A conformal approach to Carroll gravity

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ABSTRACT: We show how to take the first step in the conformal program for constructing general matter couplings to Carroll gravity. In particular, we couple a single massless electric/magnetic scalar to conformal Carroll gravity with isotropic dilatations and show how, upon gauge-fixing, we obtain a (non-conformal version of) electric/magnetic Carroll gravity. We determine the full Carroll transformation rules paying special attention to the way the so-called intrinsic torsion tensors occur in these transformation rules. A noteworthy feature in the magnetic case is that the Lagrange multiplier present in the Lagrangian gets absorbed, after coupling to conformal Carroll gravity and gauge-fixing, into one of the independent spin-connections of magnetic Carroll gravity. Our results form a convenient starting point for constructing general matter couplings to Carroll gravity. Surprisingly, we find that the same relation between dynamical matter and gravity, which forms the basis of the conformal program, does not work in the usual way in the Galilei case.

Contents

1 Introduction

Carroll symmetries have attracted a lot of attention in recent years due to a variety of reasons. First of all, they are the natural symmetries of any null surface. For this reason they play a central role in investigations of black hole horizons [1] and the null infinity of flat spacetime. Moreover, a conformal extension of Carroll symmetries can be identified with the asymptotic BMS symmetries of flat spacetime that play a fundamental role in Carroll holography [2–4]. Carroll gravity theories have also been constructed. They occur in two versions called electric Carroll gravity [5] and magnetic Carroll gravity [6].

It is natural to extend Carroll gravity to include matter couplings. One way to obtain such matter couplings is to start from a known relativistic system with matter coupled to gravity and perform a Carroll expansion [7–13]. For a single scalar coupled to conformal gravity this program has been performed at leading and sub-leading order [14]. On the other hand, in the relativistic case the conformal technique has turned out to be very efficient in constructing general matter couplings both with and without supersymmetry (see, e.g., [15]). It is the aim of this work to apply the same conformal technique directly to the Carroll case. There are, however, important differences with the relativistic case. One difference is that although there is a single conformal gravity theory, there are two types of massless Carroll scalars and correspondingly two types of Carroll gravity theories. Another important difference is that in the Carroll case there exists a special type of torsion tensor, called *intrinsic torsion*, that is independent of any spin-connection. While in general relativity non-zero torsion usually occurs only after coupling to matter, in the Carroll case these intrinsic torsion tensors can occur even in the absence of matter. Setting them to zero would lead to constraints on the geometry and therefore, they cannot be ignored. In this work we will carefully follow these intrinsic torsion tensors and in particular show how they enter the transformation rules.

The idea behind the conformal technique is based upon the observation that there is a relation between dynamical matter and gravity. Restricting to a single scalar the relation in the relativistic case is as follows. Coupling a dynamical scalar to conformal gravity and gauge-fixing dilatations by setting the scalar equal to one, one obtains the Einstein-Hilbert Lagrangian of general relativity. It also works the other way around. Replacing the Vierbein field in the Einstein-Hilbert Lagrangian by the product of a compensating scalar and a conformal Vierbein field and restricting to a flat spacetime one re-obtains the Lagrangian of a dynamical scalar. Restricting to one scalar the conformal program leads to gravity without any additional matter coupling. In this sense this is only a first step in the conformal program. Once this step is completed one may obtain non-trivial matter couplings by replacing the single scalar by a function of N scalars such that after gauge-fixing one is left with $N-1$ scalars coupled to gravity. In this work we will focus on the first step only.

Not every gravity theory gives rise within the conformal program to dynamical matter. For instance, in the relativistic case the Weyl tensor squared is already invariant under local dilatations and therefore cannot produce dynamical matter. We will find another example of this in the Carroll case. Remarkably, we find that the conformal program also sometimes does not work the other way around: it is not always possible to couple dynamical matter to any non-Lorentzian conformal gravity theory following the structure of the underlying non-Lorentzian conformal algebra. We will see that it works for conformal Carroll gravity but that it does not work in the usual way for conformal Galilei gravity. We note that even in the Carroll case the coupling is non-trivial and only works due to some 'lucky' coincidences.

Applying the conformal program to the Carroll case, we will find in this work that

1. Coupling an electric Carroll scalar to conformal Carroll gravity leads, upon gauge-fixing, to a non-conformal version of electric Carroll gravity.

2. Coupling a magnetic Carroll scalar to conformal Carroll gravity leads, upon gauge-fixing, to magnetic Carroll gravity.

It is a bit ironic that the conformal program relates the more standard electric Carroll scalar to the more exotic electric Carroll gravity[1] whereas it relates the more exotic magnetic Carroll scalar[2] to the more standard magnetic Carroll gravity theory.[3]

This work is organized as follows. For the convenience of the reader we first review in section 2 the conformal technique in the relativistic case. To prepare the reader for the Carroll case, we use this relativistic case to show how general torsion (in this case not intrinsic torsion) can arise in the transformation rules once it is included. Furthermore, we emphasize that there are two ways of writing down the Lagrangian for dynamical matter that are equivalent in the sense that they differ by a total derivative but that, nevertheless, one of them leads to simplifications in the calculations. Using the insights obtained from the

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[1]For instance, electric Carroll gravity is built from intrinsic torsion tensors and therefore does not contain spin-connection fields.

[2]The Lagrangian of a magnetic Carroll scalar contains an additional Lagrange multiplier field. As we will see, this field disappears from the Lagrangian after coupling to conformal Carroll gravity and gauge-fixing since it gets absorbed by a specific component of the independent Carroll boost spin-connection field.

[3]The magnetic Carroll gravity theory contains a term that is similar to the Einstein-Hilbert term involving spin-connections.

relativistic case, we perform in section 3 the conformal program to a single Carroll scalar both in the electric and magnetic case. We discuss the corresponding Carroll gravity theories including the electric Carroll gravity theory that does not give rise to dynamical matter. Next, in section 4 we discuss the issues that arise if one wishes to apply the conformal program to the Galilei case. Along the way, we will introduce a new Galilei gravity theory that contains an additional Lagrange multiplier field. Finally, in the Conclusions we discuss a few extensions of the present work.

2 A conformal approach to general relativity

The purpose of this section is to explain the techniques that we wish to apply to Carroll gravity, in the familiar context of general relativity. In particular, we will show how the Einstein-Hilbert (EH) action can be obtained from that of a conformally coupled compensating scalar (i.e., a massless scalar coupled to suitable gauge fields of the conformal algebra), by gauge-fixing those conformal symmetries that are not part of the Poincaré algebra. These superfluous conformal symmetries are the dilatations, as well as the special conformal transformations whose role we will clarify. The scalar field is called *compensating*, since its conformally coupled action can be re-obtained from the EH action by substituting the Vielbein in the latter with an appropriate product of a conformal Vielbein and the compensating scalar. The scalar thus compensates for the non-invariance of the conformal Vielbein under dilatations so that their product produces the scale-invariant Vielbein of general relativity.

Before discussing this compensating mechanism we first discuss in subsection 2.1 the gauging of the conformal algebra[4] and in subsection 2.2 how to obtain a minimal representation of conformal algebra gauge fields. In the final subsection 2.3 we will use this information to show how general relativity emerges from a conformally coupled compensating scalar.

2.1 Gauging the conformal algebra

The relativistic conformal algebra is spanned by the generators of translations $P_{\hat{A}}$, Lorentz transformations $M_{\hat{A}\hat{B}} = -M_{\hat{B}\hat{A}}$, special conformal transformations $K_{\hat{A}}$ and dilatations D. Here, the indices $\hat{A}$, $\hat{B}$ take on the values $0, 1, \cdots, D-1$.[5] The non-zero commutation relations of the conformal algebra are given by

$$\begin{aligned}
[M_{\hat{A}\hat{B}}, M_{\hat{C}\hat{D}}] &= 4\,\eta_{[\hat{A}[\hat{C}}M_{\hat{D}]\hat{B}]}\,, & [P_{\hat{A}}, M_{\hat{B}\hat{C}}] &= 2\,\eta_{\hat{A}[\hat{B}}P_{\hat{C}]}\,, \\
[K_{\hat{A}}, M_{\hat{B}\hat{C}}] &= 2\,\eta_{\hat{A}[\hat{B}}K_{\hat{C}]}\,, & [P_{\hat{A}}, K_{\hat{B}}] &= 2\,\eta_{\hat{A}\hat{B}}D + 2\,M_{\hat{A}\hat{B}}\,, \\
[D, P_{\hat{A}}] &= P_{\hat{A}}\,, & [D, K_{\hat{A}}] &= -K_{\hat{A}}\,,
\end{aligned} \tag{2.1}$$

where $\eta_{\hat{A}\hat{B}}$ is the Minkowski metric for which we adopt the mostly plus signature convention. Introducing gauge fields $E_\mu{}^{\hat{A}}$ (called the Vielbein), $\Omega_\mu{}^{\hat{A}\hat{B}}$ (called the spin-connection), $F_\mu{}^{\hat{A}}$ and B_μ, associated to respectively the generators $P_{\hat{A}}$, $M_{\hat{A}\hat{B}}$, $K_{\hat{A}}$ and D, the above commutation

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[4]For early literature on gauging space-time symmetry algebras, see e.g., [16, 17].

[5]In this section, we use a hatted index $\hat{A}$, since the unhatted index A will be used as a spatial index in the Carrollian case.

relations give rise to the following infinitesimal gauge transformation rules:

$$\delta_0 E_\mu{}^{\hat{A}} = \partial_\mu \zeta^{\hat{A}} - \zeta^{\hat{B}} \, \Omega_{\mu\hat{B}}{}^{\hat{A}} + \zeta^{\hat{A}} \, B_\mu - \Lambda^{\hat{A}}{}_{\hat{B}} \, E_\mu{}^{\hat{B}} - \Lambda_D \, E_\mu{}^{\hat{A}} \,,$$

$$\delta_0 \Omega_\mu{}^{\hat{A}\hat{B}} = \partial_\mu \Lambda^{\hat{A}\hat{B}} - 2 \, \Lambda^{[\hat{A}|}{}_{\hat{C}} \, \Omega_\mu{}^{\hat{C}|\hat{B}]} - 4 \, \Lambda_K{}^{[\hat{A}} \, E_\mu{}^{\hat{B}]} - 4 \, \zeta^{[\hat{A}} \, F_\mu{}^{\hat{B}]} \,,$$

$$\delta_0 F_\mu{}^{\hat{A}} = \partial_\mu \Lambda_K{}^{\hat{A}} - \Lambda_{K\hat{B}} \, \Omega_\mu{}^{\hat{B}\hat{A}} - \Lambda_K{}^{\hat{A}} \, B_\mu - \Lambda^{\hat{A}}{}_{\hat{B}} \, F_\mu{}^{\hat{B}} + \Lambda_D \, F_\mu{}^{\hat{A}} \,,$$

$$\delta_0 B_\mu = \partial_\mu \Lambda_D + 2 \, \Lambda_K{}^{\hat{A}} \, E_{\mu\hat{A}} - 2 \, \zeta^{\hat{A}} \, F_{\mu\hat{A}} \,. \tag{2.2}$$

Here, $\zeta^{\hat{A}}$, Λ_D, $\Lambda^{\hat{A}\hat{B}}$ and $\Lambda_K{}^{\hat{A}}$ are the parameters of translations, dilatations, Lorentz and special conformal transformations, respectively. In what follows, we will often refer to dilatations, Lorentz and special conformal transformations as *homogeneous conformal transformations*. The action of the infinitesimal homogeneous conformal transformations of (2.2) will be denoted by the symbol $\delta_{\text{hom},0}$. We use the subscript "0" on δ_0 and $\delta_{\text{hom},0}$ to indicate that these are not the final conformal transformations that are used to reconstruct the EH gravity action. Instead, the transformation rules used in the conformal approach are a modification of (2.2) with extra terms, as will be explained in the next section 2.2.

The curvatures that are defined in the following way

$$R_{\mu\nu}(P^{\hat{A}}) \equiv 2 \, \partial_{[\mu} E_{\nu]}{}^{\hat{A}} + 2 \, \Omega_{[\mu}{}^{\hat{A}\hat{B}} \, E_{\nu]\hat{B}} + 2 \, B_{[\mu} \, E_{\nu]}{}^{\hat{A}} \,,$$

$$R_{\mu\nu}(M^{\hat{A}\hat{B}}) \equiv 2 \, \partial_{[\mu} \Omega_{\nu]}{}^{\hat{A}\hat{B}} + 2 \, \Omega_{[\mu|}{}^{[\hat{A}|}{}_{\hat{C}} \, \Omega_{|\nu]}{}^{\hat{C}|\hat{B}]} + 8 \, F_{[\mu}{}^{[\hat{A}} \, E_{\nu]}{}^{\hat{B}]} \,,$$

$$R_{\mu\nu}(K^{\hat{A}}) \equiv 2 \partial_{[\mu} F_{\nu]}{}^{\hat{A}} + 2 \, \Omega_{[\mu}{}^{\hat{A}\hat{B}} \, F_{\nu]\hat{B}} - 2 \, B_{[\mu} \, F_{\nu]}{}^{\hat{A}} \,,$$

$$R_{\mu\nu}(D) \equiv 2 \, \partial_{[\mu} B_{\nu]} - 4 \, F_{[\mu}{}^{\hat{A}} \, E_{\nu]\hat{A}} \,, \tag{2.3}$$

have the property that they transform into each other covariantly (i.e. without the derivative of a parameter) under (2.2). Explicitly, their transformations under (2.2) are given by:

$$\delta_0 R_{\mu\nu}(P^{\hat{A}}) = -\zeta^{\hat{B}} \, R_{\mu\nu}(M_{\hat{B}}{}^{\hat{A}}) + \zeta^{\hat{A}} \, R_{\mu\nu}(D) - \Lambda^{\hat{A}}{}_{\hat{B}} \, R_{\mu\nu}(P^{\hat{B}}) - \Lambda_D \, R_{\mu\nu}(P^{\hat{A}}) \,,$$

$$\delta_0 R_{\mu\nu}(M^{\hat{A}\hat{B}}) = -2 \, \Lambda^{[\hat{A}|}{}_{\hat{C}} \, R_{\mu\nu}(M^{\hat{C}|\hat{B}]}) - 4 \, \Lambda_K{}^{[\hat{A}} \, R_{\mu\nu}(P^{\hat{B}]}) - 4 \, \zeta^{[\hat{A}} \, R_{\mu\nu}(K^{\hat{B}]}) \,,$$

$$\delta_0 R_{\mu\nu}(K^{\hat{A}}) = -\Lambda_{K\hat{B}} \, R_{\mu\nu}(M^{\hat{B}\hat{A}}) - \Lambda_K{}^{\hat{A}} \, R_{\mu\nu}(D) - \Lambda^{\hat{A}}{}_{\hat{B}} \, R_{\mu\nu}(K^{\hat{B}}) + \Lambda_D \, R_{\mu\nu}(K^{\hat{A}}) \,,$$

$$\delta_0 R_{\mu\nu}(D) = 2 \, \Lambda_K{}^{\hat{A}} \, R_{\mu\nu}(P_{\hat{A}}) - 2 \, \zeta^{\hat{A}} \, R_{\mu\nu}(K_{\hat{A}}) \,. \tag{2.4}$$

The covariant curvatures (2.3) are not independent in the sense that they satisfy the following Bianchi identities:

$$D_{[\mu} R_{\nu\rho]}(P^{\hat{A}}) - R_{[\mu\nu}(M^{\hat{A}\hat{B}}) E_{\rho]\hat{B}} - R_{[\mu\nu}(D) E_{\rho]}{}^{\hat{A}} = 0 \,,$$

$$D_{[\mu} R_{\nu\rho]}(M^{\hat{A}\hat{B}}) - 4 R_{[\mu\nu}(K^{[\hat{A}}) E_{\rho]}{}^{\hat{B}]} - 4 R_{[\mu\nu}(P^{[\hat{A}}) F_{\rho]}{}^{\hat{B}]} = 0 \,,$$

$$D_{[\mu} R_{\nu\rho]}(K^{\hat{A}}) - R_{[\mu\nu}(M^{\hat{A}\hat{B}}) F_{\rho]\hat{B}} + R_{[\mu\nu}(D) F_{\rho]}{}^{\hat{A}} = 0 \,,$$

$$\partial_{[\mu} R_{\nu\rho]}(D) + 2 R_{[\mu\nu}(K^{\hat{A}}) E_{\rho]\hat{A}} - 2 R_{[\mu\nu}(P^{\hat{A}}) F_{\rho]\hat{A}} = 0 \,. \tag{2.5}$$

Here we have used derivatives D_μ that are covariant with respect to Lorentz transformations and dilatations but not with respect to special conformal transformations.

2.2 Imposing constraints

In order to retrieve EH gravity via the conformal approach, the gauge fields and curvatures of the previous subsection must be supplemented with an additional ingredient. The main reason for this is that the multiplet of gauge fields (2.2) is not minimal, in the sense that it realizes the conformal algebra with more independent gauge fields than are needed for the construction of gravitational theories. The extra ingredient that is required to reduce the number of independent gauge fields consists of imposing the following two constraints on the curvatures:

$$R_{\mu\nu}(P^{\hat{A}}) = T_{\mu\nu}{}^{\hat{A}} \qquad \text{and} \qquad R_{\mu\hat{B}}(M^{\hat{A}\hat{B}}) = 0\,. \tag{2.6}$$

The tensor $T_{\mu\nu}{}^{\hat{A}}$ can be identified as the torsion of an affine connection $\Gamma_{\mu\nu}{}^{\rho}$ that is introduced via the Vielbein postulate

$$\partial_{\mu}E_{\nu}{}^{\hat{A}} + \Omega_{\mu}{}^{\hat{A}\hat{B}}E_{\nu\hat{B}} + B_{\mu}E_{\nu}{}^{\hat{A}} - \Gamma_{\mu\nu}{}^{\rho}E_{\rho}{}^{\hat{A}} = 0\,. \tag{2.7}$$

Comparing the result of anti-symmetrizing this Vielbein postulate in $[\mu\nu]$ with the first constraint of (2.6) shows that indeed:

$$T_{\mu\nu}{}^{\hat{A}} = 2\Gamma_{[\mu\nu]}{}^{\rho}E_{\rho}{}^{\hat{A}}\,. \tag{2.8}$$

For this reason, we will refer to $T_{\mu\nu}{}^{\hat{A}}$ as the torsion in what follows. Note that the inclusion of the term involving B_{μ} in the Vielbein postulate (2.7) ensures that the affine connection $\Gamma_{\mu\nu}{}^{\rho}$ is dilatation invariant. It is however not metric-compatible. Indeed, from (2.7) one finds that the metric $g_{\mu\nu} = E_{\mu}{}^{\hat{A}}E_{\nu\hat{A}}$ obeys

$$\nabla_{\mu}g_{\nu\rho} \equiv \partial_{\mu}g_{\nu\rho} - 2\Gamma_{\mu(\nu}{}^{\sigma}g_{\rho)\sigma} = -2B_{\mu}g_{\nu\rho}\,. \tag{2.9}$$

The geometric structure introduced above thus constitutes that of a Cartan-Weyl space-time[6] with Weyl connection given by $-2B_{\mu}$.

Typically, discussions of the relativistic conformal method only consider the case where the torsion $T_{\mu\nu}{}^{\hat{A}}$ vanishes, as this allows one to recover general relativity in the final step of the conformal approach. Including torsion in relativistic gravity is usually done by coupling to (fermionic) matter. Unlike Lorentzian geometry however, in its non-Lorentzian analogue part of the torsion is intrinsic. This intrinsic torsion can be present in non-Lorentzian gravity theories, even in the absence of matter couplings. Moreover, setting it to zero amounts to imposing constraints on the geometry. Choosing zero torsion will therefore be too restrictive in extensions of the conformal approach to Carroll gravity. In order to anticipate the Carrollian case, we will thus momentarily keep the torsion arbitrary to illustrate its effects, in particular how it deforms the conformal transformation rules. We will switch

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[6]A Cartan-Weyl space-time is endowed with a metric $g_{\mu\nu}$ and a connection $\Gamma_{\mu\nu}{}^{\rho}$ that is not metric-compatible, but for which a trace-free part of the non-metricity tensor $Q_{\mu\nu\rho} \equiv -\nabla_{\mu}g_{\nu\rho}$ vanishes:

$$Q_{\mu\nu\rho} - \frac{1}{D}Q_{\mu\sigma}{}^{\sigma}g_{\nu\rho} = 0\,.$$

The trace $-(1/D)Q_{\mu\nu}{}^{\nu}$ transforms like a gauge field under conformal rescalings (dilatations) of the metric and is called the Weyl connection.

focus to the usual relativistic zero torsion case in the next section, when discussing how EH gravity arises from the conformal approach.

The constraints (2.6) are often called "conventional", since they can be used to solve the spin-connection $\Omega_\mu{}^{\hat{A}\hat{B}}$ and the special conformal gauge field $F_\mu{}^{\hat{A}}$ in terms of the Vielbein $E_\mu{}^{\hat{A}}$, the dilatation gauge field B_μ and the torsion $T_{\mu\nu}{}^{\hat{A}}$ as follows:

$$\Omega_\mu{}^{\hat{A}\hat{B}}(E,B,T) = \Omega_\mu{}^{\hat{A}\hat{B}}(E,T) + 2\,E_\mu{}^{[\hat{A}}E^{\hat{B}]\nu}B_\nu\,,$$

$$F_\mu{}^{\hat{A}}(E,B,T) = -\frac{1}{2(D-2)}\left[R'_{\mu\hat{B}}(M^{\hat{A}\hat{B}}) - \frac{1}{2(D-1)}\,E_\mu{}^{\hat{A}}\,R'_{\hat{B}\hat{C}}(M^{\hat{B}\hat{C}})\right]\,, \qquad (2.10)$$

where $\Omega_\mu{}^{\hat{A}\hat{B}}(E,T)$ is the usual torsionful spin-connection

$$\Omega_\mu{}^{\hat{A}\hat{B}}(E,T) = E^{[\hat{A}|\nu}\left(2\partial_{[\mu}E_{\nu]}{}^{|\hat{B}]} - T_{\mu\nu}{}^{|\hat{B}]}\right) - \frac{1}{2}E_{\mu\hat{C}}\,E^{\hat{A}\nu}\,E^{\hat{B}\rho}\left(2\partial_{[\nu}E_{\rho]}{}^{\hat{C}} - T_{\nu\rho}{}^{\hat{C}}\right)\,, \quad (2.11)$$

and $R'_{\mu\nu}(M^{\hat{A}\hat{B}})$ is the Lorentz transformation curvature with the term containing the special conformal gauge field deleted:

$$R'_{\mu\nu}(M^{\hat{A}\hat{B}}) = 2\,\partial_{[\mu}\Omega_{\nu]}{}^{\hat{A}\hat{B}}(E,B,T) + 2\,\Omega_{[\mu|}{}^{[\hat{A}|}{}_{\hat{C}}(E,B,T)\,\Omega_{|\nu]}{}^{\hat{C}|\hat{B}]}(E,B,T)\,. \qquad (2.12)$$

In what follows, it will be understood that the spin-connection $\Omega_\mu{}^{\hat{A}\hat{B}}$ and special conformal gauge field $F_\mu{}^{\hat{A}}$ are given by the dependent expressions $\Omega_\mu{}^{\hat{A}\hat{B}}(E,B,T)$ and $F_\mu{}^{\hat{A}}(E,B,T)$ of (2.10), whenever they appear in covariant derivatives or curvatures.

The transformation rules (2.2) often need to be modified after the conventional constraints (2.6) are imposed. Let us first consider the homogeneous conformal transformations, whose modified infinitesimal version we will denote by δ_{hom}. Their action on the independent fields $E_\mu{}^{\hat{A}}$ and B_μ is assumed to be the same as that given in (2.2). The rules $\delta_{\text{hom}}\Omega_\mu{}^{\hat{A}\hat{B}}(E,B,T)$ and $\delta_{\text{hom}}F_\mu{}^{\hat{A}}(E,B,T)$ of the dependent fields then follow from varying their explicit expressions (2.10).[7] The result of this does in general not coincide with the homologous rules given in (2.2). This happens in particular when the conventional constraints (2.6) that are used to solve for the dependent fields are not left invariant by (2.2). The transformation rules (2.2) of the dependent fields then acquire extra terms, whose role is to ensure that the constraints (2.6) are invariant under the thus modified rules.

In the present case, one finds that the homogeneous conformal transformations of the independent and dependent gauge fields are, after imposing (2.6), given by:

$$\delta_{\text{hom}}E_\mu{}^{\hat{A}} = -\Lambda^{\hat{A}}{}_{\hat{B}}\,E_\mu{}^{\hat{B}} - \Lambda_D\,E_\mu{}^{\hat{A}}\,,$$

$$\delta_{\text{hom}}B_\mu = \partial_\mu\Lambda_D + 2\,\Lambda_K{}^{\hat{A}}\,E_{\mu\hat{A}}\,,$$

$$\delta_{\text{hom}}\Omega_\mu{}^{\hat{A}\hat{B}}(E,B,T) = \partial_\mu\Lambda^{\hat{A}\hat{B}} - 2\,\Lambda^{[\hat{A}|}{}_{\hat{C}}\,\Omega_\mu{}^{\hat{C}|\hat{B}]}(E,B,T) - 4\,\Lambda_K{}^{[\hat{A}}\,E_\mu{}^{\hat{B}]}\,,$$

$$\delta_{\text{hom}}F_\mu{}^{\hat{A}}(E,B,T) = \partial_\mu\Lambda_K{}^{\hat{A}} - \Lambda_{K\hat{B}}\,\Omega_\mu{}^{\hat{B}\hat{A}}(E,B,T) - \Lambda_K{}^{\hat{A}}\,B_\mu - \Lambda^{\hat{A}}{}_{\hat{B}}\,F_\mu{}^{\hat{B}}(E,B,T)$$
$$+\,\Lambda_D\,F_\mu{}^{\hat{A}}(E,B,T) + \frac{2}{D-2}\Lambda_K{}^{\hat{C}}\left[\delta_{\hat{C}}^{[\hat{A}}T_{\mu\hat{B}}{}^{\hat{B}]} - \frac{1}{2(D-1)}E_\mu{}^{\hat{A}}T_{\hat{C}\hat{B}}{}^{\hat{B}}\right]\,.$$
$$(2.13)$$

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[7]This requires that one also assigns a homogeneous conformal transformation rule $\delta_{\text{hom}}T_{\mu\nu}{}^{\hat{A}}$ for the torsion tensor. In what follows, we will assume that $T_{\mu\nu}{}^\rho$ is invariant under homogeneous conformal transformations, so that $\delta_{\text{hom}}T_{\mu\nu}{}^{\hat{A}} = -\Lambda^{\hat{A}}{}_{\hat{B}}T_{\mu\nu}{}^{\hat{B}} - \Lambda_D T_{\mu\nu}{}^{\hat{A}}$.

Note that the first constraint of (2.6) is invariant under (2.2) and as a result the transformation rule of the dependent spin-connection $\Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)$ agrees with that of (2.2). The second constraint of (2.6) by contrast transforms non-trivially under special conformal transformations:

$$\delta_{\text{hom},0} R_{\mu\hat{B}}(M^{\hat{A}\hat{B}}) \approx -4\Lambda_K{}^{[\hat{A}} T_{\mu\hat{B}}{}^{\hat{B}]}, \tag{2.14}$$

where $\approx$ has the meaning of "equality up to the conventional constraints (2.6)". Consequently, the transformation rule of $F_\mu{}^{\hat{A}}(E, B, T)$ picks up extra special conformal transformation terms with respect to (2.2).

Let us now discuss how also the translation transformations of (2.2) are modified due to the conventional constraints (2.6). In this regard, it should be noted that translations are usually not considered as an independent symmetry, but rather as a combination of homogeneous conformal transformations and diffeomorphisms. Indeed, an infinitesimal diffeomorphism on any gauge field of the conformal algebra can be decomposed into homogeneous conformal transformations plus extra terms. These extra terms correspond to the translation rules of (2.2), supplemented with curvature- and torsion-dependent contributions and it is thus sensible to view them as modified translation symmetries. In particular, any action that is invariant under diffeomorphisms and homogeneous conformal transformations is also automatically invariant under these modified translations. Note that generically the translation rules (2.2) of both the dependent and the independent fields receive extra terms. This is unlike what happens for the homogeneous conformal transformations that are only altered for the dependent fields.

Let us illustrate the above discussion by considering the following rewriting of the action of an infinitesimal diffeomorphism δ_ξ with parameter ξ^μ on the Vielbein $E_\mu{}^{\hat{A}}$:

$$\begin{aligned}
\delta_\xi E_\mu{}^{\hat{A}} &= \partial_\mu \xi^\nu E_\nu{}^{\hat{A}} + \xi^\nu \partial_\nu E_\mu{}^{\hat{A}} = \partial_\mu(\xi^\nu E_\nu{}^{\hat{A}}) + 2\xi^\nu \partial_{[\nu} E_{\mu]}{}^{\hat{A}} \\
&= -\xi^\nu \Omega_\nu{}^{\hat{A}}{}_{\hat{B}}(E, B, T) E_\mu{}^{\hat{B}} - \xi^\nu B_\nu E_\mu{}^{\hat{A}} + \partial_\mu(\xi^\nu E_\nu{}^{\hat{A}}) + \Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)\xi^\nu E_{\nu\hat{B}} + B_\mu \xi^\nu E_\nu{}^{\hat{A}} \\
&\quad + \xi^\nu E_\nu{}^{\hat{B}} T_{\hat{B}\mu}{}^{\hat{A}}, \tag{2.15}
\end{aligned}$$

where in the last step we used the first of the conventional constraints (2.6). Comparing the last line with (2.2), we see that the first two terms correspond to homogeneous conformal transformations (namely a Lorentz transformation and dilatation with field-dependent parameters $\xi^\mu \Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)$ and $\xi^\mu B_\mu$ respectively) acting on $E_\mu{}^{\hat{A}}$. The remaining terms can then be identified with the action (2.2) of a translation on $E_\mu{}^{\hat{A}}$ with field-dependent parameter $\xi^\mu E_\mu{}^{\hat{A}}$, supplemented with an extra torsion contribution. This thus motivates defining the translation rule of $E_\mu{}^{\hat{A}}$ in the presence of the conventional constraints (2.6) as

$$\delta_\zeta E_\mu{}^{\hat{A}} = \partial_\mu \zeta^{\hat{A}} + \Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)\zeta_{\hat{B}} + B_\mu \zeta^{\hat{A}} + \zeta^{\hat{B}} T_{\hat{B}\mu}{}^{\hat{A}}. \tag{2.16}$$

More generally, a modified translation with parameter $\zeta^{\hat{A}}$ is defined as the result of subtracting the homogeneous conformal transformations (2.13) with parameters $\Lambda^{\hat{A}\hat{B}} = \zeta^{\hat{C}} E_{\hat{C}}{}^\mu \Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)$, $\Lambda_D = \zeta^{\hat{A}} E_{\hat{A}}{}^\mu B_\mu$ and $\Lambda_K{}^{\hat{A}} = \zeta^{\hat{B}} E_{\hat{B}}{}^\mu F_\mu{}^{\hat{A}}(E, B, T)$, from an infinitesimal diffeomorphism with parameter $\zeta^{\hat{A}} E_{\hat{A}}{}^\mu$. For B_μ, $\Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)$ and $F_\mu{}^{\hat{A}}(E, B, T)$,

– 7 –

this gives:

$$\delta_\zeta B_\mu = -2\zeta^{\hat{A}} F_{\mu\hat{A}}(E, B, T) + \zeta^{\hat{A}} R_{\hat{A}\mu}(D) \,,$$

$$\delta_\zeta \Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T) = -4\zeta^{[\hat{A}} F_\mu{}^{\hat{B}]}(E, B, T) + \zeta^{\hat{C}} R_{\hat{C}\mu}(M^{\hat{A}\hat{B}}) \,,$$

$$\delta_\zeta F_\mu{}^{\hat{A}}(E, B, T) = \zeta^{\hat{B}} R_{\hat{B}\mu}(K^{\hat{A}}) - \frac{2}{D-2}\zeta^{\hat{C}} F_{\hat{C}}{}^{[\hat{A}}(E, B, T) T_{\mu\hat{B}}{}^{\hat{B}]}$$
$$+ \frac{1}{(D-1)(D-2)} E_\mu{}^{\hat{A}} \zeta^{\hat{D}} F_{\hat{D}}{}^{[\hat{B}}(E, B, T) T_{\hat{B}\hat{C}}{}^{\hat{C}]} \,. \qquad (2.17)$$

By making use of the Bianchi identities (2.5), one can show explicitly that these translation rules of the dependent fields $\Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)$ and $F_\mu{}^{\hat{A}}(E, B, T)$ agree with those obtained from varying their expressions (2.10) under (2.16) and the first rule of (2.17). In checking this, one assumes that a translation of the torsion tensor $T_{\mu\nu}{}^{\hat{A}}$ is also given by the action of a diffeomorphism minus that of a Lorentz transformation and dilatation with the above mentioned field-dependent parameters:

$$\delta_\zeta T_{\mu\nu}{}^{\hat{A}} = 2\partial_{[\mu|}\left(\zeta^{\hat{B}} E_{\hat{B}}{}^\rho\right) T_{\rho|\nu]}{}^{\hat{A}} + \zeta^{\hat{B}} E_{\hat{B}}{}^\rho \partial_\rho T_{\mu\nu}{}^{\hat{A}} + \zeta^{\hat{C}} E_{\hat{C}}{}^\rho \Omega_\rho{}^{\hat{A}}{}_{\hat{B}}(E, B, T) T_{\mu\nu}{}^{\hat{B}}$$
$$+ \zeta^{\hat{B}} E_{\hat{B}}{}^\rho B_\rho T_{\mu\nu}{}^{\hat{A}} \,. \qquad (2.18)$$

The independent gauge field B_μ plays a special role since it transforms with a shift under a special conformal transformation. In fact, it is the only field that transforms under a special conformal transformation. The Vielbein field $E_\mu{}^{\hat{A}}$ is invariant while the dependent gauge fields $\Omega_\mu{}^{\hat{A}\hat{B}}(E, B, T)$ and $F_\mu{}^{\hat{A}}(E, B, T)$ only transform under special conformal transformations due to their dependence on B_μ. This means that B_μ is a Stueckelberg field that can be set to zero by fixing the special conformal transformations. The special conformal symmetry expresses the property that one can realize local dilatations without using a corresponding dilatation gauge field. The only independent gauge field left is therefore the Vielbein $E_\mu{}^{\hat{A}}$. As it stands this gauge field has D^2 field degrees of freedom. Due to the D diffeomorphisms, the $\frac{1}{2}D(D-1)$ Lorentz rotations and the dilatation the number of independent degrees of freedom is given by $D^2 - D - \frac{1}{2}D(D-1) - 1 = \frac{1}{2}(D+1)(D-2)$ which is precisely the number of helicity states of an irreducible massive spin-2 representation.

This counting of degrees of freedom can also be seen by considering the independent curvature components. Substituting the conventional constraints (2.6) into the Bianchi identities (2.5) one finds after taking traces of the first and second Bianchi identity the following additional curvature constraints (for $D \neq 2$ and $D \neq 3$):

$$R_{\hat{A}\hat{B}}(D) = \frac{1}{(D-2)}\left(3D_{[\hat{C}} T_{\hat{A}\hat{B}]}{}^{\hat{C}} + T_{\hat{A}\hat{B}}{}^{\hat{C}} T_{\hat{C}\hat{D}}{}^{\hat{D}}\right) \,, \qquad \text{and}$$

$$R_{\hat{A}\hat{B}}(K^{\hat{C}}) = \frac{1}{2(D-3)}\left(D_{\hat{D}} R_{\hat{A}\hat{B}}(M^{\hat{C}\hat{D}}) - 2T_{\hat{D}[\hat{A}}{}^{\hat{E}} R_{\hat{B}]\hat{E}}(M^{\hat{C}\hat{D}}) - 12T_{[\hat{A}\hat{B}}{}^{[\hat{C}} F_{\hat{D}]}{}^{\hat{D}]}\right)$$
$$+ \frac{1}{2(D-2)(D-3)}\left(-T_{\hat{D}\hat{E}}{}^{\hat{F}} R_{\hat{F}[\hat{A}}(M^{\hat{D}\hat{E}}) + 4T_{\hat{D}\hat{E}}{}^{[\hat{D}} F_{\hat{A}]}{}^{\hat{E}]} - 8F_{\hat{D}}{}^{[\hat{D}} T_{\hat{E}[\hat{A}}{}^{\hat{E}]}\right) \delta_{\hat{B}]}^{\hat{C}} \,. \qquad (2.19)$$

Therefore, the only independent curvature left is the Lorentz curvature. From the conventional constraints and the Bianchi identities, it follows that this Lorentz curvature satisfies a set

of algebraic and differential identities (which we refrain from giving here) that reduce the $(\frac{1}{2}D(D-1))^2$ components to $\frac{1}{2}(D+1)(D-2)$ independent ones.

In the rest of this section 2, we will set the torsion equal to zero to make contact with general relativity. The conventional constraints (2.6) are then invariant under (2.2). As a consequence, the homogeneous conformal transformation rules of the dependent spin-connection and special conformal gauge field are not modified with respect to (2.2), as is seen explicitly by setting $T_{\mu\nu}{}^{\hat{A}}$ equal to zero in (2.13). From (2.16), one sees that also the translation rule of the Vielbein coincides with that given in (2.2). The same is true for the dilatation gauge field, since the curvature contribution in its translation rule (2.17) vanishes as a consequence of Bianchi identities (see the first of (2.19) for $T_{\mu\nu}{}^{\hat{A}} = 0$). However, even in the absence of torsion, the translation rules of the spin-connection and special conformal gauge fields require modifications with curvature terms.

In what follows, we will denote the explicit solutions (2.10) for the dependent spin-connection and special conformal gauge field in the zero torsion case by $\Omega_{\mu}{}^{\hat{A}\hat{B}}(E,B)$ and $F_{\mu}{}^{\hat{A}}(E,B)$. The Levi-Civita spin-connection, obtained by setting $T_{\mu\nu}{}^{\hat{A}} = 0$ in (2.11), will be denoted by $\Omega_{\mu}{}^{\hat{A}\hat{B}}(E)$. For future reference, we note that, assuming zero torsion, we can relate $R'_{\mu\nu}(M^{\hat{A}\hat{B}})$ to the usual Riemann tensor $R_{\mu\nu}{}^{\hat{A}\hat{B}}(E)$ in the following way, by writing out its B_{μ}-dependence:

$$R'_{\mu\nu}(M^{\hat{A}\hat{B}}) = R_{\mu\nu}{}^{\hat{A}\hat{B}}(E) + 4\,E_{[\nu}{}^{\hat{A}}\,D_{\mu]}(E)B^{\hat{B}]} + 4\,E_{[\mu}{}^{[\hat{A}}\,B^{\hat{B}]}\,B_{\nu]} - 2\,E_{[\mu}{}^{[\hat{A}}\,E_{\nu]}{}^{\hat{B}]}\,B^{\hat{C}}\,B_{\hat{C}}\,. \tag{2.20}$$

Both the Riemann tensor $R_{\mu\nu}{}^{\hat{A}\hat{B}}(E)$ and the covariant derivative $D_{\mu}(E)B^{\hat{A}}$ are constructed with the usual Levi-Civita spin-connection $\Omega_{\mu}{}^{\hat{A}\hat{B}}(E)$:

$$\begin{aligned}
R_{\mu\nu}{}^{\hat{A}\hat{B}}(E) &= 2\,\partial_{[\mu}\Omega_{\nu]}{}^{\hat{A}\hat{B}}(E) + 2\,\Omega_{[\mu|}{}^{[\hat{A}|}{}_{\hat{C}}(E)\,\Omega_{|\nu]}{}^{\hat{C}|\hat{B}]}(E)\,, \\
D_{\mu}(E)B^{\hat{A}} &= \partial_{\mu}B^{\hat{A}} + \Omega_{\mu}{}^{\hat{A}\hat{B}}(E)\,B_{\hat{B}}\,.
\end{aligned} \tag{2.21}$$

Note that the covariant derivative $D_{\mu}(E)$ is covariant with respect to the Lorentz transformations only, unlike the covariant derivative D_{μ} that we used in the Bianchi identities (2.5).

2.3 Einstein-Hilbert gravity from the compensating mechanism

We will now show how the conformal approach can be used to obtain the EH Lagrangian of general relativity

$$\mathcal{L}_{\text{EH}} = \frac{1}{2\kappa^2}ER\,, \tag{2.22}$$

with κ^2 the gravitational coupling constant, in a two-step process. First, one couples a compensating scalar to the independent and dependent gauge fields of the conformal algebra that were discussed in the previous sections. The action of this scalar is determined by requiring invariance under the homogeneous conformal transformations and, manifestly, under diffeomorphisms. As discussed in the previous section, this then also guarantees invariance under translations, under the proviso that these act on the gauge fields with the modifications in (2.16) and (2.17). Einstein-Hilbert gravity is recovered next, by gauge-fixing the dilatations and special conformal transformations.

As starting point, we consider the Lagrangian of a free, massless scalar field ϕ in flat spacetime:

$$\mathcal{L}_{\text{scalar}} = -\frac{1}{2}\,\phi\,\partial^{\hat{A}}\partial_{\hat{A}}\phi\,. \tag{2.23}$$

Note that the sign in front of this Lagrangian is the opposite of the usual, physical one. This will guarantee that the EH action comes out with the correct sign. We then assume that the transformation rule of the scalar field ϕ under local homogeneous conformal transformations is given by a dilatation with scaling weight w:

$$\delta_{\text{hom}}\phi = w\,\Lambda_D\,\phi\,, \tag{2.24}$$

and we replace the ordinary derivatives in (2.23) by covariant ones, by coupling to the gauge fields of the conformal algebra, discussed in the previous sections. The first order derivative of ϕ that transforms covariantly (i.e., without a derivative of a parameter) under homogeneous conformal transformations is defined by:

$$D_{\hat{A}}\phi \equiv E_{\hat{A}}{}^{\mu}\left(\partial_{\mu} - w\,B_{\mu}\right)\phi\,. \tag{2.25}$$

Note that the second term in this covariant derivative is related to the transformation rule (2.24) by replacing the dilatation parameter Λ_D by (minus) the corresponding gauge field B_{μ}. The covariant derivative $D_{\hat{A}}\phi$ transforms as follows under homogeneous conformal transformations:

$$\delta_{\text{hom}}\left(D_{\hat{A}}\phi\right) = -\Lambda_{\hat{A}}{}^{\hat{B}}\,D_{\hat{B}}\phi + (w+1)\,\Lambda_D\,D_{\hat{A}}\phi - 2\,w\,\Lambda_{K\hat{A}}\,\phi\,. \tag{2.26}$$

As a consequence, a covariant conformal box operator on ϕ can be introduced as:

$$\Box^C\phi \equiv E^{\hat{A}\mu}\left[\partial_{\mu}\left(D_{\hat{A}}\phi\right) + \Omega_{\mu\hat{A}}{}^{\hat{B}}(E,B)\,D_{\hat{B}}\phi - (w+1)\,B_{\mu}\,D_{\hat{A}}\phi + 2\,w\,F_{\mu\hat{A}}(E,B)\,\phi\right]\,, \tag{2.27}$$

where the last three terms are obtained by replacing the gauge parameters on the right-hand-side of (2.26) by (minus) their corresponding gauge fields. Explicit computation shows that the homogeneous conformal transformation rule of $\Box^C\phi$ is given by:

$$\delta_{\text{hom}}\Box^C\phi = (w+2)\,\Lambda_D\,\Box^C\phi + (2D - 4w - 4)\,\Lambda_K{}^{\hat{A}}\,D_{\hat{A}}\phi\,. \tag{2.28}$$

Replacing the flat space-time d'Alembertian in (2.23) by $\Box^C$, one is led to the following Lagrangian of a conformally coupled scalar:

$$\mathcal{L}_{\text{conf}} = -\frac{1}{2}\,E\,\phi\,\Box^C\phi\,. \tag{2.29}$$

From the transformation rule (2.28), one sees that $\mathcal{L}_{\text{conf}}$ is not invariant under special conformal transformations unless the weight w is given by:

$$w = \frac{D}{2} - 1\,. \tag{2.30}$$

Since E, the determinant of the Vielbein $E_{\mu}{}^{\hat{A}}$, has scaling weight $-D$ under dilatations, this value of w also renders $\mathcal{L}_{\text{conf}}$ invariant under dilatations and thus under all homogeneous conformal symmetries.

The EH Lagrangian can be retrieved from $\mathcal{L}_{\text{conf}}$ by gauge-fixing the special conformal transformations and dilatations. To fix the latter, we set ϕ equal to a constant k:

$$\text{dilatation gauge-fixing:} \quad \phi = k \,. \tag{2.31}$$

Imposing this gauge-fixing condition in $\mathcal{L}_{\text{conf}}$, one obtains

$$\mathcal{L}_{\text{conf}} \xrightarrow{\phi=k} \mathcal{L} = \frac{wk^2}{2} E\, E^{\hat{A}\mu} \left[\left(\partial_\mu B_{\hat{A}} + \Omega_{\mu\hat{A}}{}^{\hat{B}}(E,B)\, B_{\hat{B}} \right) - (w+1)\, B_\mu\, B_{\hat{A}} - 2\, F_{\mu\hat{A}}(E,B) \right],$$

$$= \frac{wk^2}{2} E \left[D^{\hat{A}}(E) B_{\hat{A}} + (D - w - 2) B^{\hat{A}} B_{\hat{A}} - 2\, F_{\hat{A}}{}^{\hat{A}}(E,B) \right]. \tag{2.32}$$

The field $B_{\hat{A}}$ can not appear in $\mathcal{L}_{\text{conf}}$, since it is the only independent field in (2.32) that transforms under special conformal transformations and its change under $K_{\hat{A}}$ is given by a Stueckelberg shift $2\, \Lambda_{K\hat{A}}$.[8] Indeed, using the explicit expression for $F_\mu{}^{\hat{A}}(E,B)$ in (2.10) (with $T_{\mu\nu}{}^{\hat{A}} = 0$), along with (2.20) and (2.21), one finds that:

$$-2\, F_{\hat{A}}{}^{\hat{A}}(E,B) = \frac{R}{2(D-1)} - D^{\hat{A}}(E) B_{\hat{A}} - \frac{1}{2}(D-2) B^{\hat{A}} B_{\hat{A}} \,, \tag{2.33}$$

where $R = R_{\hat{A}\hat{B}}{}^{\hat{A}\hat{B}}(E)$ is the Ricci scalar of general relativity. The Lagrangian (2.32) can thus be rewritten as

$$\mathcal{L} = \frac{wk^2}{2} E \left[\frac{R}{2(D-1)} - \frac{1}{2}\, (2w + 2 - D)\, B^{\hat{A}} B_{\hat{A}} \right]. \tag{2.34}$$

The last term vanishes due to the parameter constraint (2.30) that guarantees conformal invariance of (2.29) and one finds that (2.32) is given by:

$$\mathcal{L} = \frac{wk^2}{4(D-1)} E\, R = \frac{k^2}{8} \frac{(D-2)}{(D-1)} E\, R \,. \tag{2.35}$$

Alternatively, one can immediately obtain (2.35) from (2.32) by gauge-fixing the special conformal transformations via the gauge choice:

$$\text{special conformal gauge-fixing:} \quad B_{\hat{A}} = 0 \,. \tag{2.36}$$

Either way, one finds that, upon choosing the constant k as:

$$k = \frac{2}{\kappa} \sqrt{\frac{(D-1)}{(D-2)}} \,, \tag{2.37}$$

(2.35) becomes the EH Lagrangian (2.22), showing that the latter can indeed be obtained as a partial gauge-fixing of the conformal matter Lagrangian (2.29).

---

[8]In particular, invariance of $\mathcal{L}_{\text{conf}}$ under special conformal transformations implies

$$0 = \delta_K \mathcal{L}_{\text{conf}} = \delta_K B_{\hat{A}} \frac{\delta \mathcal{L}_{\text{conf}}}{\delta B_{\hat{A}}} = 2\Lambda_{K\hat{A}} \frac{\delta \mathcal{L}_{\text{conf}}}{\delta B_{\hat{A}}} \qquad \Rightarrow \qquad \frac{\delta \mathcal{L}_{\text{conf}}}{\delta B_{\hat{A}}} = 0 \,,$$

so that $\mathcal{L}_{\text{conf}}$ does not depend on $B_{\hat{A}}$.

For later use, it is instructive to consider a slightly different starting point to derive the EH action from a compensating mechanism. Instead of starting from the Lagrangian (2.23), we could also start from the following Lagrangian in flat spacetime:

$$\mathcal{L}'_{\text{scalar}} = \frac{1}{2} \partial_{\hat{A}} \phi \partial^{\hat{A}} \phi \,, \tag{2.38}$$

that differs from (2.23) by a total derivative. Following the same strategy of replacing derivatives by covariant ones we end up with the following Ansatz for the Lagrangian of a scalar coupled to gauge fields of the conformal algebra:

$$\mathcal{L}'_{\text{Ansatz}} = \frac{E}{2} D_{\hat{A}} \phi \, D^{\hat{A}} \phi \,, \tag{2.39}$$

where $D_{\hat{A}} \phi$ has been defined in (2.25). However, as it stands, this Lagrangian is not conformally invariant. In particular, under special conformal transformations one has

$$\delta_K \mathcal{L}'_{\text{Ansatz}} = E \, D^{\hat{A}} \phi \, \delta_K \left(D_{\hat{A}} \phi \right) = -2w \, E \, \Lambda_{K\hat{A}} \, \phi \, D^{\hat{A}} \phi = -w \, E \, E_{\hat{A}}{}^{\mu} \, \Lambda_K{}^{\hat{A}} \, D_{\mu}(\phi^2) \,, \tag{2.40}$$

with $D_{\mu}(\phi^2) = \partial_{\mu}(\phi^2) - 2\, w\, B_{\mu}\, \phi^2$ (since ϕ^2 has dilatation weight $2w$). To see how this variation can be cancelled, one partially integrates

$$\delta_K \mathcal{L}'_{\text{Ansatz}} = w \partial_{\mu} \left(E \, E_{\hat{A}}{}^{\mu} \right) \Lambda_K{}^{\hat{A}} \, \phi^2 + w \, E \, E_{\hat{A}}{}^{\mu} \, \partial_{\mu} \Lambda_K{}^{\hat{A}} \, \phi^2 + 2\, w^2 \, E \, \Lambda_K{}^{\hat{A}} \, B_{\hat{A}} \, \phi^2$$

$$= w \, E \, E_{\hat{A}}{}^{\mu} \left(\delta_K F_{\mu}{}^{\hat{A}}(E, B) \right) \phi^2 + w \, (2w + 2 - D) \, E \, \Lambda_K{}^{\hat{A}} \, B_{\hat{A}} \, \phi^2 \,, \tag{2.41}$$

where

$$\delta_K F_{\mu}{}^{\hat{A}}(E, B) = \partial_{\mu} \Lambda_K{}^{\hat{A}} - \Lambda_{K\hat{B}} \, \Omega_{\mu}{}^{\hat{B}\hat{A}}(E, B) - \Lambda_K{}^{\hat{A}} \, B_{\mu} \,, \tag{2.42}$$

is an infinitesimal special conformal transformation of $F_{\mu}{}^{\hat{A}}(E, B)$ (see eq. (2.13) with $T_{\mu\nu}{}^{\hat{A}} = 0$). In the first line of (2.41) we dropped a total derivative term, while in going to the second line we used the identity

$$\partial_{\mu} \left(E \, E_{\hat{A}}{}^{\mu} \right) = 2 \, E \, E_{\hat{A}}{}^{\mu} \, E_{\hat{B}}{}^{\nu} \, \partial_{[\mu} E_{\nu]}{}^{\hat{B}}$$

$$= -2 \, E \, E_{\hat{A}}{}^{\mu} \, E_{\hat{B}}{}^{\nu} \, \Omega_{[\mu}{}^{\hat{B}\hat{C}}(E, B) \, E_{\nu]\hat{C}} - 2 \, E \, E_{\hat{A}}{}^{\mu} \, E_{\hat{B}}{}^{\nu} \, B_{[\mu} \, E_{\nu]}{}^{\hat{B}} \,, \tag{2.43}$$

where the first line follows from $\partial_{\mu} E = E E_{\hat{A}}{}^{\nu} \partial_{\mu} E_{\nu}{}^{\hat{A}}$ and $\partial_{\nu} E_{\hat{A}}{}^{\mu} = -E_{\hat{A}}{}^{\rho} E_{\hat{B}}{}^{\mu} \partial_{\nu} E_{\rho}{}^{\hat{B}}$ and in the second line we have used the first of the conventional constraints (2.6), with $T_{\mu\nu}{}^{\hat{A}} = 0$. Choosing the dilatation weight w as in (2.30), the last term of (2.41) vanishes. Since the remaining variation of $\mathcal{L}'_{\text{Ansatz}}$ can be cancelled by that of $-w \, E \, F_{\hat{A}}{}^{\hat{A}}(E, B) \, \phi^2$, adding this term to $\mathcal{L}'_{\text{Ansatz}}$ leads to the following conformally invariant Lagrangian $\mathcal{L}'_{\text{conf}}$:

$$\mathcal{L}'_{\text{conf}} = \mathcal{L}'_{\text{Ansatz}} - w \, E \, F_{\hat{A}}{}^{\hat{A}}(E, B) \, \phi^2 = \frac{E}{2} \left[D_{\hat{A}} \phi D^{\hat{A}} \phi - 2 \, w \, F_{\hat{A}}{}^{\hat{A}}(E, B) \, \phi^2 \right] \,. \tag{2.44}$$

By partially integrating and using (2.30) and (2.43), one can see that $\mathcal{L}'_{\text{conf}}$ is equal to the Lagrangian $\mathcal{L}_{\text{conf}}$ given in (2.29), up to a total derivative. The EH Lagrangian (2.22) is then recovered from $\mathcal{L}'_{\text{conf}}$ by adopting the dilatation and special conformal gauge-fixing conditions of eqs. (2.31) (with k again given by (2.37)) and (2.36).

This finishes our review of the conformal compensating technique in the relativistic case.

3 A conformal approach to Carroll gravity

Here, we will adapt the conformal compensating technique discussed in the previous section to construct two different Carroll gravity theories. The existence of these two theories is related to the fact that there exist two massless Carroll scalar field theories that are called 'electric' and 'magnetic' in the literature [11, 18–21]. In subsection 3.1, we will first discuss the gauging of the conformal Carroll algebra, which corresponds to an Inönü-Wigner contraction of the relativistic conformal algebra. In the next subsection 3.2, we will derive a minimal multiplet of gauge fields of the conformal Carroll algebra, by solving suitable conventional curvature constraints to express some of the gauge fields in terms of the remaining independent ones. Finally, in the last subsection 3.3, we will construct conformally coupled versions of electric and magnetic Carroll scalars and adopt gauge-fixing conditions, giving rise to two different Carroll gravity theories.

3.1 The conformal Carroll algebra and its gauging

The commutation relations of the conformal Carroll algebra can be obtained by an Inönü-Wigner contraction of the relativistic conformal algebra, given in (2.1). To perform this contraction we decompose the flat Lorentz index $\hat{A}$ as $\hat{A} = \{0, A\}$, where 0 is a flat time index and $A = 1, \cdots, D-1$ are flat spatial indices. The relativistic conformal generators decompose correspondingly as

$$M_{\hat{A}\hat{B}} \to M_{0A}, M_{AB}, \qquad P_{\hat{A}} \to P_0, P_A, \qquad K_{\hat{A}} \to K_0, K_A. \qquad (3.1)$$

Next, we redefine the conformal generators with a contraction parameter c as follows:

$$
\begin{aligned}
M_{0A} &= c^{-1} J_{0A}, & M_{AB} &= J_{AB}, & P_0 &= c^{-1} H, & P_A &= P_A, \\
K_0 &= c^{-1} K, & K_A &= K_A, & D &= D.
\end{aligned}
\qquad (3.2)
$$

The generators on the right-hand-side become those of the conformal Carroll algebra after taking the $c \to 0$ limit. In what follows, we will refer to the transformations associated to J_{AB}, J_{0A}, K/K_A and D as spatial rotations, Carroll boosts, singlet/vector special conformal transformations and dilatations, respectively. They constitute the subalgebra of homogeneous conformal Carroll transformations, whereas H and P_A correspond to time and spatial translations. Substituting the redefinitions (3.2) into the relativistic commutation relations (2.1) and taking the $c \to 0$ limit, we obtain the following non-zero commutation relations of the conformal Carroll algebra [22, 23]:

$$
\begin{aligned}
[J_{AB}, J_{CD}] &= 4\,\delta_{[A[C} J_{D]B]}, & [J_{AB}, J_{0C}] &= 2\,\delta_{C[B|} J_{0|A]} & [P_A, J_{BC}] &= 2\,\delta_{A[B} P_{C]}, \\
[P_A, J_{0B}] &= -\delta_{AB} H, & [P_A, K_B] &= 2\,(\delta_{AB} D + J_{AB}), & [K_A, J_{BC}] &= 2\,\delta_{A[B} K_{C]}, \\
[K_A, J_{0B}] &= -\delta_{AB} K, & [P_A, K] &= -2\,J_{0A}, & [H, K_A] &= 2\,J_{0A}, \\
[D, P_A] &= P_A, & [D, H] &= H, & [D, K_A] &= -K_A, \\
[D, K] &= -K.
\end{aligned}
\qquad (3.3)
$$

The conformal Carroll transformations act on the time (t) and Cartesian spatial (x^A) coordinates of flat Carroll space-time as follows:[9]

$$\delta t = a + v_A x^A + \xi t + \kappa \, x^A x_A - 2\kappa_A x^A t \,,$$
$$\delta x^A = a^A + o^A{}_B x^B + \xi x^A + \left(\kappa^A \, x^B x_B - 2\kappa_B x^B x^A \right) \,, \tag{3.4}$$

where a, a^A, o^{AB}, v^A, ξ, κ and κ^A are the constant parameters of the transformations H, P_A, J_{AB}, J_{0A}, D, K and K_A respectively. It is interesting to compare the conformal Carroll transformations (3.4) with the conformal isometries of flat Carroll space-time.[10] In the notation of (3.4), a generic such conformal isometry acts infinitesimally as

$$\delta t = \xi t - 2\kappa_A x^A t + T(x) \,,$$
$$\delta x^A = a^A + o^A{}_B x^B + \xi x^A + \left(\kappa^A \, x^B x_B - 2\kappa_B x^B x^A \right) \,. \tag{3.5}$$

Here, $T(x)$ is an arbitrary function of the spatial coordinates x^A, so that the algebra of infinitesimal conformal isometries is infinite-dimensional. It was shown in [2] that this conformal isometry algebra is isomorphic to the BMS algebra in $D+1$ dimensions, with $T(x)$ corresponding to the BMS supertranslations. The conformal Carroll transformations (3.4) then correspond to the subset of the conformal isometries (3.5) that is obtained by taking $T(x)$ as

$$T(x) = a + v_A x^A + \kappa x^A x_A \,. \tag{3.6}$$

One thus sees that the conformal Carroll algebra (3.3) admits the $(D+1)$-dimensional BMS algebra as an infinite-dimensional extension.

We then introduce the gauge fields τ_μ, $e_\mu{}^A$, $\omega_\mu{}^{AB}$, $\omega_\mu{}^{0A}$, f_μ, $g_\mu{}^A$ and b_μ, associated to the generators H, P_A, J_{AB}, J_{0A}, K, K_A and D, respectively. From the algebra (3.3), we find that these gauge fields transform under time/spatial translations, spatial rotations, Carroll boosts, singlet/vector special conformal transformations and dilatations, with respective gauge parameters ζ/ζ^A, $\lambda^{AB}, \lambda^{0A}, \lambda_K/\lambda_K{}^A$ and λ_D, as follows:

$$\delta_0 \tau_\mu = \tilde{D}_\mu \zeta - \lambda^{0A} e_{\mu A} - \lambda_D \tau_\mu \,,$$
$$\delta_0 e_\mu{}^A = \tilde{D}_\mu \zeta^A - \lambda^A{}_B e_\mu{}^B - \lambda_D e_\mu{}^A \,,$$
$$\delta_0 \omega_\mu{}^{AB} = \partial_\mu \lambda^{AB} - 2\lambda^{[A}{}_C \omega_\mu{}^{C|B]} - 4\lambda_K{}^{[A} e_\mu{}^{B]} - 4\zeta^{[A} g_\mu{}^{B]} \,,$$
$$\delta_0 \omega_\mu{}^{0A} = \partial_\mu \lambda^{0A} + \omega_\mu{}^A{}_B \lambda^{0B} - \lambda^A{}_B \omega_\mu{}^{0B} + 2\lambda_K{}^A \tau_\mu - 2\lambda_K e_\mu{}^A + 2\zeta^A f_\mu - 2\zeta g_\mu{}^A \,,$$
$$\delta_0 f_\mu = \partial_\mu \lambda_K - \lambda_K b_\mu - \lambda^{0A} g_{\mu A} + \lambda_{KA} \omega_\mu{}^{0A} + \lambda_D f_\mu \,,$$
$$\delta_0 g_\mu{}^A = \partial_\mu \lambda_K{}^A + \omega_\mu{}^A{}_B \lambda_K{}^B - \lambda_K{}^A b_\mu - \lambda^A{}_B g_\mu{}^B + \lambda_D g_\mu{}^A \,,$$
$$\delta_0 b_\mu = \partial_\mu \lambda_D + 2\lambda_K{}^A e_{\mu A} - 2\zeta^A g_{\mu A} \,, \tag{3.7}$$

_____

[9]These transformations follow from re-introducing the speed of light c in the action of the relativistic conformal transformations on Minkowski space-time coordinates and taking the limit $c \to 0$.

[10]The metric structure of a flat Carroll space-time consists of a spatial metric $h = \delta_{AB} dx^A dx^B$ and a vector field $T = \partial/\partial t$. A conformal isometry of flat Carroll space-time is then generated by a vector field X that obeys $\mathcal{L}_X h = \Omega h$ and $\mathcal{L}_X T = -\frac{\Omega}{2} T$ for some function $\Omega = \Omega(t, x^A)$, where $\mathcal{L}_X$ denotes the Lie derivative along X.

where

$$\tilde{D}_\mu\zeta \equiv \partial_\mu\zeta + \omega_\mu{}^{0A}\zeta_A + \zeta b_\mu\,, \qquad\qquad \tilde{D}_\mu\zeta^A \equiv \partial_\mu\zeta^A + \omega_\mu{}^{AB}\zeta_B + \zeta^A b_\mu\,. \qquad (3.8)$$

We use a subscript "0" on δ_0 to indicate that these are not the final conformal Carroll transformation rules that we will use to construct Carroll gravity theories. Like in the relativistic case, we will need to impose conventional constraints that will supplement these rules with extra terms and it is the thus modified transformations that will be employed in the compensating mechanism. These changes to (3.7) will be discussed in the next section 3.2. We will furthermore also make use of the symbol $\delta_{\text{hom},\,0}$ to specify the action of the homogeneous conformal Carroll transformations of (3.7).

Starting from the conformal Carroll algebra (3.3), one defines the following curvatures:

$$R_{\mu\nu}(H) = 2\,\partial_{[\mu}\tau_{\nu]} + 2\,\omega_{[\mu}{}^{0A}e_{\nu]A} + 2\,b_{[\mu}\,\tau_{\nu]}\,, \qquad (3.9\text{a})$$

$$R_{\mu\nu}(P^A) = 2\,\partial_{[\mu}e_{\nu]}{}^A + 2\,\omega_{[\mu}{}^{AB}e_{\nu]B} + 2\,b_{[\mu}\,e_{\nu]}{}^A\,, \qquad (3.9\text{b})$$

$$R_{\mu\nu}(J^{AB}) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{AB} + 2\,\omega_{[\mu|}{}^{[A|}{}_C\,\omega_{|\nu]}{}^{C|B]} + 8\,g_{[\mu}{}^{[A}e_{\nu]}{}^{B]}\,, \qquad (3.9\text{c})$$

$$R_{\mu\nu}(J^{0A}) = 2\,\partial_{[\mu}\omega_{\nu]}{}^{0A} + 2\,\omega_{[\mu|}{}^A{}_B\,\omega_{|\nu]}{}^{0B} + 4\,f_{[\mu}e_{\nu]}{}^A - 4g_{[\mu}{}^A\tau_{\nu]}\,, \qquad (3.9\text{d})$$

$$R_{\mu\nu}(K) = 2\partial_{[\mu}f_{\nu]} + 2\,\omega_{[\mu}{}^{0A}g_{\nu]A} - 2\,b_{[\mu}\,f_{\nu]}\,, \qquad (3.9\text{e})$$

$$R_{\mu\nu}(K^A) = 2\partial_{[\mu}g_{\nu]}{}^A + 2\,\omega_{[\mu}{}^{AB}g_{\nu]B} - 2\,b_{[\mu}\,g_{\nu]}{}^A\,, \qquad (3.9\text{f})$$

$$R_{\mu\nu}(D) = 2\,\partial_{[\mu}b_{\nu]} - 4\,g_{[\mu}{}^A e_{\nu]A}\,. \qquad (3.9\text{g})$$

These curvatures transform covariantly under (3.7); in particular, one has:

$$\delta_0 R_{\mu\nu}(H) = \zeta_A R_{\mu\nu}(J^{0A}) + \zeta R_{\mu\nu}(D) - \lambda^{0A}R_{\mu\nu}(P_A) - \lambda_D R_{\mu\nu}(H)\,,$$

$$\delta_0 R_{\mu\nu}(P^A) = -\zeta_B R_{\mu\nu}(J^{BA}) + \zeta^A R_{\mu\nu}(D) - \lambda^A{}_B R_{\mu\nu}(P^B) - \lambda_D R_{\mu\nu}(P^A)\,,$$

$$\delta_0 R_{\mu\nu}(J^{AB}) = -2\lambda^{[A|}{}_C R_{\mu\nu}(J^{C|B]}) - 4\lambda_K^{[A}R_{\mu\nu}(P^{B]}) - 4\zeta^{[A}R_{\mu\nu}(K^{B]})\,,$$

$$\delta_0 R_{\mu\nu}(J^{0A}) = -\lambda^{0B}R_{\mu\nu}(J_B{}^A) - \lambda^A{}_B R_{\mu\nu}(J^{0B}) + 2\lambda_K{}^A R_{\mu\nu}(H) - 2\lambda_K R_{\mu\nu}(P^A)$$
$$\qquad\qquad + 2\zeta^A R_{\mu\nu}(K) - 2\zeta R_{\mu\nu}(K^A)\,,$$

$$\delta_0 R_{\mu\nu}(K) = -\lambda_K R_{\mu\nu}(D) - \lambda^{0A}R_{\mu\nu}(K_A) + \lambda_{KA}R_{\mu\nu}(J^{0A}) + \lambda_D R_{\mu\nu}(K)\,,$$

$$\delta_0 R_{\mu\nu}(K^A) = -\lambda_K{}^B R_{\mu\nu}(J_B{}^A) - \lambda_K{}^A R_{\mu\nu}(D) - \lambda^A{}_B R_{\mu\nu}(K^B) + \lambda_D R_{\mu\nu}(K^A)\,,$$

$$\delta_0 R_{\mu\nu}(D) = 2\lambda_K{}^A R_{\mu\nu}(P_A) - 2\zeta^A R_{\mu\nu}(K_A)\,. \qquad (3.10)$$

Furthermore, they satisfy the following Bianchi identities:

$$D_{[\mu}R_{\nu\rho]}(H) + R_{[\mu\nu}(P^A)\omega_{\rho]}{}^0{}_A - R_{[\mu\nu}(J^{0A})e_{\rho]A} - R_{[\mu\nu}(D)\tau_{\rho]} = 0\,,$$

$$D_{[\mu}R_{\nu\rho]}(P^A) - R_{[\mu\nu}(J^{AB})e_{\rho]B} - R_{[\mu\nu}(D)e_{\rho]}{}^A = 0\,,$$

$$D_{[\mu}R_{\nu\rho]}(J^{AB}) - 4R_{[\mu\nu}(K^{[A})e_{\rho]}{}^{B]} - 4R_{[\mu\nu}(P^{[A})g_{\rho]}{}^{B]} = 0\,,$$

$$D_{[\mu}R_{\nu\rho]}(J^{0A}) - R_{[\mu\nu}(J^{AB})\omega_{\rho]}{}^0{}_B + 2R_{[\mu\nu}(K^A)\tau_{\rho]} - 2R_{[\mu\nu}(K)e_{\rho]}{}^A$$
$$\qquad + 2R_{[\mu\nu}(P^A)f_{\rho]} - 2R_{[\mu\nu}(H)g_{\rho]}{}^A = 0\,,$$

$$D_{[\mu}R_{\nu\rho]}(K) + R_{[\mu\nu}(K^A)\omega_{\rho]}{}^0{}_A - R_{[\mu\nu}(J^{0A})g_{\rho]A} + R_{[\mu\nu}(D)f_{\rho]} = 0\,,$$

$$D_{[\mu}R_{\nu\rho]}(K^A) - R_{[\mu\nu}(J^{AB})g_{\rho]B} + R_{[\mu\nu}(D)g_{\rho]}{}^A = 0\,,$$

$$\partial_{[\mu}R_{\nu\rho]}(D) + 2R_{[\mu\nu}(K^A)e_{\rho]A} - 2R_{[\mu\nu}(P^A)g_{\rho]A} = 0\,, \qquad (3.11)$$

where the derivatives D_μ are covariant with respect to spatial rotations and dilatations but not with respect to Carroll boosts and singlet/vector special conformal transformations since the latter do not act in a homogeneous manner.

We will often need the dual Vielbeine $(\tau^\mu, e_A{}^\mu)$ that are defined by the following duality relations:

$$\tau^\mu \tau_\mu = 1 \,, \qquad\qquad \tau^\mu e_\mu{}^A = 0 \,, \qquad\qquad \tau_\mu e_A{}^\mu = 0 \,,$$
$$e_\mu{}^A e_B{}^\mu = \delta_B^A \,, \qquad\qquad e_\mu{}^A e_A{}^\nu = \delta_\nu^\mu - \tau^\mu \tau_\nu \,. \tag{3.12}$$

These dual Vielbeine transform as follows under (3.7):

$$\delta_0 \tau^\mu = -\tau^\mu \tilde{D}_0 \zeta - e_A{}^\mu \tilde{D}_0 \zeta^A + \lambda_D \tau^\mu \,,$$
$$\delta_0 e_A{}^\mu = -\tau^\mu \tilde{D}_A \zeta - e_B{}^\mu \tilde{D}_A \zeta^B - \lambda_A{}^B e_B{}^\mu + \lambda^0{}_A \tau^\mu + \lambda_D e_A{}^\mu \,. \tag{3.13}$$

In the following, we will use τ^μ and $e_A{}^\mu$ to convert a curved index μ into flat indices 0 and A, respectively. For instance, given a one-form X_μ and a two-form $X_{\mu\nu}$ we define the different projections to flat time and spatial components as follows:

$$X_0 \equiv \tau^\mu X_\mu \,, \qquad X_A \equiv e_A{}^\mu X_\mu \,, \qquad X_{0A} \equiv \tau^\mu e_A{}^\nu X_{\mu\nu} \,, \qquad X_{AB} \equiv e_A{}^\mu e_B{}^\nu X_{\mu\nu} \,. \tag{3.14}$$

3.2 Imposing constraints

As in the relativistic case, the multiplet of gauge fields (3.7) that results from gauging the conformal Carroll algebra, contains more independent fields than are necessary in a gravitational theory. We therefore wish to impose suitable curvature constraints that allow one to solve for some fields in terms of the remaining ones and thus reduce the number of independent fields.

By inspecting the list of curvatures given in (3.9), one sees that a minimal set of independent fields can be obtained in analogy to the relativistic case, by constraining the curvatures $R_{\mu\nu}(H)$ and $R_{\mu\nu}(P^A)$ of time and spatial translations, as well as the components $R_{\mu B}(J^{AB})$ and $R_{\mu A}(J^{0A})$ of the curvatures of spatial rotations and Carroll boosts. Similar to what we discussed in section 2.2, one could impose the following constraints:

$$R_{\mu\nu}(H) = T_{\mu\nu} \,, \quad R_{\mu\nu}(P^A) = T_{\mu\nu}{}^A \,, \qquad\qquad R_{\mu B}(J^{AB}) = 0 \,, \quad R_{\mu A}(J^{0A}) = 0 \,, \tag{3.15}$$

where $T_{\mu\nu}$ and $T_{\mu\nu}{}^A$ taken together yield a Carrollian analogue of the torsion tensor. In this paper, we will not consider the case of arbitrary torsion. We will instead closely follow the reasoning that led to general relativity and put as many Carrollian torsion components equal to zero as possible. Note however that we should not require that all torsion components vanish. The reason is that some components of $R_{\mu\nu}(P^A)$ do not contain (spin-connection or dilatation) gauge field components that appear algebraically and can be solved for. In our case, this happens for the following $\frac{1}{2}D(D-1) - 1$ components of the spatial translation curvature:

$$R_0{}^{\{A}(P^{B\}}) \,, \tag{3.16}$$

where $\{AB\}$ denotes the symmetric traceless part of AB. The corresponding torsion components[11]

$$T_0{}^{\{A,B\}} \equiv \tau^\mu e^{\{A|\nu} T_{\mu\nu}{}^{|B\}}\,, \tag{3.17}$$

define the intrinsic torsion.[12] They should not be constrained to be zero from the start, since doing so is tantamount to imposing the following geometric constraints

$$R_0{}^{\{A}(P^{B\}}) = 2\tau^\mu e^{\{A|\nu}\partial_{[\mu}e_{\nu]}{}^{|B\}} = 0\,, \tag{3.18}$$

and is therefore too restrictive.

The components given in (3.17) are the only intrinsic torsion ones: all other components of $T_{\mu\nu}{}^A$ and $T_{\mu\nu}$ are not intrinsic. Setting all non-intrinsic torsion components in (3.15) equal to zero, we are led to impose the following conventional constraints

$$R_{\mu\nu}(H) = 0\,, \qquad\qquad R_{\mu\nu}(P^A) = 2\tau_{[\mu}e_{\nu]B}\,T_0{}^{\{B,A\}}\,, \tag{3.19a}$$

$$R_{\mu B}(J^{AB}) = 0\,, \qquad\qquad R_{\mu A}(J^{0A}) = 0\,. \tag{3.19b}$$

The first two of these (given in (3.19a)) represent in total

$$\frac{1}{2}D(D-1) + \frac{1}{2}D(D-1)^2 - \frac{1}{2}D(D-1) + 1 = \frac{1}{2}D^3 - D^2 + \frac{1}{2}D + 1\,, \tag{3.20}$$

constraints and can be used to solve for the following equal number of gauge field components:

$$\omega_\mu{}^{AB}(e,b) = 2\,e^{[A|\nu}\partial_{[\mu}e_{\nu]}{}^{|B]} - e_{\mu C}\,e^{A\nu}\,e^{B\rho}\partial_{[\nu}e_{\rho]}{}^C + 2\,e_\mu{}^{[A}e^{B]\nu}b_\nu\,, \tag{3.21a}$$

$$\omega_0{}^{0A}(e,\tau,b) = -2\tau^\mu e^{A\nu}\partial_{[\mu}\tau_{\nu]} + e^{A\mu}b_\mu\,, \tag{3.21b}$$

$$\omega^{[A,0|B]}(e,\tau) = -e^{A\mu}e^{B\nu}\partial_{[\mu}\tau_{\nu]}\,, \tag{3.21c}$$

$$b_0(e,\tau) = -\frac{2}{(D-1)}\tau^\mu e_A{}^\nu\partial_{[\mu}e_{\nu]}{}^A \equiv -\frac{1}{(D-1)}\mathcal{T}_{0A}{}^A\,, \tag{3.21d}$$

where we have defined $\mathcal{T}_{0A}{}^A$, corresponding to $T_{0A}{}^A$ without the b_0 gauge field. Similarly, one can show that the first conventional constraint of (3.19b) can be used to solve for the vector special conformal gauge field $g_\mu{}^A$ as follows:

$$g_\mu{}^A(e,\tau,b) = -\frac{1}{2(D-3)}\left[R'_{\mu B}\left(J^{AB}\right) - \frac{1}{2(D-2)}e_\mu{}^A R'_{BC}\left(J^{BC}\right) - \frac{1}{(D-2)}\tau_\mu R'_{0B}\left(J^{AB}\right)\right]. \tag{3.22}$$

Here, $R'_{\mu\nu}(J^{AB})$ is given by the expression (3.9c) for the curvature tensor $R_{\mu\nu}(J^{AB})$, without the term involving the vector special conformal gauge field $g_\mu{}^A$ and with $\omega_\mu{}^{AB}$ replaced by

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[11]The comma in $T_0{}^{\{A,B\}}$ indicates that the two indices to the left of the comma are flat projections of the two curved indices of $T_{\mu\nu}{}^A$.

[12]Note that we use the term 'intrinsic torsion' in a more general sense than is typically done in the mathematical literature, where this term is reserved for the torsion tensor components that do not contain spin-connection components. This is appropriate for space-times with local Carroll symmetries. Since here however, we are dealing with the conformal Carroll algebra, we will for simplicity define the word intrinsic torsion tensor as a torsion tensor that does not contain spin-connection as well as dilatation gauge fields.

the expression (3.21a). Finally, the scalar special conformal gauge field f_μ can be solved from the second conventional constraint of (3.19b) as follows:

$$f_\mu\left(e,\tau,b\right) = -\frac{1}{2(D-2)}\left[R'_{\mu A}\left(J^{0A}\right) - \frac{\tau_\mu}{(D-1)}\left(R'_{0A}\left(J^{0A}\right) + \frac{1}{2}R'_{AB}\left(J^{AB}\right)\right)\right].\quad(3.23)$$

The primed curvature $R'_{\mu\nu}(J^{0A})$ is given by the expression (3.9d) for $R_{\mu\nu}(J^{0A})$ without the terms involving f_μ and $g_\mu{}^A$ and with $\omega_\mu{}^{AB}$ and the dependent parts $\omega_0{}^{0A}$ and $\omega^{[A|,0|B]}$ of the boost connection $\omega_\mu{}^{0A}$ replaced by their expressions (3.21a), (3.21b) and (3.21c).

Summarizing: the gauge field components

$$\omega_\mu{}^{AB},\qquad \omega_0{}^{0A},\qquad \omega^{[A|,0|B]},\qquad b_0,\qquad g_\mu{}^A,\qquad f_\mu,\qquad(3.24)$$

are turned into dependent fields by virtue of the conventional constraints (3.19). In what follows, it will be understood that these field components are given by their dependent expressions (3.21a), (3.21b), (3.21c), (3.21d), (3.22), (3.23), whenever they appear explicitly or implicitly, and we will for simplicity drop the reference to the fields on which they depend. The remaining independent fields are given by:

$$\tau_\mu,\qquad e_\mu{}^A,\qquad b_A,\qquad \omega^{(A|,0|B)}.\qquad(3.25)$$

Two differences with the relativistic case are worth pointing out. First, in the Carroll case it is not possible to solve for all spin-connection components and in particular the boost spin-connection components $\omega^{(A|,0|B)}$ remain independent. The presence of these independent boost connection components is related to the fact that there is intrinsic torsion, see e.g., [24]. Secondly, not all components of the dilatation gauge field are independent; only the spatial components b_A are. Note that the dependent spin-connection fields depend only on b_A and not on the dependent b_0 component.

Let us now comment on how the conformal transformation rules of the independent and dependent fields are changed with respect to (3.7) after imposing the conventional constraints (3.19). First, we consider the homogeneous conformal Carroll transformations. As in the relativistic case, we assume that their action on the independent fields is not modified and thus corresponds to the rules given in (3.7). To determine how they act on the dependent fields, we first investigate how the conventional constraints (3.19) transform under (3.7). Of the constraints (3.19a), only the parts

$$R_{BC}(P^A) = 0,\qquad\qquad R_{0A}(H) = 0,\qquad(3.26)$$

that are used to solve for $\omega_C{}^{AB}$ and $\omega_0{}^{0,A}$, are not invariant under the homogeneous conformal Carroll transformations of (3.7), due to the fact that

$$\delta_{\mathrm{hom},0}R_{BC}(P_A) = \lambda^0{}_B T_{0\{A,C\}} - \lambda^0{}_C T_{0\{A,B\}},\qquad \delta_{\mathrm{hom},0}R_{0A}(H) = -\lambda^{0B}T_{0\{A,B\}}.\quad(3.27)$$

As a consequence, the correct homogeneous conformal Carroll transformations of $\omega_C{}^{AB}$ and $\omega_0{}^{0,A}$, that leave (3.26) invariant, are of the form

$$\delta_{\mathrm{hom}}\omega_C{}^{AB} = \delta_{\mathrm{hom},0}\omega_C{}^{AB} + \Delta\omega_C{}^{AB},\qquad\qquad \delta_{\mathrm{hom}}\omega_0{}^{0A} = \delta_{\mathrm{hom},0}\omega_0{}^{0A} + \Delta\omega_0{}^{0A},\quad(3.28)$$

where the extra contributions $\Delta\omega_C{}^{AB}$ and $\Delta\omega_0{}^{0A}$ are given by:

$$\Delta\omega_{C,AB} = \lambda^0{}_B T_{0\{A,C\}} - \lambda^0{}_A T_{0\{B,C\}}\,, \qquad\qquad \Delta\omega_0{}^{0A} = \lambda^0{}_B T_0{}^{\{A,B\}}\,. \qquad (3.29)$$

These additional transformations can alternatively be derived by varying the explicit solutions of these dependent spin-connection components. The homogeneous conformal transformation rules of the dependent gauge fields $g_\mu{}^A$ and f_μ likewise acquire extra terms due to the fact that the conventional constraints (3.19b) are not invariant under (3.7) or that their explicit expressions (3.22), (3.23) depend on $\omega_C{}^{AB}$ and $\omega_0{}^{0A}$. We will refrain from giving the modified transformations of all components of $g_\mu{}^A$ and f_μ and instead only focus on those of $g_A{}^A$ and f_0, as these are the only components that we will need. One finds that the correct transformations of $g_A{}^A$ and f_0 take the form

$$\delta_{\text{hom}} g_A{}^A = \delta_{\text{hom},0} g_A{}^A + \Delta g_A{}^A\,, \qquad\qquad \delta_{\text{hom}} f_0 = \delta_{\text{hom},0} f_0 + \Delta f_0\,, \qquad (3.30)$$

where the extra terms $\Delta g_A{}^A$ and Δf_0 are given by

$$\Delta g_A{}^A = -\frac{1}{2(D-2)} D_A \Delta\omega_B{}^{AB}\,, \qquad (3.31)$$

$$\Delta f_0 = \frac{1}{2(D-1)} D_A \Delta\omega_0{}^{0A} + \frac{1}{2(D-1)(D-2)} D_A \Delta\omega_B{}^{AB}\,, \qquad (3.32)$$

with

$$D_A \Delta\omega_B{}^{AB} \equiv \partial_A \Delta\omega_B{}^{AB} + \omega_A{}^{AD}(e,b)\Delta\omega_{B,D}{}^B - b_A \Delta\omega_B{}^{AB}\,, \qquad (3.33)$$

$$D_A \Delta\omega_0{}^{0A} \equiv \partial_A \Delta\omega_0{}^{0A} + \omega_A{}^{AB}(e,b)\Delta\omega_0{}^0{}_B - b_A \Delta\omega_0{}^{0A}\,. \qquad (3.34)$$

It turns out that only the combination $g_A{}^A + f_0$ plays a role in the calculations that follow. Since $\Delta\omega_0{}^{0A} = \Delta\omega_B{}^{AB}$, this combination does not receive any modifications in its transformation rule:

$$\Delta g_A{}^A + \Delta f_0 = 0\,. \qquad (3.35)$$

Next, we turn to how the translations of (3.7) are modified in the presence of the conventional constraints (3.19). Following the same logic as in the relativistic case, we define a time/spatial translation with parameter ζ/ζ^A as an infinitesimal diffeomorphism with parameter

$$\xi^\mu = \zeta\tau^\mu + \zeta^A e_A{}^\mu\,, \qquad (3.36)$$

minus homogeneous conformal Carroll transformations with parameters

$$\lambda^{AB} = \left(\zeta\tau^\mu + \zeta^C e_C{}^\mu\right)\omega_\mu{}^{AB}\,, \qquad \lambda^{0A} = \left(\zeta\tau^\mu + \zeta^B e_B{}^\mu\right)\omega_\mu{}^{0A}\,,$$

$$\lambda_K = \left(\zeta\tau^\mu + \zeta^A e_A{}^\mu\right) f_\mu\,, \qquad \lambda_K{}^A = \left(\zeta\tau^\mu + \zeta^B e_B{}^\mu\right) g_\mu{}^A\,,$$

$$\lambda_D = \left(\zeta\tau^\mu + \zeta^A e_A{}^\mu\right) b_\mu\,. \qquad (3.37)$$

For the independent fields, this yields the following rules:

$$\delta_\zeta \tau_\mu = \tilde{D}_\mu \zeta \,, \qquad\qquad \delta_\zeta e_\mu{}^A = \tilde{D}_\mu \zeta^A - \zeta_B \tau_\mu T_0{}^{\{B,A\}} + \zeta e_{\mu B} T_0{}^{\{B,A\}} \,,$$

$$\delta_\zeta b_A = -\left(\tilde{D}_A \zeta\right) b_0 - \left(\tilde{D}_A \zeta^B\right) b_B - \zeta b^B T_{0\{A,B\}} - 2\zeta_B g_A{}^B + \zeta R_{0A}(D)$$
$$\qquad\qquad + \zeta^B R_{BA}(D) \,,$$

$$\delta_\zeta \omega^{(A,0|B)} = -\left(\tilde{D}^{(A|}\zeta\right) \omega_0{}^{0|B)} - \left(\tilde{D}^{(A|}\zeta^C\right) \omega_C{}^{0|B)} - \frac{1}{2}\zeta T_0{}^{\{B,C\}}\omega_C{}^{0A} - \frac{1}{2}\zeta T_0{}^{\{A,C\}}\omega_C{}^{0B}$$
$$\qquad\qquad + 2\zeta^{(A} f^{B)} - 2\zeta g^{(A,B)} + \zeta_C R^{C(A|}(J^{0|B)}) + \zeta R_0{}^{(A|}(J^{0|B)}) \,. \tag{3.38}$$

We will refrain from giving the translation rules of the dependent fields. They can be found using the same prescription, or by varying their explicit expressions under (3.38).

Finally, we mention the following useful formulae for later use:

$$e_A{}^\mu g_\mu{}^A = -\frac{1}{4(D-2)} R'_{AB}(J^{AB}) \,, \tag{3.39}$$

$$\tau^\mu f_\mu = -\frac{1}{2(D-1)} R'_{0A}(J^{0A}) + \frac{1}{4(D-1)(D-2)} R'_{AB}(J^{AB}) \,, \tag{3.40}$$

$$e_A{}^\mu g_\mu{}^A + \tau^\mu f_\mu = -\frac{1}{4(D-1)} \left[R'_{AB}(J^{AB}) + 2R'_{0A}(J^{0A}) \right] \,. \tag{3.41}$$

3.3 Carroll gravity from a compensating mechanism

In this subsection we will couple the conformal Carroll gravity theory that we constructed in the previous subsection to a massless electric and magnetic Carroll scalar field, respectively. Furthermore, we will show how, upon gauge-fixing, this leads to a special electric Carroll gravity theory and a magnetic Carroll gravity theory. We first discuss the electric case.

The electric case. Our starting point is the following Lagrangian for a massless electric Carroll scalar field:[13]

$$\mathcal{L}_{\text{electric scalar}} = -\frac{1}{2}\phi \partial_t \partial_t \phi \,. \tag{3.42}$$

As in the relativistic case, we will render this action invariant under local homogeneous conformal Carroll transformations (as well as, manifestly, under diffeomorphisms). We assume that the transformation rule of the scalar field ϕ under homogeneous conformal Carroll transformations is given by a dilatation with weight w as follows

$$\delta_{\text{hom}}\phi = w\,\lambda_D\,\phi\,. \tag{3.43}$$

As a first step we replace the time derivative $\partial_t\phi$ of ϕ by a covariant derivative $D_0\phi$:

$$\partial_t \phi \rightarrow D_0\phi \equiv \tau^\mu \left(\partial_\mu - w\,b_\mu\right)\phi\,. \tag{3.44}$$

Note that the second term in the above covariant derivative is related to the transformation rule (3.43) by replacing the dilatation parameter λ_D by the corresponding gauge field b_μ.

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[13] Note that this action is invariant under the Carrollian conformal isometries/BMS transformations (3.5).

In a second step we consider the transformation rule of the covariant derivative $D_0\phi$ under the homogeneous conformal Carroll transformations:

$$\delta_{\text{hom}}(D_0\phi) = (w+1)\lambda_D D_0\phi\,. \qquad (3.45)$$

This transformation rule shows that the second covariant time derivative $D_0 D_0\phi$ of ϕ is given by

$$D_0 D_0\phi \equiv \tau^\mu \left[\partial_\mu (D_0\phi) - (w+1) b_\mu D_0\phi\right]\,. \qquad (3.46)$$

Using the transformation rules (3.13) of τ^μ and the transformation rule (3.7) of b_μ we derive that

$$\delta_{\text{hom}}(D_0 D_0\phi) = (w+2)\lambda_D D_0 D_0\phi\,. \qquad (3.47)$$

We now consider the following Lagrangian describing the coupling of the electric Carroll scalar to conformal Carroll gauge fields:

$$\mathcal{L}_{\text{electric coupling}} = -\frac{1}{2} e\phi D_0 D_0\phi\,. \qquad (3.48)$$

Here, $e = \det(\tau_\mu, e_\mu{}^A)$ which transforms under dilatations as

$$\delta_{\text{hom}} e = -D\lambda_D e\,. \qquad (3.49)$$

Combining the transformation rules (3.43), (3.47) with (3.49) we find that the Lagrangian (3.48) transforms as

$$\delta_{\text{hom}}\mathcal{L}_{\text{electric coupling}} = -\frac{1}{2}[-D + 2w + 2]\lambda_D\, e\phi D_0 D_0\phi\,. \qquad (3.50)$$

This shows that the proposed Lagrangian is invariant under dilatations provided we take the scaling weight w to be

$$w = \frac{D-2}{2}\,. \qquad (3.51)$$

Having constructed an invariant Lagrangian we can now gauge-fix the dilatations by imposing the gauge condition $\phi = 1$ and obtain the following Lagrangian:

$$\mathcal{L}_{\text{electric Carroll}} = \frac{w^2}{2} e\, b_0^2\,. \qquad (3.52)$$

In deriving this, we partially integrated and used the identity

$$\partial_\mu (e\tau^\mu) = 2\, e\, \tau^\mu e_A{}^\nu \partial_{[\mu} e_{\nu]}{}^A = -(D-1)\, e\, b_0\,, \qquad (3.53)$$

where the last equality follows from the conventional constraint $R_{0A}(P^A) = 0$. Substituting the expression for the dependent dilatation gauge field given in (3.21d) we find a Lagrangian describing a particular form of electric Carroll gravity:[14]

$$\mathcal{L}_{\text{electric Carroll}} = \frac{w^2}{2(D-1)^2}\, e\, \mathcal{T}_{0A,}{}^A \mathcal{T}_{0B,}{}^B\,. \qquad (3.54)$$

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[14]We use here a calligraphic notation to indicate that $\mathcal{T}_{0A,}{}^A$ is an intrinsic torsion tensor with respect to the Carroll algebra but not with respect to the conformal Carroll algebra.

Usually, the literature considers a more general electric Carroll gravity theory [5], whose Lagrangian is given by

$$\mathcal{L} \sim e \left(K_{AB} K^{AB} - K_A{}^A K_B{}^B \right),$$ (3.55)

where K_{AB} are the spatial components of the extrinsic curvature tensor

$$K_{\mu\nu} \equiv \tau^\rho \partial_\rho h_{\mu\nu} + 2 \partial_{(\mu} \tau^\rho h_{\nu)\rho}, \qquad \text{with} \quad h_{\mu\nu} = e_\mu{}^A e_{\nu A}.$$ (3.56)

Note that $\tau^\mu K_{\mu\nu} = 0$, so that K_{AB} are the only non-vanishing flat components of the extrinsic curvature. From $\tau^\mu e_\mu{}^A = 0$, one can see that K_{AB} is related to the intrinsic torsion (with respect to the Carroll algebra) as follows:

$$K_{AB} = 2 \mathcal{T}_{0(A,B)} = 2 T_{0\{A,B\}} + \frac{2}{(D-1)} \delta_{AB} \mathcal{T}_{0C}{}^C.$$ (3.57)

The electric Carroll gravity theory of [5] can thus be written as

$$\mathcal{L} \sim e \left[T_0{}^{\{A,B\}} T_{0\{A,B\}} - \left(\frac{D-2}{D-1} \right) \mathcal{T}_{0A}{}^A \mathcal{T}_{0B}{}^B \right].$$ (3.58)

Comparing with (3.54), we see that the conformal approach only reproduces the second term in this Lagrangian. The first term

$$\mathcal{L}_{\text{conformal Carroll}} \sim e \, T_0{}^{\{A,B\}} T_{0\{A,B\}},$$ (3.59)

transforms homogeneously under local dilatations (but does not contain a dilatation gauge field). This "conformal electric Carroll gravity" Lagrangian thus constitutes an example of a Carroll gravity that cannot be obtained by starting from a dynamical matter Lagrangian in the conformal approach.

The magnetic case. In this case, the Lagrangian that we wish to make invariant under local homogeneous conformal Carroll transformations (and diffeomorphisms) is that of a massless magnetic Carroll scalar [18, 25]:

$$\mathcal{L}_{\text{magnetic scalar}} = \pi \partial_t \phi - \frac{1}{2} \partial^A \phi \partial_A \phi,$$ (3.60)

where π is an independent Lagrange multiplier field. This Lagrangian is invariant under the constant Carroll boost transformations

$$\delta\pi = \lambda^{0A} \partial_A \phi, \qquad\qquad \delta \partial_A \phi = \lambda^0{}_A \partial_t \phi,$$ (3.61)

and under the constant transformation

$$\delta\pi = \alpha \lambda_K \phi,$$ (3.62)

for any α.

As before, we assume that the scalar field ϕ transforms under local homogeneous conformal Carroll transformations with a dilatation with scaling weight w. We already explained in the electric case how the time derivative $\partial_t \phi$ of ϕ can be extended to the covariant derivative

$D_0\phi$ defined in eq. (3.44) and how this covariant derivative transforms according to eq. (3.45) under the homogeneous Carroll transformations. Similarly, we extend the spatial derivative $\partial_A \phi$ of ϕ to the following covariant derivative:

$$D_A \phi \equiv e_A{}^\mu \left(\partial_\mu - w\, b_\mu \right) \phi\,. \tag{3.63}$$

We find that under the homogeneous conformal Carroll transformations this covariant derivative transforms as follows:

$$\delta_{\text{hom}} \left(D_A \phi \right) = -\lambda_A{}^B\, D_B \phi + \lambda^0{}_A\, D_0 \phi + (w+1)\, \lambda_D\, D_A \phi - 2\, w\, \lambda_{KA}\, \phi\,. \tag{3.64}$$

We now propose the following Ansatz Lagrangian describing the coupling of a massless magnetic Carroll scalar to conformal Carroll gauge fields:

$$\mathcal{L}_{\text{Ansatz}} = e\pi D_0 \phi - \frac{1}{2} e D^A \phi D_A \phi\,. \tag{3.65}$$

This Ansatz Lagrangian is invariant under spatial rotations. Requiring it to be invariant under dilatations and Carroll boosts requires us to take

$$\delta\pi = \lambda^{0A} D_A \phi + \tfrac{1}{2} D \lambda_D \pi \qquad \text{and} \qquad w = \tfrac{1}{2}(D-2)\,. \tag{3.66}$$

We next consider the vector special conformal transformations. We find that the Ansatz lagrangian is not invariant under these transformations:

$$\delta\mathcal{L}_{\text{Ansatz}} = 2w e \lambda_K{}^A \phi D_A \phi\,. \tag{3.67}$$

To cancel this term, we perform a partial differentiation. Using the identity

$$\partial_\mu \left(e\, e_A{}^\mu \right) = 2\, e\, e_A{}^\mu e_B{}^\nu \partial_{[\mu} e_{\nu]}{}^B - 2\, e\, \tau^\mu e_A{}^\nu \partial_{[\mu} \tau_{\nu]}\,, \tag{3.68}$$

together with the conventional constraints

$$R_{AB}(P^B) = R_{0A}(H) = 0\,, \tag{3.69}$$

we find that

$$\delta\mathcal{L}_{\text{Ansatz}} = -w e\, \omega_0{}_,{}^{0A} \lambda_{KA} \phi^2 - w e\, e_A{}^\mu \left(D_\mu \lambda_K{}^A \right) \phi^2\,, \tag{3.70}$$

with $\qquad e_A{}^\mu \left(D_\mu \lambda_K{}^A \right) \equiv e_A{}^\mu \left(\partial_\mu \lambda_K{}^A + \omega_\mu{}^A{}_B \lambda_K{}^B - \lambda_K{}^A b_\mu \right)\,.$

Both terms can be canceled by adding two terms that involve the gauge fields of the scalar and vector special conformal transformation which, remarkably, does not break the boost symmetry. Doing this, we end up with the following modified Lagrangian describing the coupling of a massless magnetic Carroll scalar to conformal Carroll gauge fields:

$$\mathcal{L}_{\text{magnetic coupling}} = e\pi D_0 \phi - \frac{e}{2} D^A \phi D_A \phi + w e g_A{}^A \phi^2 + w e f_0 \phi^2\,. \tag{3.71}$$

This Lagrangian is invariant under spatial rotations, boosts, dilatations and the vector special conformal transformations.

We still need to verify that the Lagrangian (3.71) is also invariant under the scalar special conformal transformations. Allowing a K-variation $\delta_K \pi$ of the Lagrange multiplier π, we find

$$\delta_K \mathcal{L}_{\text{magnetic coupling}} = e \delta_K \pi D_0 \phi + w e \left(D_0 \lambda_K \right) \phi^2 , \qquad (3.72)$$

$$\text{with} \qquad D_0 \lambda_K \equiv \tau^\mu \left(\partial_\mu \lambda_K - \lambda_K b_\mu \right) .$$

We now partially differentiate the covariant time derivative in the second term. Using (3.53), we obtain invariance under local K-transformations provided we take

$$\delta_K \pi = (D - 2) \lambda_K \phi . \qquad (3.73)$$

This local K-transformation reduces to the global K-transformation (3.62) for $\alpha = D - 2$.

We now fix the dilatations by imposing $\phi = 1$. The invariance of the Lagrangian (3.71) under the vector special conformal transformations combined with the fact that all dependent gauge fields transform under these transformations due to their dependence on b_A guarantees that all b_A-dependent terms will cancel out. One is then left with the following Lagrangian for magnetic Carroll gravity:

$$\mathcal{L}_{\text{magnetic Carroll}} = \frac{1}{2} \frac{D-2}{D-1} e \left\{ \pi \, \mathcal{T}_{0A,}{}^A - \frac{1}{4} \left[R'_{AB}(J^{AB})(e,\tau) + 2 R'_{0A}(J^{0A})(e,\tau) \right] \right\} , \quad (3.74)$$

where $R'(e,\tau)$ are curvature tensors with respect to the (non-conformal) Carroll algebra. At first sight, this expression differs from the result for magnetic Carroll gravity given in [6] due to the presence of the first term. However, as pointed out in [24], the curvature terms of (3.74) contain many more Lagrange multipliers. These are the independent spin-connection components $\omega^{(A|,0|B)}$ that impose the geometric constraints:

$$\mathcal{T}_0{}^{(A,B)} = 0 . \qquad (3.75)$$

In particular, this implies that the first term in (3.74) can be absorbed by a redefinition of the Lagrange multiplier field $\omega_{A,}{}^{0A}$. This is supported by the fact that the Lagrangian (3.74) is invariant under the K-transformation:

$$\delta \pi = (D - 2) \lambda_K , \qquad\qquad \delta \omega_{A,}{}^{0A} = -2 \lambda_K , \qquad (3.76)$$

that can be gauge-fixed by setting $\pi = 0$. We thus end up with the following Lagrangian for magnetic Carroll gravity [6]:

$$\mathcal{L}_{\text{magnetic Carroll}} = -\frac{1}{8} \frac{D-2}{D-1} e \left[R'_{AB}(J^{AB})(e,\tau) + 2 R'_{0A}(J^{0A})(e,\tau) \right] . \qquad (3.77)$$

Alternatively, one could gauge-fix the K-transformations by imposing $\omega_{A,}{}^{0A} = 0$. In that case, due to compensating transformations, the Lagrange multiplier field π would behave as a special boost spin-connection component.

The upshot of the above paragraph is then that the Lagrangian (3.77) is the same as the magnetic Carroll gravity Lagrangian of [6] in the second order formulation (i.e., where the dependent spin-connection components $\omega_\mu{}^{AB}$, $\omega_0{}^{0A}$ and $\omega^{[A|,0|B]}$ have been replaced by their

explicit expressions (3.21a), (3.21b) and (3.21c)). In particular, the boost curvature term in the second order Lagrangian (3.77) still contains the independent spin-connection components $\omega^{(A|,0|B)}$. As in [6], upon writing out the contributions that involve these independent spin-connection components explicitly, one finds that the $\omega^{(A|,0|B)}$ appear linearly as Lagrange multipliers that impose the intrinsic torsion constraint (3.75).

4 The curious case of Galilei gravity

Similar to the Carroll case, there exist two Galilei gravity theories at the two-derivative level. The Vielbeine $(\tau_\mu, e_\mu{}^A)$ (with $A = 1, \cdots, D-1$) that appear in these theories transform under local spatial rotations (with parameters $\lambda^{AB} = -\lambda^{BA}$) and local Galilean boosts (with parameters λ^{0A}) as

$$\delta\tau_\mu = 0\,, \qquad\qquad \delta e_\mu{}^A = -\lambda^A{}_B e_\mu{}^B - \lambda^{0A}\tau_\mu\,. \qquad (4.1)$$

One also introduces dual Vielbeine $(\tau^\mu, e_A{}^\mu)$ via the relations (3.12). Their transformation rules under spatial rotations and Galilean boosts are given by

$$\delta\tau^\mu = \lambda^{0A} e_A{}^\mu\,, \qquad\qquad \delta e_A{}^\mu = -\lambda_A{}^B e_B{}^\mu\,. \qquad (4.2)$$

The rules (3.14) can then again be used to turn curved (μ) indices into flat ones (0 and A). The first Galilei gravity theory appears as the leading order term in a $1/c^2$-expansion of general relativity, similar to electric Carroll gravity. For this reason, we will refer to it as 'electric Galilei gravity'. Its action is proportional to[15]

$$\int \mathrm{d}^D x\, e\, \tau^{AB}\tau_{AB}\,, \qquad (4.3)$$

where $\tau_{\mu\nu} = 2\partial_{[\mu}\tau_{\nu]}$. Likewise, the second Galilei gravity theory corresponds to the subleading term in a $1/c^2$-expansion of general relativity. Since this is analogous to magnetic Carroll gravity, we will refer to it as 'magnetic' Galilei gravity. Its action is proportional to

$$\int \mathrm{d}^D x\, e\, \left[e_A{}^\mu e_B{}^\nu R_{\mu\nu}(J^{AB}) + A^{AB}\tau_{AB}\right]\,. \qquad (4.4)$$

Here, A_{AB} is a Lagrange multiplier field that imposes the constraint $\tau_{AB} = 0$ and

$$R_{\mu\nu}(J^{AB}) \equiv 2\partial_{[\mu}\omega_{\nu]}{}^{AB}(e,\tau) + 2\omega_{[\mu|}{}^A{}_C(e,\tau)\omega_{|\nu]}{}^{CB}(e,\tau)\,, \qquad \text{with} \qquad (4.5)$$

$$\omega_\mu{}^{AB}(e,\tau) \equiv 2e^{[A|\nu}\partial_{[\mu}e_{\nu]}{}^{|B]} - e_{\mu C}e^{A\nu}e^{B\rho}\partial_{[\nu}e_{\rho]}{}^C - \frac{2}{D-3}e_\mu{}^{[A}\tau_0{}^{B]}\,. \qquad (4.6)$$

Above, we saw that both electric and magnetic Carroll gravity can be obtained by applying the conformal approach to the conformal Carroll algebra and compensating electric and magnetic Carroll scalars. It is then natural to ask whether electric and magnetic Galilei gravity can be obtained in an analogous manner, starting from the Galilean conformal algebra,

---

[15]Note that this action resembles that of magnetic Galilean electromagnetism, upon interpreting the clock form τ_μ as the electromagnetic potential. For this reason, one could also sensibly refer to it as magnetic Galilei gravity. Here however, we refer to it as "electric" to stress that it appears as the leading order term in an expansion of general relativity, in analogy to electric Carroll gravity.

the Galilean counterpart of the conformal Carroll algebra (3.3). In this regard, it is useful to note that both electric and magnetic Galilei gravity are already invariant under local scale symmetries. These are however anisotropic rescalings (with parameter λ), given by:

for electric Galilei gravity: $\qquad \delta\tau_\mu = \frac{1}{3}(D-5)\lambda\tau_\mu\,, \qquad\qquad \delta e_\mu{}^A = -\lambda e_\mu{}^A\,,$ (4.7)

for magnetic Galilei gravity: $\qquad \delta\tau_\mu = (D-3)\lambda\tau_\mu\,, \qquad\qquad\quad \delta e_\mu{}^A = -\lambda e_\mu{}^A\,,$

$$\delta A_{AB} = -(D-3)\lambda A_{AB}\,. \tag{4.8}$$

Neither electric nor magnetic Galilei gravity are invariant under the isotropic rescalings

$$\delta\tau_\mu = -\lambda_D\tau_\mu\,, \qquad\qquad \delta e_\mu{}^A = -\lambda_D e_\mu{}^A\,, \tag{4.9}$$

that constitute the dilatation symmetry of the Galilean conformal algebra. The Lagrangian of electric Carroll gravity does however transform covariantly, i.e., without a derivative of the parameter, under such isotropic local dilatations:

$$\delta\left(e\tau^{AB}\tau_{AB}\right) = -(D-2)\lambda_D\, e\tau^{AB}\tau_{AB}\,. \tag{4.10}$$

This implies in particular that the electric Galilei action (4.3) can not be obtained by applying the conformal approach to the Galilean conformal algebra and a *dynamical* compensating scalar. Indeed, in order to render the electric Galilei action (4.3) invariant under the Galilean conformal algebra, and in particular the dilatations (4.9), it suffices to couple it to a non-dynamical compensating scalar ϕ with dilatation weight $w = (D-2)/2$ as follows

$$\int \mathrm{d}^D x\, e\, \phi^2 \tau^{AB}\tau_{AB}\,. \tag{4.11}$$

The electric Galilei gravity action (4.3) is then retrieved as usual by imposing the dilatation gauge fixing $\phi = 1$.

The magnetic Galilei action (4.4) does not simply rescale covariantly under the Galilean conformal dilatations (4.9). In particular, applying such an isotropic dilatation to the first term of (4.4), leads to $\partial_A\lambda_D\partial^A\lambda_D$ terms. While this suggests that it might be possible to obtain magnetic Galilei gravity from the conformal approach applied to the Galilean conformal algebra and a compensating magnetic Galilei scalar with action

$$\int \mathrm{d}^D x\left(-\frac{1}{2}\partial_A\phi\partial^A\phi\right)\,, \tag{4.12}$$

we will now show that this is not the case, by explicitly going through the procedure.

Like the conformal Carroll algebra, the Galilean conformal algebra is an Inönü-Wigner contraction of the relativistic conformal algebra (2.1). This contraction is obtained by splitting the index $\hat{A}$ into 0 and $A = 1, \cdots, D-1$, and subsequently redefining the generators of (2.1) with a contraction parameter c as follows

$$M_{0A} \to c\, G_A\,, \qquad M_{AB} \to J_{AB}\,, \qquad P_0 \to c^{-1}H\,, \qquad P_A \to P_A\,,$$
$$K_0 \to c\, K\,, \qquad K_A \to c^2\, K_A\,, \qquad D \to D\,, \tag{4.13}$$

and by finally taking the $c \to \infty$ limit. The non-zero commutation relations of the resulting Galilean conformal algebra are then given by (see e.g., [26])

$$
\begin{aligned}
[J_{AB}, J_{CD}] &= 4\delta_{[A[C} J_{D]B]}, & [J_{AB}, G_C] &= -2\delta_{C[A} G_{B]}, \\
[H, G_A] &= -P_A, & [K, G_A] &= -K_A, \\
[J_{AB}, P_C] &= -2\delta_{C[A} P_{B]}, & [J_{AB}, K_C] &= -2\delta_{C[A} K_{B]}, \\
[H, K] &= -2D, & [D, H] &= H, \\
[H, K_A] &= 2G_A, & [D, P_A] &= P_A, \\
[P_A, K] &= -2G_A, & [D, K] &= -K, \\
[D, K_A] &= -K_A.
\end{aligned}
\tag{4.14}
$$

In what follows we will be mostly interested in the homogeneous Galilean conformal transformations, that consist of spatial rotations J_{AB}, Galilean boosts G_A, dilatations D and the special conformal transformations K and K_A.

Introducing gauge fields τ_μ, $e_\mu{}^A$, $\omega_\mu{}^{AB}$, $\omega_\mu{}^{0A}$, b_μ, f_μ and $g_\mu{}^A$ (associated to H, P_A, J_{AB}, G_A, D, K and K_A respectively), the commutation relations (4.14) give the following homogeneous Galilean conformal transformation rules:[16]

$$
\begin{aligned}
\delta\tau_\mu &= -\lambda_D \tau_\mu, \\
\delta e_\mu{}^A &= -\lambda^A{}_B e_\mu{}^B - \lambda^{0A}\tau_\mu - \lambda_D e_\mu{}^A, \\
\delta\omega_\mu{}^{AB} &= \partial_\mu \lambda^{AB} - 2\lambda^{[A|C|}\omega_{\mu C}{}^{B]}, \\
\delta\omega_\mu{}^{0A} &= \partial_\mu \lambda^{0A} - \lambda^A{}_B \omega_\mu{}^{0B} - \lambda^{0B}\omega_{\mu B}{}^A + 2\lambda_K{}^A \tau_\mu - 2\lambda_K e_\mu{}^A, \\
\delta b_\mu &= \partial_\mu \lambda_D - 2\lambda_K \tau_\mu, \\
\delta f_\mu &= \partial_\mu \lambda_K + \lambda_D f_\mu - \lambda_K b_\mu, \\
\delta g_\mu{}^A &= \partial_\mu \lambda_K{}^A - \lambda_K{}^B \omega_{\mu B}{}^A - \lambda^A{}_B g_\mu{}^B + \lambda_K \omega_\mu{}^A - \lambda^{0A} f_\mu - \lambda_K{}^A b_\mu + \lambda_D g_\mu{}^A.
\end{aligned}
\tag{4.15}
$$

Here, λ^{AB}, λ^{0A}, λ_D, λ_K and $\lambda_K{}^A$ are the parameters corresponding to the generators J_{AB}, G_A, D, K and K_A respectively. Note that the dual Vielbeine τ^μ and $e_A{}^\mu$ then transform as follows:

$$
\delta\tau^\mu = \lambda^{0A} e_A{}^\mu + \lambda_D \tau^\mu, \qquad \delta e_A{}^\mu = -\lambda_A{}^B e_B{}^\mu + \lambda_D e_A{}^\mu.
\tag{4.16}
$$

The curvatures that are covariant with respect to the transformations (4.15) read

$$
\begin{aligned}
R_{\mu\nu}(H) &= 2\partial_{[\mu}\tau_{\nu]} + 2b_{[\mu}\tau_{\nu]}, \\
R_{\mu\nu}(P^A) &= 2\partial_{[\mu}e_{\nu]}{}^A + 2\omega_{[\mu}{}^{AB}e_{\nu]B} + 2\omega_{[\mu}{}^{0A}\tau_{\nu]} + 2b_{[\mu}e_{\nu]}{}^A, \\
R_{\mu\nu}(J^{AB}) &= 2\partial_{[\mu}\omega_{\nu]}{}^{AB} + 2\omega_{[\mu}{}^{[A|C|}\omega_{\nu]C}{}^{B]}, \\
R_{\mu\nu}(G^A) &= 2\partial_{[\mu}\omega_{\nu]}{}^{0A} + 2\omega_{[\mu}{}^{AB}\omega_{\nu]}{}^0{}_B - 4g_{[\mu}{}^A\tau_{\nu]} + 4f_{[\mu}e_{\nu]}{}^A, \\
R_{\mu\nu}(D) &= 2\partial_{[\mu}b_{\nu]} + 4f_{[\mu}\tau_{\nu]}, \\
R_{\mu\nu}(K) &= 2\partial_{[\mu}f_{\nu]} + 2f_{[\mu}b_{\nu]}, \\
R_{\mu\nu}(K^A) &= 2\partial_{[\mu}g_{\nu]}{}^A + 2g_{[\mu}{}^B\omega_{\nu]B}{}^A - 2f_{[\mu}\omega_{\nu]}{}^{0A} + 2g_{[\mu}{}^A b_{\nu]}.
\end{aligned}
\tag{4.17}
$$

---

[16]Strictly speaking, we should denote these transformations as $\delta_{\mathrm{hom}, 0}$ to be in line with the notation of the previous sections. Here however, we will not go through all details of the conformal approach, but only focus on those aspects that are relevant for the case at hand. For our purposes, it will not be necessary to distinguish between e.g., $\delta_{\mathrm{hom}, 0}$ and δ_{hom} and we will simply use δ everywhere.

Note that in contrast to the Carroll case, the conformal Galilean curvature $R_{\mu\nu}(J^{AB})$ does not contain any component of a gauge field other than the spatial spin-connection $\omega_\mu{}^{AB}$. This already indicates that the term $e e_A{}^\mu e_B{}^\nu R_{\mu\nu}(J^{AB})$ of magnetic Galilei gravity can not appear as a component of a gauge field that becomes dependent in applying the conformal approach to the Galilean conformal algebra.

In order to see what theory we do get from this approach, when applied to the flat space magnetic Galilei scalar Lagrangian (4.12), we promote the latter to a curved space Lagrangian that is invariant under local homogeneous Galilean conformal transformations. In doing this, we assume that the compensating scalar ϕ only transforms with weight w under local dilatations:

$$\delta\phi = w\lambda_D\phi \,. \tag{4.18}$$

The proper covariantization of $\partial_A\phi$ is then

$$D_A\phi \equiv (\partial_A - w b_A(e,\tau))\phi \,. \tag{4.19}$$

Here, we have written the (spatial part of the) dilatation gauge field as $b_A(e,\tau)$ to indicate that this is a dependent gauge field, as it can be solved from the conventional constraint

$$R_{0A}(H) \equiv 0 \,. \tag{4.20}$$

Explicitly, one finds the following expression for b_A from this constraint:

$$b_A(e,\tau) = \tau_{0A} \,. \tag{4.21}$$

Under homogeneous Galilean conformal transformations $b_A(e,\tau)$ transforms as

$$\delta b_A(e,\tau) = \partial_A\lambda_D + \lambda_D b_A(e,\tau) - \lambda_A{}^B b_B(e,\tau) + \lambda^{0B}\tau_{BA} \,. \tag{4.22}$$

As can be seen from (4.15), the last term does not follow from the Galilean conformal algebra. The presence of this extra boost transformation is a consequence of the fact that the constraint (4.20) is not invariant under the transformation rules (4.15), but rather one has

$$\delta R_{0A}(H) = \lambda^{0B} R_{BA}(H) = \lambda^{0B}\tau_{BA} \,. \tag{4.23}$$

Under homogeneous Galilean conformal transformations, $D_A\phi$ transforms as

$$\delta D_A\phi = -\lambda_A{}^B D_B\phi + (w+1)\lambda_D D_A\phi - w\lambda^{0B}\tau_{BA}\phi \,. \tag{4.24}$$

The appropriate generalization of (4.12) that we wish to consider is then given by:

$$\mathcal{L}_{\mathrm{magn},0} = -\frac{e}{2}D^A\phi D_A\phi \,, \tag{4.25}$$

where $e = \det(\tau_\mu, e_\mu{}^A)$. Since

$$\delta e = -D\lambda_D e \,, \tag{4.26}$$

one finds that the Lagrangian (4.25) is dilatation invariant if the weight w is chosen to satisfy

$$w = \frac{D}{2} - 1 \,. \tag{4.27}$$

Assuming this holds, one finds that under Galilean boosts

$$\delta\mathcal{L}_{\text{magn},\,0} = we\lambda^{0A}\left(D^B\phi\right)\tau_{AB}\phi = \frac{e}{2}w\lambda^{0A}D^B(\phi^2)\tau_{AB}\,. \tag{4.28}$$

Note that this situation is different from the magnetic Carroll case, where the non-invariance of the (boost-invariant) Ansatz Lagrangian (3.65) under vector special conformal transformations could be cured by adding a specific combination of special conformal gauge field components that is still invariant under Carroll boosts. Here, we no longer have invariance under boosts and in order to solve this, we introduce an extra Lagrange multiplier field $\pi_{AB} = \pi_{[AB]}$ that transforms as

$$\delta\pi_{AB} = -\frac{w}{2}\lambda^0{}_{[A}D_{B]}(\phi^2) - 2\lambda_{[A}{}^C\pi_{|C|B]} + (2w+1)\lambda_D\pi_{AB}\,, \tag{4.29}$$

and consider the Lagrangian

$$\mathcal{L}_{\text{magn}} = \mathcal{L}_{\text{magn},\,0} + e\pi^{AB}\tau_{AB} = -\frac{e}{2}D^A\phi D_A\phi + e\pi^{AB}\tau_{AB}\,. \tag{4.30}$$

Since e and τ_{AB} have dilatation weight $-D$ and 1 respectively, the last term is dilatation invariant by virtue of (4.27). The boost transformation of π_{AB} has moreover been chosen such that $\mathcal{L}_{\text{magn}}$ is boost invariant.

To pass to a Lagrangian that is invariant under only homogeneous Galilean transformations, we fix the superfluous conformal ones. The only one that acts non-trivially on the fields in $\mathcal{L}_{\text{magn}}$ is the dilatation symmetry and it is easily fixed by imposing the gauge

$$\phi = 1\,. \tag{4.31}$$

With this gauge fixing condition, the Lagrangian $\mathcal{L}_{\text{magn}}$ becomes[17]

$$\mathcal{L}_{\text{magn, fixed}} = -\frac{e}{2}w^2 b^A(e,\tau)b_A(e,\tau) + e\pi^{AB}\tau_{AB} = -\frac{e}{2}w^2\tau_0{}^A\tau_{0A} + e\pi^{AB}\tau_{AB}\,, \tag{4.32}$$

while the transformation rule (4.29) becomes

$$\delta'\pi_{AB} = w^2\lambda_{[A}b_{B]}(e,\tau) - 2\lambda_{[A}{}^C\pi_{|C|B]} = w^2\lambda_{[A}\tau_{|0|B]} - 2\lambda_{[A}{}^C\pi_{|C|B]}\,. \tag{4.33}$$

One thus sees that, even though the conformal approach, applied to (4.12), leads to the novel Galilean invariant action (4.32), it does not reproduce the magnetic Galilei gravity action (4.4). This concludes our discussion of the Galilean case.

---

[17]Note that this Lagrangian resembles that of electric Galilean electromagnetism, if one identifies the clock form τ_μ with the electromagnetic potential. For this reason, one could also call it an electric Lagrangian instead of a magnetic one. Here, we prefer to denote it as magnetic to contrast with the Lagrangian (4.3) and to emphasize that (4.32) is obtained from gauge-fixing a conformally invariant version of the Lagrangian for a magnetic Galilei scalar (4.12).

5 Discussion

In this paper we applied the conformal technique to a single Carroll scalar and showed its relation, upon gauge-fixing, with Carroll gravity. In particular, we showed that a massless electric scalar leads to a particular form of electric Carroll gravity while a massless magnetic scalar leads to magnetic Carroll gravity. On the way we showed that other gravitational invariants involving the intrinsic torsion tensor exist as well but they were not related to dynamical matter. In the Galilei case we found other examples where the conformal program did not work as in general relativity, in the sense that one can couple Galilean scalars to Galilean conformal gravity but not by following the rules of the conformal Galilei algebra in the same way that we did for the conformal Carroll algebra. This already shows from the mere fact that in the Galilei case we cannot construct dependent gauge fields for the Galilei special conformal transformations, a feature that played a crucial role in the Carroll calculation. Another peculiar feature of Galilei gravity invariants is that some of them are invariant under local an-isotropic dilatations.

Starting from the results of this work one may obtain non-trivial couplings of Carroll gravity to matter by replacing the single scalar we have been using in this work by a function of N scalars of which only one is used for gauge-fixing the local dilatations. One could also consider Carroll gravity coupled to fermions like in [27].

There are several generalizations that come to mind. First of all, the conformal technique has been heavily applied with great success in the supergravity literature [15]. One may extend the results of this paper to the supersymmetric case and construct first examples of a Carroll supergravity theory. Another natural generalization is to consider conformal Carroll algebras with an-isotropic dilatations such as the ones considered in [23]. These algebras lead to a different scaling weight for the longitudinal and transverse Vierbeine. This suggests a realization of conformal Carroll symmetries in terms of a scalar field theory with different powers of time and spatial derivatives. Such Lagrangians have appeared in the study of fractons [28–30] and spacetime subsymmetries [31, 32]. Yet another generalization is to consider extended objects instead of particles alone. For a p-brane, this requires a more general decomposition of the relativistic flat index into $p + 1$ longitudinal and $D - p - 1$ transverse directions. One could then also consider the geometry underlying Carroll strings [33, 34].

Finally, the Carroll and Galilei algebras are just two examples of a non-Lorentzian algebra. One could apply the conformal technique to other non-Lorentzian algebras as well. For the Schrödinger algebra, this has already been done in [35]. Another interesting option, with a view on fractons [29, 30], is to consider the Aristotelian algebra which is an intersection of the Carroll and Galilei algebra breaking the boost symmetry. It would be interesting to understand, for the case of non-Lorentzian algebras, when the conformal program works and when not. We hope to come back to some of these issues and generalizations soon.

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References

[1] L. Donnay and C. Marteau, *Carrollian Physics at the Black Hole Horizon*, *Class. Quant. Grav.* **36** (2019) 165002 [arXiv:1903.09654] [INSPIRE].

[2] C. Duval, G.W. Gibbons and P.A. Horvathy, *Conformal Carroll groups and BMS symmetry*, *Class. Quant. Grav.* **31** (2014) 092001 [arXiv:1402.5894] [INSPIRE].

[3] A. Bagchi, P. Dhivakar and S. Dutta, *Holography in flat spacetimes: the case for Carroll*, *JHEP* **08** (2024) 144 [arXiv:2311.11246] [INSPIRE].

[4] L. Donnay, *Celestial holography: An asymptotic symmetry perspective*, *Phys. Rept.* **1073** (2024) 1 [arXiv:2310.12922] [INSPIRE].

[5] M. Henneaux, *Geometry of Zero Signature Space-times*, *Bull. Soc. Math. Belg.* **31** (1979) 47 [INSPIRE].

[6] E. Bergshoeff et al., *Carroll versus Galilei Gravity*, *JHEP* **03** (2017) 165 [arXiv:1701.06156] [INSPIRE].

[7] G. Dautcourt, *On the ultrarelativistic limit of general relativity*, *Acta Phys. Polon. B* **29** (1998) 1047 [gr-qc/9801093] [INSPIRE].

[8] J. Gomis, A. Kleinschmidt, J. Palmkvist and P. Salgado-Rebolledo, *Newton-Hooke/Carrollian expansions of (A)dS and Chern-Simons gravity*, *JHEP* **02** (2020) 009 [arXiv:1912.07564] [INSPIRE].

[9] D. Grumiller, J. Hartong, S. Prohazka and J. Salzer, *Limits of JT gravity*, *JHEP* **02** (2021) 134 [arXiv:2011.13870] [INSPIRE].

[10] J. Gomis, D. Hidalgo and P. Salgado-Rebolledo, *Non-relativistic and Carrollian limits of Jackiw-Teitelboim gravity*, *JHEP* **05** (2021) 162 [arXiv:2011.15053] [INSPIRE].

[11] D. Hansen, N.A. Obers, G. Oling and B.T. Søgaard, *Carroll Expansion of General Relativity*, *SciPost Phys.* **13** (2022) 055 [arXiv:2112.12684] [INSPIRE].

[12] F. Ecker et al., *Carroll black holes*, *SciPost Phys.* **15** (2023) 245 [arXiv:2308.10947] [INSPIRE].

[13] L. Ciambelli and D. Grumiller, *Carroll geodesics*, *Eur. Phys. J. C* **84** (2024) 933 [arXiv:2311.04112] [INSPIRE].

[14] S. Baiguera, G. Oling, W. Sybesma and B.T. Søgaard, *Conformal Carroll scalars with boosts*, *SciPost Phys.* **14** (2023) 086 [arXiv:2207.03468] [INSPIRE].

[15] D.Z. Freedman and A. Van Proeyen, *Supergravity*, Cambridge Univ. Press, Cambridge, U.K. (2012) [DOI:10.1017/CBO9781139026833] [INSPIRE].

[16] A.H. Chamseddine and P.C. West, *Supergravity as a Gauge Theory of Supersymmetry*, *Nucl. Phys. B* **129** (1977) 39 [INSPIRE].

[17] M. Kaku, P.K. Townsend and P. van Nieuwenhuizen, *Gauge Theory of the Conformal and Superconformal Group*, *Phys. Lett. B* **69** (1977) 304 [INSPIRE].

[18] M. Henneaux and P. Salgado-Rebolledo, *Carroll contractions of Lorentz-invariant theories*, *JHEP* **11** (2021) 180 [arXiv:2109.06708] [INSPIRE].

[19] A. Pérez, *Asymptotic symmetries in Carrollian theories of gravity*, *JHEP* **12** (2021) 173 [arXiv:2110.15834] [INSPIRE].

[20] A. Pérez, *Asymptotic symmetries in Carrollian theories of gravity with a negative cosmological constant*, *JHEP* **09** (2022) 044 [arXiv:2202.08768] [INSPIRE].

[21] E. Bergshoeff, J. Figueroa-O'Farrill and J. Gomis, *A non-lorentzian primer*, *SciPost Phys. Lect. Notes* **69** (2023) 1 [arXiv:2206.12177] [INSPIRE].

[22] K. Nguyen and P. West, *Carrollian Conformal Fields and Flat Holography*, *Universe* **9** (2023) 385 [arXiv:2305.02884] [INSPIRE].

[23] H. Afshar, X. Bekaert and M. Najafizadeh, *Classification of conformal carroll algebras*, *JHEP* **12** (2024) 148 [arXiv:2409.19953] [INSPIRE].

[24] E. Bergshoeff et al., *p-brane Galilean and Carrollian geometries and gravities*, *J. Phys. A* **57** (2024) 245205 [arXiv:2308.12852] [INSPIRE].

[25] J. de Boer et al., *Carroll Symmetry, Dark Energy and Inflation*, *Front. in Phys.* **10** (2022) 810405 [arXiv:2110.02319] [INSPIRE].

[26] A. Bagchi and R. Gopakumar, *Galilean Conformal Algebras and AdS/CFT*, *JHEP* **07** (2009) 037 [arXiv:0902.1385] [INSPIRE].

[27] E.A. Bergshoeff et al., *Carroll fermions*, *SciPost Phys.* **16** (2024) 153 [arXiv:2312.00745] [INSPIRE].

[28] M. Pretko, X. Chen and Y. You, *Fracton Phases of Matter*, *Int. J. Mod. Phys. A* **35** (2020) 2030003 [arXiv:2001.01722] [INSPIRE].

[29] L. Bidussi et al., *Fractons, dipole symmetries and curved spacetime*, *SciPost Phys.* **12** (2022) 205 [arXiv:2111.03668] [INSPIRE].

[30] J. Hartong et al., *Fractons on curved spacetime in $2+1$ dimensions*, *SciPost Phys.* **18** (2025) 022 [arXiv:2409.04525] [INSPIRE].

[31] S.A. Baig et al., *Spacetime Subsystem Symmetries*, arXiv:2303.15590 [INSPIRE].

[32] O. Kasikci, M. Ozkan and Y. Pang, *Carrollian origin of spacetime subsystem symmetry*, *Phys. Rev. D* **108** (2023) 045020 [arXiv:2304.11331] [INSPIRE].

[33] B. Cardona, J. Gomis and J.M. Pons, *Dynamics of Carroll Strings*, *JHEP* **07** (2016) 050 [arXiv:1605.05483] [INSPIRE].

[34] M. Harksen, D. Hidalgo, W. Sybesma and L. Thorlacius, *Carroll strings with an extended symmetry algebra*, *JHEP* **05** (2024) 206 [arXiv:2403.01984] [INSPIRE].

[35] H.R. Afshar et al., *A Schrödinger approach to Newton-Cartan and Hořava-Lifshitz gravities*, *JHEP* **04** (2016) 145 [arXiv:1512.06277] [INSPIRE].