefficient, we determine the third and final condition from supersymmetric invariance of the interaction sector, which reads

$$\Pi^{\rho[\sigma}\partial_\rho R^{\kappa\lambda]}{}_{(\mu\nu)} + R^{\rho[\sigma}{}_{(\mu\nu)}\partial_\rho\Pi^{\kappa\lambda]} - 2\overline{\Gamma}^{[\sigma|\rho}{}_{(\mu|}R^{|\kappa\lambda]}{}_{\rho|\nu)} = 0\,. \tag{4.56}$$

This condition bears a striking resemblance with the Bianchi identity verified by the basic curvature, which reads

$$\Pi^{[\sigma|\rho}\partial_\rho S^{|\kappa\lambda]}{}_{\mu\nu} + S^{[\sigma|\rho}{}_{\mu\nu}\partial_\rho\Pi^{|\kappa\lambda]} + \overline{\Gamma}^{[\sigma|\rho}{}_\nu S^{|\kappa\lambda]}{}_{\mu\rho} - \overline{\Gamma}^{[\sigma|}{}_\mu{}^\rho S^{|\kappa\lambda]}{}_{\rho\nu} = 0\,, \tag{4.57}$$

and differs from (4.56) only by the last term involving the component of the $T^*M$-on-$TM$ part of the basic connection (4.53). Comparing the latter with (4.51), the component of the $T^*M$-on-$T^*M$ part of the basic connection, one finds that they are related by

$$\overline{\Gamma}^\mu{}_\lambda{}^\nu = -\overline{\Gamma}^{\mu\nu}{}_\lambda - \Pi^{\nu\alpha}T_{\alpha\lambda}{}^\mu\,, \tag{4.58}$$

with $T_{\alpha\beta}{}^\gamma = 2\,\Gamma^\gamma_{[\alpha\beta]}$ the torsion of $\nabla$, and hence the Bianchi identity for the basic curvature can be re-written as

$$\Pi^{[\sigma|\rho}\partial_\rho S^{|\kappa\lambda]}{}_{\mu\nu} + S^{[\sigma|\rho}{}_{\mu\nu}\partial_\rho\Pi^{|\kappa\lambda]} + \overline{\Gamma}^{[\sigma|\rho}{}_\nu S^{|\kappa\lambda]}{}_{\mu\rho} + \overline{\Gamma}^{[\sigma|\rho}{}_\mu S^{|\kappa\lambda]}{}_{\rho\nu} + \Pi^{\alpha\beta}T_{\beta\mu}{}^{[\sigma}S^{\kappa\lambda]}{}_{\alpha\nu} = 0\,. \tag{4.59}$$

Knowing that $R$ is opposite to the basic curvature, up to terms in the kernel, we conclude that we may take the coefficient of the quartic term in the action to be[15]

$$R^{\kappa\lambda}{}_{\mu\nu} = -S^{\kappa\lambda}{}_{(\mu\nu)}\,, \tag{4.60}$$

upon also imposing that the torsion of $\nabla$ belongs to the kernel of $\Pi^\sharp$,

$$\Pi^{\mu\rho}T_{\rho\nu}{}^\sigma = 0 \qquad \Longleftrightarrow \qquad T^\nabla(\Pi^\sharp(\eta), -) = 0\,, \quad \forall\eta \in \Gamma(T^*M)\,, \tag{4.61}$$

thereby implying the previously encountered condition (4.50). Note that, in light of eq. (4.58), this condition amounts to requiring that the basic connection is indeed formed of a pair of

---

[15]Let us remark that the symmetric part of the basic curvature for the cotangent Lie algebroid (in its lower indices) is given by

$$S^{\kappa\lambda}{}_{(\mu\nu)} = \mathring{S}^{\kappa\lambda}{}_{\mu\nu} - \tfrac{1}{2}\,\Pi^{\rho\sigma}\,T^{[\kappa}_{\rho\mu}\,T^{\lambda]}_{\sigma\nu}\,,$$

where $\Gamma^\lambda_{\mu\nu} = \mathring{\Gamma}^\lambda_{\mu\nu} + \tfrac{1}{2}\,T^\lambda_{\mu\nu}$ with $\mathring{\Gamma}^\lambda_{\mu\nu}$ the symmetric, i.e. torsionless, part of the connection $\nabla$, and $\mathring{S}$ denotes the basic curvature with respect to this torsion-free connection.

dual connections. Equivalently, this amounts to the requirement that the basic connection preserves the non-degenerate bilinear form on the complex $TM \oplus T^*M$ that is the canonical pairing between the tangent and cotangent bundle, i.e.

$$\Pi^\sharp(\eta)\langle X, \eta'\rangle = \langle \overline{\nabla}_\eta X, \eta'\rangle + \langle X, \overline{\nabla}_\eta \eta'\rangle \,, \qquad X \in \Gamma(TM)\,,\ \eta, \eta' \in \Gamma(T^*M)\,, \qquad (4.62)$$

where on the right hand side, the first term is the $T^*M$-on-$TM$ part of the basic connection, and the second term its $T^*M$-on-$T^*M$ part. A direct computation shows that the above identity holds if and only if (4.61) is satisfied.

Summarizing the discussion on supersymmetry and keeping in mind the identified geometrical conditions, the homological vector field that generates supersymmetry transformations is

$$\mathcal{Q}_S = \Pi^{\mu\nu}\theta_\mu \frac{\partial}{\partial x^\nu} - \tfrac{1}{2}\partial_\mu \Pi^{\kappa\lambda}\theta_\kappa \theta_\lambda \frac{\partial}{\partial \theta_\mu} +$$
$$+ \left(\partial_\rho \Pi^{\nu\mu}a_\mu \theta_\nu - \tfrac{1}{2}\partial_\rho\partial_\nu \Pi^{\kappa\lambda}\theta_\kappa \theta_\lambda \chi^\nu\right)\frac{\partial}{\partial a_\rho} + \left(\Pi^{\mu\rho}a_\mu - \partial_\mu\Pi^{\rho\nu}\chi^\mu\theta_\nu\right)\frac{\partial}{\partial \chi^\rho}\,. \qquad (4.63)$$

Indeed, besides being homological, this $(0,1)$ vector field is also compatible with the symplectic structure, since it satisfies the corresponding consistency conditions. We write the supersymmetry transformations of the various fields explicitly:

$$\delta_S X^\mu = -\Pi^{\mu\nu}\theta_\nu \,, \qquad (4.64a)$$

$$\delta_S \theta_\mu = -\tfrac{1}{2}\,\partial_\mu \Pi^{\kappa\lambda}\theta_\kappa \theta_\lambda \,, \qquad (4.64b)$$

$$\delta_S A_\mu = \partial_\mu \Pi^{\kappa\lambda}A_\lambda \theta_\kappa - \tfrac{1}{2}\partial_\mu\partial_\nu \Pi^{\kappa\lambda}\theta_\kappa \theta_\lambda \chi^\nu \,, \qquad (4.64c)$$

$$\delta_S \chi^\mu = -\Pi^{\mu\nu}A_\nu - \partial_\rho \Pi^{\mu\nu}\chi^\rho \theta_\nu \,. \qquad (4.64d)$$

When we redefine the bosonic 1-form to $A^\nabla$, the supersymmetry transformations for $A^\nabla$ and $\chi^\mu$ are given in terms of the components (4.51) and (4.53) of the basic connection and they become

$$\delta_S A^\nabla_\mu = -\overline{\Gamma}^\rho{}_\mu{}^\nu A^\nabla_\nu \theta_\rho - \tfrac{1}{2}S^{\kappa\lambda}{}_{\mu\nu}\theta_\kappa \theta_\lambda \chi^\nu \,, \qquad (4.65a)$$

$$\delta_S \chi^\mu = -\Pi^{\mu\nu}A^\nabla_\nu + \overline{\Gamma}^{\rho\mu}{}_\nu \chi^\nu \theta_\rho \,. \qquad (4.65b)$$

In this form it becomes clear that the Poisson-supersymmetry generating homological vector field corresponds precisely to the structure operator for the coadjoint representation (up to homotopy) of the cotangent Lie algebroid.

## 4.5 Lie algebroid models and anchor supersymmetry

We now return to a more general context with the purpose of determining the common characteristics of the models described in sections 4.3 and 4.4, and investigating whether other examples can be constructed and under which conditions. To this end, we consider any Lie algebroid $E$, giving rise to a Poisson supermanifold $E[0,1]$ and we take $T^*[1,0]E[0,1]$ as target space of the sigma model. The previous two examples correspond to the "extremal" cases of tangent and cotangent Lie algebroids. To keep the discussion more general, we include an order 0 component in $\mathcal{P}^{ab}$ and $V^a$, both of which were vanishing in the examples

above. Nevertheless, we still assume that the connection is the induced one and that $\mathcal{P}^{\mu\nu}$ has only degree 0 components given by the Poisson structure. Hence we choose:

$$\mathcal{P}^{\mu\nu} = \Pi^{\mu\nu}, \tag{4.66a}$$

$$\mathcal{P}^{\mu a} = -\Pi^{\mu\nu}\Gamma^a_{\nu b}\theta_b, \tag{4.66b}$$

$$\mathcal{P}^{ab} = g^{ab} + \tfrac{1}{2}\big(R_{cd}{}^{ab} + 2\Pi^{\rho\sigma}\Gamma^{(a}_{\rho c}\Gamma^{b)}_{\sigma d}\big)\theta^c\theta^d, \tag{4.66c}$$

for some $R_{cd}{}^{ab} = R_{[cd]}{}^{(ab)}$ to be determined by supersymmetry and for a symmetric tensor $g$ with components $g^{ab}$, not necessarily nondegenerate.[16] Note that for vanishing $g$ the Poisson structure is not just even but actually of degree 0, a feature shared by the examples in sections 4.3 and 4.4. On the other hand, when $\Pi^{\mu\nu}$ and $R_{cd}{}^{ab}$ vanish and the only nonvanishing component is $g$, the Poisson structure has degree $-2$. We will mention an example of this type at the end of this section. In general, we already know that for gauge invariance of the theory, the induced connection must satisfy the metricity condition (3.30) with respect to $g^{ab}$. Moreover, in the basis that corresponds to the splitting of the supermanifold, for these choices we have

$$\mathcal{P}^{\mu a}_{\nabla} = 0 \qquad \text{and} \qquad \mathcal{P}^{ab}_{\nabla} = g^{ab} + \tfrac{1}{2}\theta^c\theta^d R_{cd}{}^{ab}. \tag{4.67}$$

Furthermore, we know from the graded Jacobi identity that the following algebraic and differential Bianchi identities hold due to (3.39a) and (3.39b), respectively:

$$g^{d(e}R_{cd}{}^{ab)} = 0, \tag{4.68a}$$

$$\Pi^{\mu\nu}\nabla_\nu R_{cd}{}^{ab} = 0. \tag{4.68b}$$

These, together with $R^\nabla(\Pi^\sharp, \Pi^\sharp) = 0$, are all we need for gauge invariance of the model, based on the detailed analysis of section 3.

We now turn to the supersymmetry of the model. Regarding the kinetic sector there is nothing new to add with respect to the general case; its invariance under supersymmetry fixes most of the coefficients in $\mathcal{Q}_S$ as in (4.7). The two yet undetermined coefficients, $t_a{}^\mu$ and $V^a$ may be expanded in powers of $\theta^2$, as explained in section 2. We will not deal with the most general case, and instead we will restrict up to second order and assume that these two coefficients take the form

$$t_a{}^\mu(x, \theta^2) = t_a{}^\mu(x), \tag{4.69a}$$

$$V^a(x, \theta^2) = \mathring{V}^a(x) + \tfrac{1}{2}C_{bc}{}^a(x)\theta^b\theta^c. \tag{4.69b}$$

In plain words, $t_a{}^\mu$ is only a function of the bosonic scalar field, and all powers equal to or higher than $\theta^2$ vanish. Geometrically, these are the components of the map $t : E \to TM$, the anchor of the Lie algebroid $E \twoheadrightarrow M$. The coefficient $V^a$ contains a constant and a quadratic term in $\theta$, which respectively correspond to a section $\mathring{V} \in \Gamma(E)$ with components $\mathring{V}^a$, and to the structure coefficients of the Lie bracket on sections of $E$. We can now analyze order by order in $\theta$ the conditions for supersymmetry in the interaction sector of the model.

---

[16]Although the position of indices suggests that we should call this an inverse metric $g^{-1}$, we refrain from using this notation because we have not assumed nondegeneracy and indeed in both previous examples we had a completely degenerate metric.

**Order 0.** At lowest order we get two equations:

$$\tfrac{1}{2}t_a{}^\rho \partial_\rho \Pi^{\mu\nu} + \Pi^{\rho[\mu}\big(\partial_\rho t_a{}^{\nu]} - \Gamma_{\rho a}^b t_b{}^{\nu]}\big) = 0\,, \tag{4.70a}$$

$$\Pi^{\mu\nu}\nabla_\mu \mathring{V}^a + g^{ab}t_b{}^\nu = 0\,. \tag{4.70b}$$

To clarify their geometrical meaning, let us first recall what these conditions amounted to in the two previous examples. The second condition was trivial in both cases, since both $g$ and $\mathring{V}$ were zero. The first condition had the same geometric meaning for both the tangent and the cotangent case: it said that the Poisson structure $\Pi$ is covariantly constant with respect to the basic $E$-on-$TM$ connection in each case. Since we identify the map $t$ with the anchor of the Lie algebroid, we can write the first condition in terms of the coefficients of this connection given in (2.19) as

$$\tfrac{1}{2}t_a{}^\rho \partial_\rho \Pi^{\mu\nu} - \Pi^{\rho[\mu}\overline{\Gamma}_{a\rho}^{\nu]} = 0\,. \tag{4.71}$$

This is precisely the condition of covariant constancy of the Poisson structure on the base with respect to the basic $E$-on-$TM$ connection. Turning to the second condition, we observe that the right-hand side of (4.70b) is the composition of the map $t : E \to TM$ and the metric $g$, which gives rise to the induced map $g^\sharp : E^* \to E$. The left-hand side is the induced connection acting on the section $\mathring{V}$. In total, the geometric form of the two lowest order conditions is

$$\overline{\nabla}^E \Pi = 0\,, \tag{4.72a}$$

$$\overset{\bullet}{\nabla}\mathring{V} = t \circ g^\sharp\,. \tag{4.72b}$$

Note that the covariant constancy of the Poisson bivector $\Pi$ with respect to the basic connection can be re-written as

$$\overline{\nabla}^E_{t(e)}\Pi \equiv \mathcal{L}_{t(e)}\Pi - t_\wedge(\overset{\bullet}{\nabla}e) = 0\,, \qquad e \in \Gamma(E)\,, \tag{4.73}$$

where

$$t_\wedge : \Gamma(TM) \otimes \Gamma(E) \xrightarrow{-\otimes t} \Gamma(TM) \otimes \Gamma(TM) \xrightarrow{\ \wedge\ } \Gamma(\wedge^2 TM) \tag{4.74}$$

denotes the application of the anchor map on the second factor of the tensor product, then antisymmetrization. In components, it reads

$$\mathcal{L}_{t_a}\Pi^{\mu\nu} = 2\,\Pi^{[\mu|\lambda}\Gamma_{\lambda a}^b t_b{}^{|\nu]}\,, \tag{4.75}$$

i.e. the Lie derivative of the Poisson bivector $\Pi$ along the distribution defined by the anchor of $E$ does not vanish, but is instead encoded by the covariant derivative of this distribution by the induced $T^*M$-on-$E$ connection.

**Order 1.** There is only one condition at this order. It reads

$$\Pi^{\mu\nu}\Gamma_{\nu c}^{(a}\partial_\mu \mathring{V}^{b)} - \tfrac{1}{2}t_c{}^\mu \partial_\mu g^{ab} - \tfrac{1}{2}\mathring{V}^d \mathcal{P}_{dc}{}^{ab} - g^{d(a}C_{dc}{}^{b)} = 0\,. \tag{4.76}$$

It is trivially satisfied in the previous examples where both $\mathring{V}$ and $g$ were zero. Using the order 0 result, we can write this condition in a more transparent and covariant form,

$$\overline{\nabla}_c g^{ab} + R_{cd}{}^{ab}\mathring{V}^d = 0 \qquad \Longleftrightarrow \qquad \overline{\nabla}^E g = \imath_{\mathring{V}}R\,, \tag{4.77}$$

where $\imath_{\mathring{V}}$ denotes the interior product with $\mathring{V}$.

**Order 2.** Under the assumptions we made, there is also only one condition at order 2. This is:

$$\Pi^{\mu\nu} S_{ab\,\nu}{}^c - R_{ab}{}^{cd} t_d{}^\mu = 0\,, \qquad (4.78)$$

which relates the tensorial part $R_{ab}{}^{cd}$ of the quadratic piece of the Poisson structure on $E[0,1]$ to the basic curvature $S_{ab\mu}{}^c$. When evaluated on a pair of sections $e, e' \in \Gamma(E)$, a 1-form $\eta \in \Omega^1(M)$ and a section of the dual vector bundle $e^* \in \Gamma(E^*)$, the above identity reads

$$\langle S(e,e')\Pi^\sharp(\eta), e^* \rangle = \langle R(e,e'), t^*(\eta) \otimes e^* \rangle\,, \qquad (4.79)$$

where $\langle \cdot, \cdot \rangle$ denote the canonical pairing between sections of $E$ and its dual $E^*$, and $t^* : \Gamma(T^*M) \to \Gamma(E^*)$ is the dual map of the anchor. The fact that these two tensors are related by contraction with the Poisson bivector $\Pi$ of the base manifold $M$ and the anchor $\rho$ of the Lie algebroid $E$ makes it clear that the previous two examples, $E = TM$ and $E = T^*M$, are singled out in this framework. Indeed, for the tangent bundle, the anchor is the identity so that this condition can be read as a *definition* of the tensor $R_{ab}{}^{cd}$. In the cotangent bundle case, the anchor is the base Poisson bivector, so that one can factor it and read the previous condition as a definition of the tensor $R_{ab}{}^{cd}$ again, *up to terms in the kernel of* $\Pi^\sharp$.

There is one further possibility that allows for $R_{ab}{}^{cd}$ to be determined. If the anchor $t$ is *invertible*, which means that $E \cong TM$ — the vector bundle $E$ is isomorphic to the tangent bundle $TM$ as a vector bundle over $M$ — then the previous identity can also be read as a definition of the tensor $R$. Note that this implies that $E$ is isomorphic to $TM$ as a *Lie algebroid*, since the anchor is also required to be a Lie algebra morphism between sections of $E$ and vector fields on the base.

As an example, one may consider any endomorphism $J \in \Gamma\big(\mathrm{End}(TM)\big)$ of the tangent bundle whose Nijenhuis tensor,

$$N_J : \Gamma(TM) \otimes \Gamma(TM) \longrightarrow \Gamma(TM)\,, \qquad (4.80)$$

defined on a pair of vector fields $X, Y \in \Gamma(TM)$ as

$$N_J(X,Y) := -J^2[X,Y] + J\big([J(X),Y] + [X,J(Y)]\big) - [J(X),J(Y)]\,, \qquad (4.81)$$

vanishes, and construct a Lie algebroid structure on $TM$, by taking $J$ as its anchor, and

$$[X,Y]_J := [J(X),Y] + [X,J(Y)] - J([X,Y])\,, \qquad X, Y \in \Gamma(TM)\,, \qquad (4.82)$$

for its Lie bracket [62]. The latter obeys the Jacobi identity as a consequence of the vanishing of the Nijenhuis tensor for $J$. Moreover, as long as $J$ is invertible, then $(TM, J, [\cdot, \cdot]_J)$ is a Lie algebroid with bijective anchor. Examples of such invertible endomorphisms of particular interest are complex structures on $M$, namely those endomorphisms $J$ which square to minus the identity, $J^2 = -\mathbf{1}_{TM}$, and whose Nijenhuis tensor vanishes.[17]

More generally, since the condition (4.79) relates the evaluation of the tensor $R$ on sections of $\mathrm{Im}(t^*) \otimes E^*$ to the composition of the basic curvature and the sharp-morphism, this identity

---

[17]Note that generalized complex structures, introduced by Hitchin [63], are at the heart of a number of two-dimensional Poisson sigma models, see for instance [64–69] and references therein.

becomes sufficient to determine $R$ if the dual anchor $t^*$ is surjective, or equivalently if the anchor $t$ is injective. Such Lie algebroids amount to being given an *involutive* distribution on $M$, corresponding to the image of the anchor.

Action Lie algebroids provide examples where the anchor can be injective, or even bijective, depending on the setup. Suppose that a Lie algebra $\mathfrak{g}$ acts on a manifold $M$, that is, there exists a Lie algebra morphism $\mathfrak{g} \longrightarrow \Gamma(TM)$ sending elements of $\mathfrak{g}$ to vector fields on $M$. The trivial bundle $M \times \mathfrak{g}$ is then endowed with a Lie algebroid structure with anchor

$$ t : \Gamma(M \times \mathfrak{g}) \cong \mathscr{C}^\infty(M) \otimes \mathfrak{g} \longrightarrow \Gamma(TM) \,, \tag{4.83} $$

being simply the $\mathscr{C}^\infty(M)$-linear extension of the Lie algebra action, and bracket

$$ [f \otimes \xi, g \otimes \xi'] = f\, g \otimes [\xi, \xi']_{\mathfrak{g}} + f\, (t(\xi)g) \otimes \xi' - g\, (t(\xi')f) \otimes \xi \,, \tag{4.84} $$

for $f, g \in \mathscr{C}^\infty(M)$ and $\xi, \xi' \in \mathfrak{g}$. The action of $\mathfrak{g}$ on $M$ being (locally) free is equivalent to the anchor $t$ being injective. Put differently, at every point of $M$, the representation of $\mathfrak{g}$ is *faithful* (i.e. no element is represented trivially). Another, closely related source of examples are gauge Lie algebroids (see e.g. [70] for more details concerning such Lie algebroids).

**Order 3.** Finally, there exists a single order 3 consistency condition that must be satisfied for the theory to have supersymmetric invariance, which reads

$$ t_{[a|}{}^\mu\, \partial_\mu R_{|bc]}{}^{de} + C_{[ab}{}^\times R_{c]\times}{}^{de} + 2R_{[ab}{}^{\times(d}\, \overline{\Gamma}{}^{e)}_{c]\times} = 0 \,, \tag{4.85} $$

which is nothing but the component form of the 'Bianchi-like' identity

$$ \mathrm{d}_{\overline{\nabla}^E} R = 0 \,. \tag{4.86} $$

Put differently, the tensorial part of $\mathcal{P}^{ab}$ that is $R \in \Gamma(\wedge^2 E^* \otimes S^2 E)$, which can be thought of as a cochain in the complex $\Omega(E, SE)$ of forms on $E$ valued in its symmetric algebra, is annihilated by the differential $\mathrm{d}_{\overline{\nabla}^E}$ associated with the basic connection. The resemblance with the Bianchi identity for the basic curvature is again noticeable, although it is important to keep in mind that the tensor $R$ does not take values in the same bundle as the basic curvature.

**Closure of the supersymmetry.** Apart from the invariance of the action under supersymmetry transformations, we would like these transformations to close into an algebra. For this reason, we impose in addition that the vector field $\mathcal{Q}_S$ is homological. Under the assumptions of the present section,[18] we obtain an additional set of conditions on the undetermined coefficients. At order 0 we obtain

$$ t_a{}^\mu \mathring{V}^a = 0 \qquad \Longleftrightarrow \qquad t(\mathring{V}) = 0 \,, \tag{4.87} $$

i.e. that $\mathring{V}$ is an $E$-section in the kernel of the anchor map. On top of that, a last condition appears at order 1,

$$ t_b{}^\mu \partial_\mu \mathring{V}^a + C_{bc}{}^a \mathring{V}^c = 0 \,, \tag{4.88} $$

---

[18]Assumptions that, in particular, include that $E$ be a Lie algebroid.

which, upon using the previous condition, can be re-written as

$$t_b{}^\mu \partial_\mu \mathring{V}^a + (C_{bc}{}^a + t_c{}^\mu \Gamma_{\mu b}{}^a) \mathring{V}^c \equiv \overline{\nabla}_b \mathring{V}^a = 0 \,, \tag{4.89}$$

i.e. the section $\mathring{V} \in \Gamma(E)$ is covariantly constant with respect to the basic connection,

$$\overline{\nabla}^E \mathring{V} = 0 \,. \tag{4.90}$$

Note that, under the change of coordinates $(x^\mu, \theta^a, a_\mu, \chi_a) \to (x^\mu, \theta^a, a_\mu^\nabla, \chi_a)$ where, as before

$$a_\mu^\nabla = a_\mu + \Gamma_{\mu a}^b \, \theta^a \chi_b \,, \tag{4.91}$$

the homological vector field $\mathcal{Q}_S$ takes the form

$$\mathcal{Q}_S = \mathcal{Q}_S^{(-1)} + \mathcal{Q}_S^{(0)} + \mathcal{Q}_S^{(+1)} + \mathcal{Q}_S^{(+2)} \,, \tag{4.92}$$

with

$$\mathcal{Q}_S^{(-1)} = \mathring{V}^a \, \tfrac{\partial}{\partial \theta^a} \,, \tag{4.93a}$$

$$\mathcal{Q}_S^{(0)} = \nabla_\mu \mathring{V}^a \, \chi_a \, \tfrac{\partial}{\partial a_\mu^\nabla} - a_\mu^\nabla \, t_a{}^\mu \, \tfrac{\partial}{\partial \chi_a} \,, \tag{4.93b}$$

$$\mathcal{Q}_S^{(1)} = \theta^a \, t_a{}^\mu \, \tfrac{\partial}{\partial x^\mu} - \tfrac{1}{2} \theta^a \theta^b \, C_{ab}{}^c \, \tfrac{\partial}{\partial \theta^c} + \theta^a \, \overline{\Gamma}_{ab}^c \, \chi_c \, \tfrac{\partial}{\partial \chi_b} + \theta^a \, \overline{\Gamma}_{a\mu}^\nu(x) \, a_\nu^\nabla \, \tfrac{\partial}{\partial a_\mu^\nabla} \tag{4.93c}$$

$$\mathcal{Q}_S^{(2)} = -\tfrac{1}{2} \theta^a \theta^b \, \chi_c \, S_{ab\,\mu}{}^c \, \tfrac{\partial}{\partial a_\mu^\nabla} \,. \tag{4.93d}$$

We can recognize, in the above expression, the homological vector field associated with the coadjoint representation up to homotopy of the Lie algebroid $E$ spelled out in (2.44) previously, together with the interior product with $\mathring{V}$, as the piece $\mathcal{Q}_S^{(-1)}$, and contraction with the covariant derivative of $\mathring{V}$ as part of the piece $\mathcal{Q}_S^{(0)}$.

To summarize, subject to the above conditions on the geometrical data we have obtained a class of supersymmetric Poisson sigma models based on a Lie algebroid $E$ with "anchor supersymmetry", meaning that the supersymmetry transformations are controlled by the anchor of the Lie algebroid and the structure constants of its Lie bracket, together with the section $\mathring{V}$. This includes the case when the anchor vanishes altogether, as for example in the case of the Lie algebroid that corresponds to a bundle of Lie algebras. A simple yet nontrivial model where this is the case would be to consider a degree $-2$ Poisson structure, in other words by keeping only the metric $g$ in the super-Poisson bivector. This leads to the simple action

$$S[X, A, \theta, \chi] = \int A_\mu \wedge dX^\mu + \chi_a \wedge d\theta^a + \tfrac{1}{2} g^{ab}(X) \, \chi_a \wedge \chi_b \,. \tag{4.94}$$

If we consider in addition that $g$ is nondegenerate, then the compatibility conditions of the supersymmetry with the gauge symmetry result in the supersymmetry transformations

$$\delta_S X^\mu = 0 \,, \tag{4.95a}$$

$$\delta_S \theta^a = \mathring{V}^a(X) - \tfrac{1}{2} C_{bc}{}^a \, \theta^b \theta^c \,, \tag{4.95b}$$

$$\delta_S A_\mu = \partial_\mu \mathring{V}^a(X) \, \chi_a \,, \tag{4.95c}$$

$$\delta_S \chi_a = -C_{ab}{}^c \, \theta^b \chi_c \,, \tag{4.95d}$$

for arbitrary $\mathring{V} \in \Gamma(X^*E)$, where $C_{ab}{}^c$ are the structure constants of the pointwise bracket from the fiber Lie algebras, and $g$ is an ad*-invariant fiberwise metric as a consequence of (4.76).

# 5 Conclusions and outlook

In this paper, we revisited the supersymmetric version of the Poisson sigma model, originally proposed in [16], from the point of view of NQ *supermanifolds*. The latter being equipped with two gradings, over $\mathbb{Z}$ and over $\mathbb{Z}_2$, two kinds of homological vector fields can be considered, which we identify as controlling the (bosonic and fermionic) gauge symmetry and the (global) supersymmetry of the model, respectively. Apart from being homological, gauge and supersymmetry invariance of the kinetic sector requires compatibility with the graded symplectic structure of the target space. Invariance of the interaction sector under global supersymmetry transformations imposes the additional condition that the two homological vector fields are graded-commuting. We have analysed this set of conditions in detail throughout the paper and explained the geometric data they entail. Our main conclusions are summarized as follows:

- There exists a distinguished class of supersymmetric Poisson sigma models, where all the above conditions are met, for supermanifolds originating from Lie algebroids. The class comprises (i) the differential Poisson sigma model described in ref. [14], based on the canonical tangent Lie algebroid, (section 4.3) (ii) a set of models with invertible anchor map (section 4.5), and (iii) a new model, the contravariant supersymmetric Poisson sigma model, based on the cotangent Lie algebroid over the body of the Poisson supermanifold (section 4.4).

- The contravariant model is itself distinguished within this class as the single case where the anchor map is not necessarily invertible, instead being identified with the map induced by the Poisson structure on the body of the supermanifold, which is itself part of the full Poisson structure on the total supermanifold. Since this controls the supersymmetry transformations, this model exhibits a property we call Poisson supersymmetry, which is generically nonlinear.

- The common characteristic of all these models is their underlying mathematical structure. The graded-commutativity of the two homological vector fields implies that the Poisson structure on the body of the supermanifold is covariantly constant with respect to the basic $E$-connection on the chain complex $E \xrightarrow{\rho} TM$. The quartic term in the fermions has a coefficient directly related to the basic curvature tensor of an ordinary connection on the Lie algebroid. This led us to the conclusion that in all cases the supersymmetry-generating vector field is precisely identified with the coadjoint representation up to homotopy of the associated Lie algebroid.

**An outlook towards quantization.** The path integral quantization of the supersymmetric Poisson sigma model considered in [14], and of the new example presented in section 4.4, would be particularly interesting to understand, as one may expect them to produce a deformation quantization of the algebra of differential forms or polyvectors, respectively. This possibility has intriguing implications, starting with the fact that there seem to be several ways of obtaining a deformation quantization of such structures. For the sake of definiteness, let us focus on the case of differential forms. They appear as functions on the

parity-shifted tangent bundle, which can be made into a Poisson supermanifold, as explained in [14] and reviewed in section 4.3. The super-Poisson structure (4.38) is however not the most general one can consider on $\Pi TM$, as there is in principle no reason to discard terms of higher order in the fermionic coordinates — we made this assumption merely to simplify our analysis. This begs the question: what is the dependency of the quantization of $\Omega(M)$ on the a priori different super-Poisson structures it admits? In the case of ordinary Poisson manifold, the star-product obtained by deformation quantization only depends on the equivalence class of the Poisson bivector under (formal) diffeomorphisms [1]. In fact, this extends to the deformation quantization of *homotopy Poisson manifolds* obtained as a corollary of the *relative formality* theorem proved by Cattaneo and Felder [71, 72] (see also [73] for a review), and from a different perspective, by Lyakhovich and Sharapov [74].

Recall that homotopy Poisson manifolds are $\mathbb{Z}$-graded manifolds $\mathcal{M}$ equipped with a family of polyvectors $\{\Pi_r\}$ of rank $r \geq 1$ and of degree $2 - r$ (see e.g. [75, 76]), which are in involution, meaning their Schouten-Nijenhuis bracket vanish,

$$\sum_{i+j=r} [\Pi_i, \Pi_j]_{\mathrm{SN}} = 0\,, \qquad r \geq 1\,.$$

A Poisson $\mathbb{Z}$-graded manifold is an example of homotopy Poisson manifold, with the family of polyvectors reducing to a single Poisson bivector. A slightly less trivial example would be that of a *dg-Poisson manifold*, which is a special case with homotopy Poisson structure consisting of an homological vector field $Q$, and a $Q$-invariant Poisson bivector $\Pi$, as the involution condition reads

$$Q^2 = 0\,, \qquad \mathcal{L}_Q \Pi = 0\,, \qquad [\Pi, \Pi]_{\mathrm{SN}} = 0\,.$$

This is exactly the type of structure that we considered on $\mathcal{M} \cong E[0, 1]$, where $Q$ is the restriction of $\mathcal{Q}_S$ to $\mathcal{M}$ (more specifically the pushforward of $\mathcal{Q}_S$ by the projection $T^*[1, 0]\mathcal{M} \twoheadrightarrow \mathcal{M}$), and $\Pi$ the super-Poisson bivector $\mathcal{P}$. This means that the parity-shifted tangent bundle can, in fact, also be considered as a homotopy Poisson manifold, which in particular is a $\mathbb{Z}$-graded manifold. In this case, the restriction of the Poisson bracket having the form (4.38) becomes justified as it is the most general form allowed by the requirement that it be of $\mathbb{Z}$-degree 0.

Homotopy Poisson structures on $\mathcal{M}$ are in bijection with self-commuting functions of degree 2 on $T^*[1]\mathcal{M}$ [77, 78] (in complete analogy with the result of [7]). From the AKSZ perspective, this function which contains all polyvectors of the homotopy Poisson structure, should be used as the interaction term of the sigma model. This suggests that, in the dg-Poisson case, the homological vector field should be part of the data to be quantized. More concretely, this would mean that for differential forms, not only the wedge product is deformed into a star-product, but also the de Rham differential is deformed into a new operator, subject to compatibility conditions with the star-product (in general forming a possibly curved $A_\infty$-algebra). Concerning the models discussed in this paper, this would amount to adding the Hamiltonian function for $\mathcal{Q}_S$ to the action, but this does not seem possible, due to the fact that the latter is parity odd.

To summarize, there seems to be several ways of quantizing differential forms, depending on what type of structure is used to encoded them, be it a Poisson supermanifold, with

Poisson structure possibly invariant under a parity-odd homological vector field, or a homotopy Poisson manifolds. The path integral quantization of the corresponding sigma models may shed light on these possible differences and similarities.

**An outlook towards higher dimensions.** As mentioned in the introduction, the Poisson sigma model is *the* AKSZ sigma model in two dimensions, while in three dimensions, the generic AKSZ-type model is the Courant sigma model [79, 80]. As suggested by the name, it is completely characterized by the data of a Courant algebroid. Although little is known about its quantization (with the notable exception of the works of Hofman and Park [81, 82]) the Courant sigma model has been extensively studied in recent years at the classical level (see e.g. [83–88]). A first step towards its supersymmetrization, which would couple nontrivially super Chern-Simons and super-BF theories, was taken in [8]. It would be interesting to analyze the construction of supersymmetric Courant sigma models in detail and identify the higher-dimensional analogon of the differential Poisson sigma model and its contravariant cousin. A related question is whether the fact that the coadjoint representation of a Lie algebroid appears as a supersymmetry-generating vector field in two dimensions generalizes somehow to higher dimensions. In this respect, note that representations up to homotopy for the split case of Lie 2- and $n$-algebroids were studied in ref. [89], whereas the concept of basic curvature tensor for connections on Courant algebroids and its relation to the BV/BRST formulation of *bosonic* Courant sigma models was investigated in refs. [49, 88].

## Acknowledgments

## A   General conditions

In this appendix we spell out the expanded form of the conditions that appear in the main body of the paper. In the first three subsections of the appendix, we work on a general shifted super-vector bundle $\mathcal{V}[1,0] \twoheadrightarrow \mathcal{M}$, over the supermanifold $\mathcal{M} = E[0,1]$, with coordinates $(x^\mu, \theta^a, a^m, \chi^I)$ of $\mathbb{Z} \times \mathbb{Z}_2$ degrees $(0,0), (0,1), (1,0)$ and $(1,1)$ respectively, and index ranges as explained in section 2. We report the conditions that are encoded in $[\mathcal{Q}, \mathcal{Q}] = 0$, $[\mathcal{Q}_S, \mathcal{Q}_S] = 0$ and $[\mathcal{Q}, \mathcal{Q}_S] = 0$ in that order. In the last two subsections, we specialize to $T^*[1,0]E[0,1]$, which has a canonical graded symplectic structure and report the conditions encoded in

the compatibility of $\mathcal{Q}$ and $\mathcal{Q}_S$ with it. Note that all coefficients shown are functions of $x^\mu$ and $\theta^a$, specifically quadratic in the latter, and the derivatives $\partial_\mu$ and $\partial_a$ are, respectively, with respect to $x^\mu$ and $\theta^a$.

## A.1 Homological vector field for gauge symmetries

The general vector field $\mathcal{Q}$ of $\mathbb{Z} \times \mathbb{Z}_2$ degree $(1,0)$ on the graded super-vector bundle $\mathcal{V}[1,0]$ has the following components in terms of yet undetermined functions of $x$ and $\theta^2$:

$$\mathcal{Q}^\mu = a^m \, \rho_m{}^\mu + \chi^I \theta^a \, \rho_{I\,a}{}^\mu \,, \tag{A.1a}$$

$$\mathcal{Q}^a = a^m \theta^b \, \rho_{m\,b}{}^a + \chi^I \, \rho_I{}^a \tag{A.1b}$$

$$\mathcal{Q}^p = -\tfrac{1}{2} a^m a^n \, f_{mn}{}^p - a^m \chi^I \theta^a \, f_{m\,I\,a}{}^p - \tfrac{1}{2} \chi^I \chi^J \, f_{IJ}{}^p \,, \tag{A.1c}$$

$$\mathcal{Q}^I = -\tfrac{1}{2} a^m a^n \theta^a \, f_{mn\,a}{}^I - a^m \chi^J \, f_{m\,J}{}^I - \tfrac{1}{2} \chi^J \chi^K \theta^a \, f_{JK\,a}{}^I \,, \tag{A.1d}$$

Demanding that this vector field is homological yields fourteen independent conditions. For completeness, we report them below, even though they are not so illuminating without any further assumptions, such as restricting to the case of $\mathcal{V} = T^*\mathcal{M}$ as we eventually assume in the text. From $\mathcal{Q}^2 x^\mu = 0$ we obtain three conditions:

$$\rho_{[m}{}^\nu \partial_\nu \rho_{n]}{}^\mu + \theta^a \rho_{[m|a}{}^b \partial_b \rho_{|n]}{}^\mu - \tfrac{1}{2} f_{mn}{}^p \rho_p{}^\mu - \tfrac{1}{2} \theta^a \theta^b f_{mn[a|}{}^I \rho_{I|b]}{}^\mu = 0 \,, \tag{A.2a}$$

$$\rho_I{}^a \partial_a \rho_m{}^\mu - \theta^a \left( \rho_m{}^\nu \partial_\nu \rho_{Ia}{}^\mu + \rho_{ma}{}^b \rho_{Ib}{}^\mu - \rho_{Ia}{}^\nu \partial_\nu \rho_m{}^\mu - f_{mIa}{}^p \rho_p{}^\mu - f_{mI}{}^J \rho_{Ja}{}^\mu \right) \tag{A.2b}$$

$$+ \theta^a \theta^b \rho_{ma}{}^c \partial_c \rho_{Ib}{}^\mu = 0 \,,$$

$$\rho_{(I}{}^c \rho_{J)c}{}^\mu - \tfrac{1}{2} f_{IJ}{}^m \rho_m{}^\mu - \theta^a \rho_{(I}{}^b \partial_b \rho_{J)a}{}^\mu + \theta^a \theta^b \left( \rho_{(I|a}{}^\nu \partial_\nu \rho_{|J)b}{}^\mu - \tfrac{1}{2} f_{IJa}{}^K \rho_{Kb}{}^\mu \right) = 0 \,, \tag{A.2c}$$

where (anti)symmetrizations are with weight 1, and they refer only to the two surrounded indices in the present set of equations. Next, from $\mathcal{Q}^2 \theta^a = 0$, we obtain three additional conditions:

$$\theta^b \left( \rho_{[m}{}^\mu \partial_\mu \rho_{n]b}{}^a + \rho_{[m|b}{}^c \rho_{|n]c}{}^a - \tfrac{1}{2} f_{mn}{}^p \rho_{pb}{}^a - \tfrac{1}{2} f_{mnb}{}^I \rho_I{}^a \right) - \theta^b \theta^c \rho_{[m|b}{}^d \partial_d \rho_{|n]c}{}^a = 0 \,, \tag{A.3a}$$

$$\rho_m{}^\mu \partial_\mu \rho_I{}^a - \rho_I{}^b \rho_{mb}{}^a - f_{mI}{}^J \rho_J{}^a + \theta^b \left( \rho_I{}^c \partial_c \rho_{mb}{}^a + \rho_{mb}{}^c \partial_c \rho_I{}^a \right) \tag{A.3b}$$

$$- \theta^b \theta^c \left( \rho_{Ib}{}^\mu \partial_\mu \rho_{mc}{}^a + f_{mIb}{}^p \rho_{pc}{}^a \right) = 0 \,,$$

$$\rho_{(J}{}^b \partial_b \rho_{I)}{}^a + \theta^b \left( \rho_{(I|b}{}^\mu \partial_\mu \rho_{|J)}{}^a - \tfrac{1}{2} f_{IJ}{}^m \rho_{mb}{}^a - \tfrac{1}{2} f_{IJb}{}^K \rho_K{}^a \right) = 0 \,. \tag{A.3c}$$

The next set of independent conditions is obtained from $\mathcal{Q}^2 a^m = 0$ and it comprises four equations as follows:

$$\rho_{[n}{}^\mu \partial_\mu f_{pq]}{}^m + f_{[np}{}^r f_{q]r}{}^m + \theta^a \rho_{[n|a}{}^b \partial_b f_{|pq]}{}^m + \theta^a \theta^b f_{[np|a}{}^I f_{|q]Ib}{}^m = 0, \tag{A.4a}$$

$$\rho_I{}^a \partial_a f_{np}{}^m + \theta^a \big(2\rho_{[n}{}^\mu \partial_\mu f_{p]Ia}{}^m + 2\rho_{[n|a}{}^b f_{|p]Ib}{}^m + \rho_{Ia}{}^\mu \partial_\mu f_{np}{}^m - 2f_{[n|Ia}{}^q f_{p]q}{}^m \tag{A.4b}$$

$$- 2f_{[n|I}{}^J f_{|p]Ja}{}^m - f_{np}{}^q f_{qIa}{}^m - f_{npa}{}^J f_{IJ}{}^m\big) - 2\theta^a \theta^b \rho_{[n|a}{}^c \partial_c f_{|p]Ib}{}^m = 0,$$

$$2\rho_{(I|}{}^a f_{n|J)a}{}^m - \rho_n{}^\mu \partial_\mu f_{IJ}{}^m + 2f_{n(I}{}^K f_{J)K}{}^m - f_{IJ}{}^p f_{np}{}^m \tag{A.4c}$$

$$- \theta^a \big(\rho_{na}{}^b \partial_b f_{IJ}{}^m + 2\rho_{(I|}{}^b \partial_b f_{n|J)a}{}^m\big) + 2\theta^a \theta^b \big(\rho_{(I|a}{}^\mu \partial_\mu f_{n|J)b}{}^m + f_{n(I|a}{}^p f_{p|J)b}{}^m - \tfrac{1}{2} f_{IJa}{}^K f_{nKb}{}^m\big) = 0,$$

$$\rho_{(I}{}^a \partial_a f_{JK)}{}^m + \theta^a \big(\rho_{(I|a}{}^\mu \partial_\mu f_{|JK)}{}^m - f_{(IJ|}{}^n f_{n|K)a}{}^m - f_{(IJ|a}{}^L f_{|K)L}{}^m\big) = 0. \tag{A.4d}$$

Finally, a last set of four conditions stems from $\mathcal{Q}^2 \chi^I = 0$, and reads

$$\theta^a \big(\rho_{[m}{}^\mu \partial_\mu f_{np]a}{}^I + \rho_{[m|a}{}^b f_{|np]b}{}^I + f_{[mn}{}^q f_{p]qa}{}^I + f_{[mn|a}{}^J f_{|p]J}{}^I\big) - \theta^a \theta^b \rho_{[m|a}{}^c \partial_c f_{|np]b}{}^I = 0, \tag{A.5a}$$

$$2\rho_{[m}{}^\mu \partial_\mu f_{n]J}{}^I + \rho_J{}^a f_{mna}{}^I - 2f_{[m|J}{}^K f_{|n]K}{}^I - f_{mn}{}^p f_{pJ}{}^I + \theta^a \big(2\rho_{[m|a}{}^b \partial_b f_{|n]J}{}^I - \rho_J{}^b \partial_b f_{mna}{}^I\big)$$

$$+ \theta^a \theta^b \big(\rho_{Ja}{}^\mu \partial_\mu f_{mnb}{}^I - 2f_{[m|Ja}{}^p f_{|n]pb}{}^I - f_{mna}{}^K f_{JKb}{}^I\big) = 0, \tag{A.5b}$$

$$2\rho_{(J|}{}^a \partial_a f_{m|K)}{}^I + \theta^a \big(2\rho_{(J|a}{}^\mu \partial_\mu f_{m|K)}{}^I - \rho_m{}^\mu \partial_\mu f_{JKa}{}^I - \rho_{ma}{}^b f_{JKb}{}^I + 2f_{m(J|a}{}^n f_{n|K)}{}^I$$

$$+ 2f_{m(J}{}^L f_{K)La}{}^I - f_{JK}{}^n f_{mna}{}^I - f_{JKa}{}^L f_{mL}{}^I\big) + \theta^a \theta^b \rho_{ma}{}^c \partial_c f_{JKb}{}^I = 0, \tag{A.5c}$$

$$\rho_{(J}{}^a f_{KL)a}{}^I - f_{(JK|}{}^m f_{m|L)}{}^I - \theta^a \rho_{(J|}{}^b \partial_b f_{|KL)a}{}^I + \theta^a \theta^b \big(\rho_{(J|a}{}^\mu \partial_\mu f_{|KL)b}{}^I - f_{(JK|a}{}^M f_{|L)Mb}{}^I\big) = 0. \tag{A.5d}$$

## A.2 Homological vector field for supersymmetry

We recall from the main text that the general vector field $\mathcal{Q}_S$ of degree $(0,1)$ in the chosen coordinate system has components

$$\mathcal{Q}_S^\mu = \theta^a t_a{}^\mu, \quad \mathcal{Q}_S^a = V^a, \quad \mathcal{Q}_S^m = a^n \theta^a U_{na}{}^m + \chi^I W_I{}^m, \quad \mathcal{Q}_S^I = a^m Y_m{}^I + \chi^J \theta^a Z_{Ja}{}^I, \tag{A.6}$$

with all undetermined coefficients being functions of $x$ and $\theta^2$. Requiring that this vector field is homological gives the following six conditions, each of them organised in increasing order of $\theta$:

$$V^a t_a{}^\mu - \theta^a V^b \partial_b t_a{}^\mu - \theta^a \theta^b t_b{}^\nu \partial_\nu t_a{}^\mu = 0, \tag{A.7a}$$

$$V^b \partial_b V^a + \theta^b t_b{}^\mu \partial_\mu V^a = 0, \tag{A.7b}$$

$$Y_n{}^I W_I{}^m - V^a U_{na}{}^m + \theta^a V^b \partial_b U_{na}{}^m - \theta^a \theta^b \big(U_{nb}{}^l U_{la}{}^m - t_b{}^\lambda \partial_\lambda U_{na}{}^m\big) = 0, \tag{A.7c}$$

$$V^a \partial_a W_I{}^m + \theta^a \big(W_I{}^n U_{na}{}^m + Z_{Ia}{}^J W_J{}^m + t_a{}^\mu \partial_\mu W_I{}^m\big) = 0, \tag{A.7d}$$

$$V^a \partial_a Y_m{}^I - \theta^a \big(U_{ma}{}^n Y_n{}^I - t_a{}^\mu \partial_\mu Y_m{}^I + Y_m{}^J Z_{Ja}{}^I\big) = 0, \tag{A.7e}$$

$$V^a Z_{Ja}{}^I + W_J{}^m Y_m{}^I - \theta^a V^b \partial_b Z_{Ja}{}^I - \theta^a \theta^b \big(Z_{Jb}{}^K Z_{Ka}{}^I + t_b{}^\mu \partial_\mu Z_{Ja}{}^I\big) = 0. \tag{A.7f}$$

## A.3 Commutator of gauge symmetry and supersymmetry

Next we examine under what conditions the two homological vector fields $\mathcal{Q}$ and $\mathcal{Q}_S$ commute. Due to their grading this means that we should study the equation

$$\mathcal{Q}\mathcal{Q}_S + \mathcal{Q}_S\mathcal{Q} = 0, \tag{A.8}$$

which amounts to

$$0 = \rho_A{}^\beta \partial_\beta V^\alpha + (-1)^{|A|} V^\beta \partial_\beta \rho_A{}^\alpha + U_A{}^B \rho_B{}^\alpha, \tag{A.9a}$$

$$0 = (-1)^{|C|(|B|+1)} \left( \rho_B{}^\alpha \partial_\alpha U_C{}^A - U_B{}^D f_{CD}{}^A \right) + (-1)^{|B|} \left( \rho_C{}^\alpha \partial_\alpha U_B{}^A - U_C{}^D f_{BD}{}^A \right) \tag{A.9b}$$
$$- f_{BC}{}^D U_D{}^A - (-1)^{|B|+|C|} V^\alpha \partial_\alpha f_{BC}{}^A,$$

in the condensed notation of section 2. This gives a set of ten conditions, which we organize as follows. From the action on $x^\mu$, we obtain

$$V^a \partial_a \rho_m{}^\mu - \theta^b \left( \rho_{mb}{}^a t_a{}^\mu + \rho_m{}^\nu \partial_\nu t_b{}^\mu + U_{mb}{}^n \rho_n{}^\mu - t_b{}^\nu \partial_\nu \rho_m{}^\mu + Y_m{}^I \rho_{Ib}{}^\mu \right) \tag{A.10a}$$
$$- \theta^a \theta^b \rho_{mb}{}^c \partial_c t_a{}^\mu = 0,$$

$$\rho_I{}^a t_a{}^\mu + W_I{}^m \rho_m{}^\mu + V^a \rho_{Ia}{}^\mu - \theta^a \left( \rho_I{}^b \partial_b t_a{}^\mu + V^b \partial_b \rho_{Ia}{}^\mu \right) \tag{A.10b}$$
$$- \theta^a \theta^b \left( \rho_{Ib}{}^\nu \partial_\nu t_a{}^\mu + Z_{Ib}{}^J \rho_{Ja}{}^\mu + t_b{}^\nu \partial_\nu \rho_{Ia}{}^\mu \right) = 0.$$

From the action on $\theta^a$, we obtain two additional conditions:

$$\rho_m{}^\mu \partial_\mu V^a - V^b \rho_{mb}{}^a + Y_m{}^I \rho_I{}^a + \theta^b \left( \rho_{mb}{}^c \partial_c V^a + V^c \partial_c \rho_{mb}{}^a \right) \tag{A.11a}$$
$$- \theta^b \theta^c \left( U_{mc}{}^m \rho_{mb}{}^a - t_c{}^\mu \partial_\mu \rho_{mb}{}^a \right) = 0,$$

$$\rho_I{}^b \partial_b V^a + V^b \partial_b \rho_I{}^a + \theta^b \left( \rho_{Ib}{}^\mu \partial_\mu V^a + Z_{Ib}{}^J \rho_J{}^a + W_I{}^m \rho_{mb}{}^a + t_b{}^\mu \partial_\mu \rho_I{}^a \right) = 0. \tag{A.11b}$$

The action on $a^m$ gives three more equations:

$$V^a \partial_a f_{np}{}^m + \theta^a \left( f_{np}{}^q U_{qa}{}^m - 2\rho_{[n|a}{}^b U_{|p]b}{}^m - 2\rho_{[n}{}^\mu \partial_\mu U_{p]a}{}^m + f_{npa}{}^I W_I{}^m \right. \tag{A.12a}$$
$$\left. + 2 U_{[n|a}{}^q f_{|p]q}{}^m + t_a{}^\mu \partial_\mu f_{np}{}^m + 2 Y_{[n}{}^I f_{p]Ia}{}^m \right) + 2\theta^a \theta^b \rho_{[n|a}{}^c \partial_c U_{|p]b}{}^m = 0,$$

$$\rho_n{}^\mu \partial_\mu W_I{}^m - \rho_I{}^a U_{na}{}^m + V^a f_{nIa}{}^m - f_{nI}{}^J W_J{}^m + W_I{}^p f_{np}{}^m - Y_n{}^J f_{IJ}{}^m \tag{A.12b}$$
$$+ \theta^a \left( \rho_I{}^b \partial_b U_{ma}{}^m + \rho_{na}{}^b \partial_b W_I{}^m - V^b \partial_b f_{nIa}{}^m \right)$$
$$+ \theta^a \theta^b \left( t_a{}^\mu \partial_\mu f_{nIb}{}^m - \rho_{Ia}{}^\mu \partial_\mu U_{nb}{}^m - f_{nIa}{}^p U_{pb}{}^m - U_{na}{}^p f_{pIb}{}^m + Z_{Ia}{}^J f_{nJb}{}^m \right) = 0,$$

$$2\rho_{(I}{}^a \partial_a W_{J)}{}^m - V^a \partial_a f_{IJ}{}^m - \theta^a \left( f_{IJ}{}^n U_{na}{}^m + f_{IJa}{}^K W_K{}^m - 2\rho_{(I|a}{}^\mu \partial_\mu W_{|J)}{}^m \right. \tag{A.12c}$$
$$\left. + 2 W_{(I|}{}^n f_{n|J)a}{}^m + 2 Z_{(I|a}{}^K f_{|J)K}{}^m + t_a{}^\mu \partial_\mu f_{IJ}{}^m \right) = 0.$$

Finally, the action on $\chi^I$ gives the final three equations:

$$2\rho_{[m}{}^\mu \partial_\mu Y_{n]}{}^I - V^a f_{mna}{}^I - f_{mn}{}^p Y_p{}^I - 2Y_{[m}{}^J f_{n]J}{}^I + \theta^a \left(2\rho_{[m|a}{}^b \partial_b Y_{|n]}{}^I + V^b \partial_b f_{mna}{}^I\right) \tag{A.13a}$$
$$- \theta^a \theta^b \left(f_{mna}{}^J Z_{Jb}{}^I - 2U_{[m|a}{}^p f_{|n]pb}{}^I - t_a{}^\mu \partial_\mu f_{mnb}{}^I\right) = 0,$$

$$\rho_J{}^a \partial_a Y_m{}^I - V^a \partial_a f_{mJ}{}^I - \theta^a \left(\rho_m{}^\mu \partial_\mu Z_{Ja} + \rho_{ma}{}^b Z_{Jb}{}^I - \rho_{Ja}{}^\mu \partial_\mu Y_m{}^I + t_a{}^\mu \partial_\mu f_{mJ}{}^I\right) \tag{A.13b}$$
$$+ W_J{}^n f_{mna}{}^I + Z_{Ja}{}^K f_{mK}{}^I - U_{ma}{}^n f_{nJ}{}^I - Y_m{}^K f_{JKa}{}^I - f_{mJa}{}^n Y_n{}^I - f_{mJ}{}^K Z_{Ka}{}^I\right)$$
$$+ \theta^a \theta^b \rho_{ma}{}^c \partial_c Z_{Jb}{}^I = 0,$$

$$2\rho_{(J}{}^a Z_{K)a}{}^I - f_{JK}{}^m Y_m{}^I - 2W_{(J|}{}^m f_{m|K)}{}^I - V^a f_{JKa}{}^I \tag{A.13c}$$
$$- \theta^a \left(2\rho_{(J}{}^b \partial_b Z_{K)a}{}^I - V^b \partial_b f_{JKa}{}^I\right)$$
$$+ \theta^a \theta^b \left(2\rho_{(J|a}{}^\mu \partial_\mu Z_{|K)b}{}^I - 2Z_{(J|a}{}^L f_{|K)Lb}{}^I - f_{JKa}{}^L Z_{Lb}{}^I - t_a{}^\mu \partial_\mu f_{JKb}{}^I\right) = 0.$$

## A.4 Compatibility of gauge symmetry and symplectic form

We consider now the symplectic case $T^*[1,0]E[0,1]$, recalling that the symplectic form of $\mathbb{Z}$-degree 1 is taken to be

$$\omega = \mathrm{d}a_\mu \wedge \mathrm{d}x^\mu + \mathrm{d}\chi_a \wedge \mathrm{d}\theta^a, \tag{A.14}$$

in therms of the coordinates $x^\mu, \theta^a$ and the conjugate momenta $a_\mu, \chi_a$. The compatibility of the homological vector field $\mathcal{Q}$ that generates the gauge symmetries of the supersymmetric Poisson sigma model and the symplectic form reads

$$\mathcal{L}_\mathcal{Q}\omega = 0. \tag{A.15}$$

In detail, this imposes the following conditions on the coefficients that appear in $\mathcal{Q}$:

$$\rho^{\mu\nu} = -\rho^{\nu\mu}, \qquad \rho^{a\mu} = \rho^{\mu a}, \qquad \rho^{ab} = \rho^{ba}, \tag{A.16a}$$

$$f^{\kappa\lambda}{}_\mu = \partial_\mu \rho^{\kappa\lambda}, \qquad f^{ab}{}_\mu = -\partial_\mu \rho^{ab}, \qquad f^{\kappa b}{}_a = \partial_a \rho^{\kappa b}, \tag{A.16b}$$

$$\theta^b f^{\kappa a}{}_{b\mu} = -\partial_\mu \rho^{\kappa a}, \qquad \theta^b f^{\kappa\lambda}{}_{ba} = \partial_a \rho^{\kappa\lambda}, \qquad \theta^d f^{bc}{}_{da} = -\partial_a \rho^{bc}. \tag{A.16c}$$

These correspond to the more compact eq. (3.3) of the main text.

## A.5 Compatibility of supersymmetry and symplectic form

Finally, we ask whether the homological vector field $\mathcal{Q}_S$ associated to supersymmetry transformations is compatible with the symplectic form $\omega$, i.e.

$$\mathcal{L}_{\mathcal{Q}_S}\omega = 0. \tag{A.17}$$

This gives rise to the following independent conditions:

$$Y^\mu{}_a = -t_a{}^\mu + \theta^b \partial_a t_b{}^\mu, \qquad U^\mu{}_{a\nu} = \partial_\nu t_a{}^\mu, \qquad W^a{}_\mu = \partial_\mu V^a, \qquad \theta^c Z^a{}_{cb} = \partial_b V^a. \tag{A.18}$$

Evidently this fixes all additional coefficients in terms of the basic coefficients $t_a{}^\mu$ and $V^a$.

## References

[1] M. Kontsevich, *Deformation quantization of Poisson manifolds. 1*, *Lett. Math. Phys.* **66** (2003) 157 [`q-alg/9709040`] [INSPIRE].

[2] M. Alexandrov, A. Schwarz, O. Zaboronsky and M. Kontsevich, *The geometry of the master equation and topological quantum field theory*, *Int. J. Mod. Phys. A* **12** (1997) 1405 [`hep-th/9502010`] [INSPIRE].

[3] A.S. Cattaneo and G. Felder, *A path integral approach to the Kontsevich quantization formula*, *Commun. Math. Phys.* **212** (2000) 591 [`math/9902090`] [INSPIRE].

[4] A.S. Schwarz, *Geometry of Batalin-Vilkovisky quantization*, *Commun. Math. Phys.* **155** (1993) 249 [`hep-th/9205088`] [INSPIRE].

[5] A.S. Cattaneo and G. Felder, *On the AKSZ formulation of the Poisson sigma model*, *Lett. Math. Phys.* **56** (2001) 163 [`math/0102108`] [INSPIRE].

[6] P. Ševera, *Some title containing the words "homotopy" and "symplectic", e.g. this one*, [`math/0105080`] [INSPIRE].

[7] D. Roytenberg, *On the structure of graded symplectic supermanifolds and Courant algebroids*, in the proceedings of the *Workshop on Quantization, Deformations, and New Homological and Categorical Methods in Mathematical Physics*, Manchester, U.K., July 07–13 (2001) [`math/0203110`] [INSPIRE].

[8] V. Salnikov, *Supersymmetrization: AKSZ and beyond?*, *Russ. J. Math. Phys.* **27** (2020) 517 [`arXiv:1608.07457`] [INSPIRE].

[9] O. Hulík, J. Svoboda and F. Valach, *Super AKSZ construction, integral forms, and the 2-dimensional $\mathcal{N} = (1,1)$ sigma model*, *JHEP* **11** (2022) 160 [`arXiv:2209.07210`] [INSPIRE].

[10] I. Chepelev and C. Ciocarlie, *A path integral approach to noncommutative superspace*, *JHEP* **06** (2003) 031 [`hep-th/0304118`] [INSPIRE].

[11] D. Borthwick, S. Klimek, A. Lesniewski and M. Rinaldi, *Super Toeplitz operators and non-perturbative deformation quantization of supermanifolds*, *Commun. Math. Phys.* **153** (1993) 49.

[12] I.V. Tyutin, *The general form of the star product on the Grassman algebra*, *Theor. Math. Phys.* **127** (2001) 619 [`hep-th/0101046`] [INSPIRE].

[13] S. Ferrara, M.A. Lledó and O. Maciá, *Supersymmetry in noncommutative superspaces*, *JHEP* **09** (2003) 068 [`hep-th/0307039`] [INSPIRE].

[14] C. Arias, N. Boulanger, P. Sundell and A. Torres-Gomez, *2D sigma models and differential Poisson algebras*, *JHEP* **08** (2015) 095 [`arXiv:1503.05625`] [INSPIRE].

[15] C. Arias, P. Sundell and A. Torres-Gomez, *Differential Poisson Sigma Models with Extended Supersymmetry*, arXiv:1607.00727 [InSPIRE].

[16] N. Ikeda, *Two-dimensional gravity and nonlinear gauge theory*, *Annals Phys.* **235** (1994) 435 [hep-th/9312059] [InSPIRE].

[17] P. Schaller and T. Strobl, *Poisson structure induced (topological) field theories*, *Mod. Phys. Lett. A* **9** (1994) 3129 [hep-th/9405110] [InSPIRE].

[18] N. Ikeda, *Gauge theory based on nonlinear Lie superalgebras and structure of 2-D dilaton supergravity*, *Int. J. Mod. Phys. A* **9** (1994) 1137 [InSPIRE].

[19] T. Strobl, *Target superspace in 2-D dilatonic supergravity*, *Phys. Lett. B* **460** (1999) 87 [hep-th/9906230] [InSPIRE].

[20] M. Ertl, W. Kummer and T. Strobl, *General two-dimensional supergravity from Poisson superalgebras*, *JHEP* **01** (2001) 042 [hep-th/0012219] [InSPIRE].

[21] W. Kummer, M. Ertl and T. Strobl, *All minimal supergravity extensions of 2-D dilaton theories*, *PoS* **HEP2001** (2001) 234 [hep-th/0110032] [InSPIRE].

[22] L. Bergamin, D. Grumiller and W. Kummer, *Quantization of 2-D dilaton supergravity with matter*, *JHEP* **05** (2004) 060 [hep-th/0404004] [InSPIRE].

[23] C.A. Abad and M. Crainic, *Representations up to homotopy of Lie algebroids*, *J. Reine Angew. Math. (Crelles Journal)* **2012** (2012) 91 [arXiv:0901.0319].

[24] R.A. Mehta, *Supergroupoids, double structures, and equivariant cohomology*, math/0605356.

[25] R.A. Mehta, *Q-algebroids and their cohomology*, *J. Symplectic Geom.* **7** (2009) 263 [math/0703234] [InSPIRE].

[26] A. Kotov and T. Strobl, *Characteristic classes associated to Q-bundles*, *Int. J. Geom. Meth. Mod. Phys.* **12** (2014) 1550006 [arXiv:0711.4106] [InSPIRE].

[27] M. Batchelor, *The Structure of Supermanifolds*, *Trans. Am. Math. Soc.* **253** (1979) 329.

[28] M. Batchelor, *Two approaches to supermanifolds*, *Trans. Am. Math. Soc.* **258** (1980) 257.

[29] K.C.H. Mackenzie, *General Theory of Lie Groupoids and Lie Algebroids*, Cambridge University Press (2005) [DOI:10.1017/cbo9781107325883].

[30] A.Y. Vaintrob, *Lie algebroids and homological vector fields*, *Russ. Math. Surv.* **52** (1997) 428.

[31] M. Grigoriev and A. Kotov, *Gauge PDE and AKSZ-type Sigma Models*, *Fortsch. Phys.* **67** (2019) 1910007 [arXiv:1903.02820] [InSPIRE].

[32] M. Grigoriev, *Presymplectic gauge PDEs and Lagrangian BV formalism beyond jet-bundles*, *Contemp. Math.* **788** (2023) 111 [arXiv:2212.11350] [InSPIRE].

[33] M. Grigoriev and D. Rudinsky, *Weak Gauge PDEs*, *SIGMA* **21** (2025) 096 [arXiv:2408.08287] [InSPIRE].

[34] I. Dneprov, M. Grigoriev and V. Gritzaenko, *Presymplectic minimal models of local gauge theories*, *J. Phys. A* **57** (2024) 335402 [arXiv:2402.03240] [InSPIRE].

[35] F. Bonechi, J. Qiu and M. Zabzine, *Wilson Lines from Representations of NQ-Manifolds*, *Int. Math. Res. Not.* **2014** (2014) 2440 [arXiv:1108.5358] [InSPIRE].

[36] P. Batakidis and S. Lavau, *Atiyah classes of Lie algebroid homotopy modules*, arXiv:2406.05204.

[37] A. Blaom, *Geometric structures as deformed infinitesimal symmetries*, *Trans. Am. Math. Soc.* **358** (2006) 3651 [`math/0404313`].

[38] A.D. Blaom, *Lie algebroids and Cartan's method of equivalence*, *Trans. Am. Math. Soc.* **364** (2012) 3071 [`math/0509071`].

[39] G. Bonavolontà and N. Poncin, *On the category of Lie n-algebroids*, *J. Geom. Phys.* **73** (2013) 70 [`arXiv:1207.3590`] [InSPIRE].

[40] F. Cantrijn and L.A. Ibort, *Introduction to Poisson supermanifolds*, *Differ. Geom. Appl.* **1** (1991) 133.

[41] E. Boffo and P. Schupp, *Deformed graded Poisson structures, Generalized Geometry and Supergravity*, *JHEP* **01** (2020) 007 [`arXiv:1903.09112`] [InSPIRE].

[42] E.J. Dubuc and A. Kock, *On 1-form classifiers*, *Commun. Algebra* **12** (1984) 1471.

[43] D. Carchedi and D. Roytenberg, *Homological Algebra for Superalgebras of Differentiable Functions*, `arXiv:1212.3745`.

[44] N. Reshetikhin, A. Voronov and A. Weinstein, *Semiquantum geometry*, *J. Math. Sci.* **82** (1996) 3255 [`q-alg/9606007`].

[45] H. Bursztyn, *Poisson Vector Bundles, Contravariant Connections and Deformations*, *Prog. Theor. Phys. Suppl.* **144** (2001) 26.

[46] E. Hawkins, *Noncommutative rigidity*, `math/0211203` [InSPIRE].

[47] E.J. Beggs and S. Majid, *Semiclassical differential structures*, `math/0306273` [InSPIRE].

[48] E.J. Beggs and S. Majid, *Poisso7-1iemannian geometry*, *J. Geom. Phys.* **114** (2017) 450 [InSPIRE].

[49] A. Chatzistavrakidis and L. Jonke, *Basic curvature & the Atiyah cocycle in gauge theory*, *J. Phys. A* **57** (2024) 465401 [`arXiv:2302.04956`] [InSPIRE].

[50] C.-S. Chu and P.-M. Ho, *Poisson algebra of differential forms*, *Int. J. Mod. Phys. A* **12** (1997) 5573 [`q-alg/9612031`] [InSPIRE].

[51] S. McCurdy and B. Zumino, *Covariant Star Product for Exterior Differential Forms on Symplectic Manifolds*, *AIP Conf. Proc.* **1200** (2010) 204 [`arXiv:0910.0459`] [InSPIRE].

[52] L. Baulieu, A.S. Losev and N.A. Nekrasov, *Target space symmetries in topological theories. 1*, *JHEP* **02** (2002) 021 [`hep-th/0106042`] [InSPIRE].

[53] N. Ikeda and T. Strobl, *BV and BFV for the H-twisted Poisson sigma model*, *Annales Henri Poincare* **22** (2021) 1267 [`arXiv:1912.13511`] [InSPIRE].

[54] M. Grützmann and T. Strobl, *General Yan7-1ills type gauge theories for p-form gauge fields: From physics-based ideas to a mathematical framework or From Bianchi identities to twisted Courant algebroids*, *Int. J. Geom. Meth. Mod. Phys.* **12** (2014) 1550009 [`arXiv:1407.6759`] [InSPIRE].

[55] N. Boulanger, C. Iazeolla and P. Sundell, *Unfolding Mixed-Symmetry Fields in AdS and the BMV Conjecture: I. General Formalism*, *JHEP* **07** (2009) 013 [`arXiv:0812.3615`] [InSPIRE].

[56] D. Montano, K. Aoki and J. Sonnenschein, *Topological Supergravity in Two-dimensions*, *Phys. Lett. B* **247** (1990) 64 [InSPIRE].

[57] M. Cárdenas et al., *Boundary theories for dilaton supergravity in 2D*, *JHEP* **11** (2018) 077 [`arXiv:1809.07208`] [InSPIRE].

[58] A. Chatzistavrakidis, *Topological field theories induced by twisted R-Poisson structure in any dimension*, *JHEP* **09** (2021) 045 [`arXiv:2106.01067`] [ɪɴSPIRE].

[59] A. Chatzistavrakidis, N. Ikeda and G. Šimunić, *The BV action of 3D twisted R-Poisson sigma models*, *JHEP* **10** (2022) 002 [`arXiv:2206.03683`] [ɪɴSPIRE].

[60] A. Chatzistavrakidis, T. Kodžoman and Z. Škoda, *Brane mechanics and gapped Lie n-algebroids*, *JHEP* **08** (2024) 231 [`arXiv:2404.14126`] [ɪɴSPIRE].

[61] N. Ikeda, *Higher Dimensional Lie Algebroid Sigma Model with WZ Term*, *Universe* **7** (2021) 391 [`arXiv:2109.02858`] [ɪɴSPIRE].

[62] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-nijenhuis structures*, *Annales Henri Poincare* **53** (1990) 35. [`https://www.numdam.org/item/AIHPA_1990__53_1_35_0/`].

[63] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math. Oxford Ser.* **54** (2003) 281 [`math/0209099`] [ɪɴSPIRE].

[64] U. Lindstrom, R. Minasian, A. Tomasiello and M. Zabzine, *Generalized complex manifolds and supersymmetry*, *Commun. Math. Phys.* **257** (2005) 235 [`hep-th/0405085`] [ɪɴSPIRE].

[65] R. Zucchini, *A sigma model field theoretic realization of Hitchin's generalized complex geometry*, *JHEP* **11** (2004) 045 [`hep-th/0409181`] [ɪɴSPIRE].

[66] U. Lindstrom, M. Rocek, R. von Unge and M. Zabzine, *Generalized Kahler geometry and manifest N = (2,2) supersymmetric nonlinear sigma-models*, *JHEP* **07** (2005) 067 [`hep-th/0411186`] [ɪɴSPIRE].

[67] R. Zucchini, *Generalized complex geometry, generalized branes and the Hitchin sigma model*, *JHEP* **03** (2005) 022 [`hep-th/0501062`] [ɪɴSPIRE].

[68] V. Pestun, *Topological strings in generalized complex space*, *Adv. Theor. Math. Phys.* **11** (2007) 399 [`hep-th/0603145`] [ɪɴSPIRE].

[69] R. Zucchini, *The Hitchin Model, Poisson-quasi-Nijenhuis Geometry and Symmetry Reduction*, *JHEP* **10** (2007) 075 [`arXiv:0706.1289`] [ɪɴSPIRE].

[70] X. Bekaert, *Geometric tool kit for higher spin gravity (Part II): An introduction to Lie algebroids and their enveloping algebras*, *Int. J. Mod. Phys. A* **38** (2023) 2330013 [`arXiv:2308.00724`] [ɪɴSPIRE].

[71] A.S. Cattaneo and G. Felder, *Coisotropic submanifolds in Poisson geometry and branes in the Poisson sigma model*, *Lett. Math. Phys.* **69** (2004) 157 [`math/0309180`] [ɪɴSPIRE].

[72] A.S. Cattaneo and G. Felder, *Relative formality theorem and quantisation of coisotropic submanifolds*, `math/0501540` [ɪɴSPIRE].

[73] A.S. Cattaneo, *Deformation Quantization and Reduction*, *Contemp. Math.* **450** (2008) 79 [`math/0701378`] [ɪɴSPIRE].

[74] S.L. Lyakhovich and A.A. Sharapov, *BRST theory without Hamiltonian and Lagrangian*, *JHEP* **03** (2005) 011 [`hep-th/0411247`] [ɪɴSPIRE].

[75] F. Schätz, *Coisotropic submanifolds and the BFV-complex*, Ph.D. thesis, University of Zurich, Zurich, Switzerland (2009) [`DOI:https://doi.org/10.5167/uzh-30756`].

[76] R. Bandiera, Z. Chen, M. Stiénon and P. Xu, *Shifted Derived Poisson Manifolds Associated with Lie Pairs*, *Commun. Math. Phys.* **375** (2019) 1717 [`arXiv:1712.00665`].

[77] T. Voronov, *Graded manifolds and drinfeld doubles for Lie bialgebroids*, `math/0105237` [ɪɴSPIRE].

[78] R.A. Mehta, *On homotopy Poisson actions and reduction of symplectic Q-manifolds*, *Differ. Geom. Appl.* **29** (2011) 319 [`arXiv:1009.1280`].

[79] N. Ikeda, *Chern-Simons gauge theory coupled with BF theory*, *Int. J. Mod. Phys. A* **18** (2003) 2689 [`hep-th/0203043`] [InSPIRE].

[80] D. Roytenberg, *AKSZ-BV Formalism and Courant Algebroid-induced Topological Field Theories*, *Lett. Math. Phys.* **79** (2007) 143 [`hep-th/0608150`] [InSPIRE].

[81] C. Hofman and J.-S. Park, *BV quantization of topological open membranes*, *Commun. Math. Phys.* **249** (2004) 249 [`hep-th/0209214`] [InSPIRE].

[82] C. Hofman and J.-S. Park, *Topological open membranes*, `hep-th/0209148` [InSPIRE].

[83] A.S. Cattaneo, J. Qiu and M. Zabzine, *2D and 3D topological field theories for generalized complex geometry*, *Adv. Theor. Math. Phys.* **14** (2010) 695 [`arXiv:0911.0993`] [InSPIRE].

[84] T. Bessho, M.A. Heller, N. Ikeda and S. Watamura, *Topological Membranes, Current Algebras and H-flu7-1-flux Duality based on Courant Algebroids*, *JHEP* **04** (2016) 170 [`arXiv:1511.03425`] [InSPIRE].

[85] P. Ševera, *Poisson-Lie T-duality as a boundary phenomenon of Chern-Simons theory*, *JHEP* **05** (2016) 044 [`arXiv:1602.05126`] [InSPIRE].

[86] D. Mylonas, P. Schupp and R.J. Szabo, *Membrane Sigma-Models and Quantization of Non-Geometric Flux Backgrounds*, *JHEP* **09** (2012) 012 [`arXiv:1207.0926`] [InSPIRE].

[87] A. Chatzistavrakidis, L. Jonke, F.S. Khoo and R.J. Szabo, *Double Field Theory and Membrane Sigma-Models*, *JHEP* **07** (2018) 015 [`arXiv:1802.07003`] [InSPIRE].

[88] A. Chatzistavrakidis, N. Ikeda and L. Jonke, *Geometric BV for twisted Courant sigma models and the BRST power finesse*, *JHEP* **07** (2024) 115 [`arXiv:2401.00425`] [InSPIRE].

[89] M. Jotz, R.A. Mehta and T. Papantonis, *Modules and representations up to homotopy of Lie n-algebroids*, *J. Homotopy Relat. Struct.* **18** (2023) 23 [`arXiv:2001.01101`].