Journal of Physics: Conference Series **3152** (2025) 012033 doi:10.1088/1742-6596/3152/1/012033

# Poisson Q-supermanifolds and differential Poisson sigma-models

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**Abstract.** We perform a systematic construction of supersymmetric Poisson sigma-models with an additional global supersymmetry in the geometric approach of Poisson Q-supermanifolds. Our analysis unifies Jackiw-Teitelboim supergravity and differential Poisson sigma-models exhibiting de Rham supersymmetry and reveals the existence of new models in the same family, such as a model with Lichnerowicz–Poisson supersymmetry, one based on a symmetric bilinear form and a class of models with super-Poisson structure of inhomogeneous degree. We emphasize the relation of the coadjoint representation of Lie algebroids to the global supersymmetry of these sigma-models.

## 1 Introduction and motivation

The Poisson sigma-model (PSM) is a 2D topological field theory that found many applications in theoretical and mathematical physics in the past three decades [1, 2]. Notably, it packages various 2D gravity models in a single framework [3], its BV quantization reproduces the Kontsevich formula for the noncommutative $\star$-product in the deformation quantization of Poisson manifolds [4], and at the same time it is the simplest example of the geometric approach to the BV formalism introduced in [5] (see also [6]). Its relation to the BV formalism may be traced in the fact that the PSM is arguably the simplest gauge theory with an open gauge algebra, namely one that closes only on the stationary surface, which was one of the original motivations to introduce this formalism [7].

Supersymmetric PSMs were also constructed and their inclusion of 2D $\mathcal{N} = 1$ dilaton supergravity was emphasized [1]. More recently, a different type of supersymmetric PSM based on a differential Poisson algebra and exhibiting an additional global supersymmetry was proposed as a model that could account for the deformation quantization of the algebra of differential forms on a Poisson manifold [8], see also [9].

In this contribution, which is based on [10], we discuss a systematic approach to explain the structure and the relation between the aforementioned supersymmetric PSMs and to place these and other related models under the same geometrical roof. We use Poisson Q-supermanifolds to determine consistency conditions that allow for the construction of such models with different types of global supersymmetry. This is closely related to the question of when is a given Lie algebroid structure compatible with a Poisson bracket of given degree.[1] In the case of degree 0 super-Poisson bracket, solutions to these consistency

[1]Recall that for a Poisson bracket of degree $-1$, a compatible Lie algebroid gives rise to a Lie bialgebroid. Here we focus on Poisson brackets of even degree instead. We thank D. Roytenberg for discussions on this point.

Journal of Physics: Conference Series **3152** (2025) 012033 doi:10.1088/1742-6596/3152/1/012033

conditions single out essentially two cases, giving rise to PSMs with global supersymmetry controlled by the anchor map of a Lie algebroid. In one case, we recover the model constructed in [8] (and a class of accompanying models corresponding to Lie algebroids with invertible anchor map), whereas in a second case we find a new model where the global supersymmetry is induced by the Poisson structure on the body of the target supermanifold through the Lichnerowicz–Poisson differential. This "Poisson supersymmetry" is a feature that to our knowledge has not been encountered before in any field theory. Cases corresponding to degree $-2$ and also higher-order super-Poisson brackets are also briefly discussed. We also highlight that structural compatibility corresponds to the notion of Q-bundle [11] and the closely related notion of representations up to homotopy for Lie algebroids [12]. In particular, it turns out that the different types of global supersymmetry exhibited in the models we discuss are tied together as the coadjoint representation of different Lie algebroids, which is indeed a representation up to homotopy.

## 2 General structure of Poisson Q-supermanifolds

We consider supermanifolds $\mathcal{M}$ with super-Poisson structure $\mathcal{P}$, hence Poisson supermanifolds $(\mathcal{M},\mathcal{P})$. Supermanifolds carry a $\mathbb{Z}_2$ grading and they can be described as parity-shifted vector bundles $\Pi E$, with $E$ a smooth vector bundle over a smooth manifold $M$ (the body of $\mathcal{M}$) [13]. A local coordinate system on $\mathcal{M}$ is $(x^\alpha)=(x^\mu,\theta^a)$, where $x^\mu$ are even, bosonic coordinates on the body and $\theta^a$ are odd, fermionic coordinates on the fiber of $E$. Their $\mathbb{Z}_2$-degree is 0 and 1, respectively. A Poisson supermanifold is endowed with a super-Poisson bracket such that its algebra of functions is a super-Poisson algebra. The bracket is

$$\{x^\alpha,x^\beta\}=\mathcal{P}^{\alpha\beta}\,, \tag{1}$$

it is graded skew-symmetric, and it satisfies the graded Jacobi identity

$$(-1)^{\gamma\alpha}\,\mathcal{P}^{\alpha\delta}\,\partial_\delta\mathcal{P}^{\beta\gamma}+(-1)^{\alpha\beta}\,\mathcal{P}^{\beta\delta}\,\partial_\delta\mathcal{P}^{\gamma\alpha}+(-1)^{\beta\gamma}\,\mathcal{P}^{\gamma\delta}\,\partial_\delta\mathcal{P}^{\alpha\beta}=0\,. \tag{2}$$

An alternative description of a Poisson supermanifold uses a homological vector field (HVF) [14]. Consider the shifted cotangent bundle $T^*[1]\mathcal{M}=T^*[1]\Pi E$, where the shift is with respect to $\mathbb{Z}$-degree. This is a NQ-supermanifold, where the N-structure refers to non-negative $\mathbb{Z}$-grading and the Q-structure to the existence of a $\mathbb{Z}$-degree 1 HVF, say $\mathcal{Q}$, namely such that

$$\mathcal{Q}^2=0\,. \tag{3}$$

This HVF is canonical and can be obtained as follows: The super-Poisson bracket $\mathcal{P}$ corresponds to a bivector field $\Pi_{\mathcal{P}}$ on $\mathcal{M}$, whose self-commutator vanishes; the adjoint operator $[\Pi_{\mathcal{P}},\,.]$ is the given HVF.

The NQ-manifold $T^*[1]\mathcal{M}$ carries a canonical symplectic structure $\omega$ of $\mathbb{Z}$-degree 1. Compatibility of $\mathcal{Q}$ with this graded symplectic form, makes Eq. (3) equivalent to (2); there is a one-to-one correspondence between Poisson supermanifolds and symplectic differential graded supermanifolds of degree 1 (Lie superalgebroids), carrying the results of [14,15] over to the category of supermanifolds. We will analyze further the graded Jacobi identity in Section 3. The homological vector field $\mathcal{Q}$ will generate the local gauge symmetries (including local supersymmetries) for supersymmetric PSMs in Section 4.

There is also a different structure on a supermanifold $\mathcal{M}$ that will be of interest for our purposes. Independently of the super-Poisson structure, we may consider that $\mathcal{M}$ is itself a Q-supermanifold. This means that there exists a HVF $q_S$ of $\mathbb{Z}_2$-degree 1 on it, namely

$$q_S^2=0\,. \tag{4}$$

We emphasize that this is associated to $\mathbb{Z}_2$ grading. Only when the $\mathbb{Z}_2$ grading is compatible with the $\mathbb{Z}$ grading this structure corresponds to Lie (super)algebroids. Without this assumption, the structure is defined only in the graded geometric world. We will also analyze the corresponding conditions in Section 3. As in the case of the super-Poisson structure, one may alternatively think in terms of the shifted cotangent bundle $T^*[1]\mathcal{M}$. In that picture, the HVF $q_S$ lifts to a HVF $\mathcal{Q}_S$ on $T^*[1]\mathcal{M}$ which is compatible with the graded symplectic form $\omega$. With this structure, $T^*[1]\mathcal{M}$ is a prototypical example of a differential super-vector bundle, one for which both the total space and the base are equipped with a HVF, the analogon of Q-bundles [11] in the category of supermanifolds.

So far we have introduced the concepts of Poisson supermanifold $(\mathcal{M},\mathcal{P})$ and Q-supermanifold $(\mathcal{M},q_S)$. For the purposes of this work, we combine them by asking for compatibility between $q_S$ and the bivector $\mathcal{P}$. This amounts to the condition

$$q_S\{f,g\}_{\mathcal{P}}=\{q_Sf,g\}_{\mathcal{P}}+(-1)^{|f|}\,\{f,q_Sg\}_{\mathcal{P}}\,,\qquad f,g\in\mathcal{C}^\infty(\mathcal{M})\,. \tag{5}$$

As with the previous conditions, we will provide a more detailed analysis in Section 3. In Section 4 we will argue that when the conditions involving $q_S$ are solvable for a given super-Poisson structure, we obtain global $\mathcal{N}=1$ supersymmetry transformations that leave the action functional of supersymmetric PSMs invariant and whose algebra closes. The compatibility condition may be expressed in terms of the homological vector fields $\mathcal{Q}$ and $\mathcal{Q}_S$ too, and it expresses their (graded) commutation, namely

$$\{\mathcal{Q},\mathcal{Q}_S\}=0\,. \tag{6}$$

To summarize, we may equivalently consider $\mathbb{Z}_2$-graded Poisson Q-supermanifolds $(\mathcal{M},\mathcal{P},q_S)$ or $\mathbb{Z}\times\mathbb{Z}_2$-graded symplectic NQ-supermanifolds $(T^*[1]\mathcal{M},\mathcal{Q},\mathcal{Q}_S,\omega)$ with all compatibility conditions satisfied. This will be the target space of supersymmetric PSMs.

## 3 Analysis of the geometrical conditions

We now look at the conditions obtained from the structural compatibility of Poisson Q-supermanifolds.[2] For clarity, we will work in local coordinates and subsequently define any geometrical structures necessary. First, we express the super-Poisson brackets as

$$\{x^\mu,x^\nu\}=\mathcal{P}^{\mu\nu}(x,\theta^2)\,,\qquad \{x^\mu,\theta^a\}=\theta^b\,\mathcal{P}^{\mu a}_{b}(x,\theta^2)\,,\qquad \{\theta^a,\theta^b\}=\mathcal{P}^{ab}(x,\theta^2)\,, \tag{7}$$

where the notation means that in general the functions appearing on the right-hand side depend on the even coordinates as well as on even powers of the odd coordinates.[3] Graded skew-symmetry of the super-Poisson bracket translates into the symmetry properties

$$\mathcal{P}^{\mu\nu}=-\mathcal{P}^{\nu\mu}\,,\qquad \mathcal{P}^{\mu a}_{b}=-\mathcal{P}^{a\mu}_{b}\,,\qquad \mathcal{P}^{ab}=\mathcal{P}^{ba}\,. \tag{8}$$

The graded Jacobi identity (2) gives four independent conditions, which read

$$0=\mathcal{P}^{[\mu|\kappa}\partial_\kappa\mathcal{P}^{\nu\lambda]}+\mathcal{P}^{[\mu|a}\partial_a\mathcal{P}^{\nu\lambda]}\,, \tag{9a}$$

$$0=-\mathcal{P}^{\kappa a}\partial_\kappa\mathcal{P}^{\mu\nu}+\mathcal{P}^{ab}\partial_b\mathcal{P}^{\mu\nu}+2\,\mathcal{P}^{[\mu|\kappa}\partial_\kappa\mathcal{P}^{\nu]a}+2\,\mathcal{P}^{[\mu|b}\partial_b\mathcal{P}^{\nu]a}\,, \tag{9b}$$

$$0=\mathcal{P}^{\mu\nu}\partial_\nu\mathcal{P}^{ab}+\mathcal{P}^{\mu c}\partial_c\mathcal{P}^{ab}+2\,\mathcal{P}^{\nu(a}\partial_\nu\mathcal{P}^{\mu|b)}-2\,\mathcal{P}^{c(a}\partial_c\mathcal{P}^{\mu|b)}\,, \tag{9c}$$

$$0=-\mathcal{P}^{\mu(a}\partial_\mu\mathcal{P}^{bc)}+\mathcal{P}^{d(a}\partial_d\mathcal{P}^{bc)}\,, \tag{9d}$$

where $\partial_a\equiv\partial/\partial\theta^a$. To analyze the condition (4) of $\mathcal{M}$ being a Q-supermanifold, we may express the vector field $q_S$ in terms of the local coordinates $(x^\alpha)$ as

$$q_S=\rho^\mu(x,\theta^2)\,\tfrac{\partial}{\partial x^\mu}+V^a(x,\theta^2)\,\tfrac{\partial}{\partial\theta^a}\,, \tag{10}$$

where $\rho^\mu=\theta^a\,\rho_a{}^\mu$. Then, condition (4) imposes

$$0=\theta^a\theta^b\,\rho_{[a}{}^\nu\partial_\nu\rho_{b]}{}^\mu+V^c\,\rho_c{}^\mu\,,\qquad 0=\theta^b\rho_b{}^\mu\,\partial_\mu V^a+V^b\partial_b V^a\,, \tag{11}$$

where the first, respectively second, equation contains only terms which are even, respectively odd, in the odd coordinate $\theta$. The final set of conditions comes from the compatibility of the super-Poisson bracket with the homological vector field, Eq. (5), whose expanded form is

$$0=\rho^\lambda\,\partial_\lambda\mathcal{P}^{\mu\nu}+V^a\,\partial_a\mathcal{P}^{\mu\nu}-2\,\partial_\lambda\rho^{[\mu|}\,\mathcal{P}^{\lambda|\nu]}+2\,\partial_a\rho^{[\mu}\,\mathcal{P}^{\nu]a}\,, \tag{12a}$$

$$0=\rho^\lambda\,\partial_\lambda\mathcal{P}^{\mu a}+V^b\,\partial_b\mathcal{P}^{\mu a}-\partial_\lambda\rho^\mu\,\mathcal{P}^{\lambda a}-\partial_\lambda V^a\,\mathcal{P}^{\mu\lambda}+\partial_b V^a\,\mathcal{P}^{\mu b}-\partial_b\rho^\mu\,\mathcal{P}^{ba}\,, \tag{12b}$$

$$0=\rho^\lambda\,\partial_\lambda\mathcal{P}^{ab}+V^c\,\partial_c\mathcal{P}^{ab}-2\,\partial_\lambda V^{(a|}\,\mathcal{P}^{\lambda|b)}+2\,\partial_c V^{(a}\,\mathcal{P}^{b)c}\,. \tag{12c}$$

The system of conditions (9), (11) and (12) is what needs to be solved so that we find a Poisson Q-supermanifold and we have the target space data of a supersymmetric PSM with gauge (super)symmetry generated by $\mathcal{P}$ and global supersymmetry generated by $q_S$.

[2] In [10] the analysis was performed with respect to the $\mathbb{Z}\times\mathbb{Z}_2$ picture, which is somewhat more tedious.

[3] Without loss of generality, we expanded the mixed component $\mathcal{P}^{\mu a}=\mathcal{P}^{\mu a}_{b}\theta^b$ to isolate its even part.

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Before presenting solutions, it is instructive to discuss some geometrical aspects, understanding that $\mathcal{M} \simeq \Pi E$. A key observation is that the leading order component of $\mathcal{P}_b^{\mu a}$ transforms as a contravariant connection on $E^*$, namely a covariant derivative of sections of $E^*$ with respect to 1-forms on the body $M$. We refer to such connections as "$T^*M$-on-$E^*$", and accordingly for linear connections with respect to other vector bundles. This connection is built-in the geometry of the super-Poisson structure and therefore does not constitute additional data. To facilitate the construction of explicit examples and preserve target space covariance when the body $M$ is a general smooth manifold (not necessarily globally flat), it is instructive to split the tangent bundle of the supermanifold $\mathcal{M}$, which is not a vector bundle over the body $M$. One advantage of this is that this results in a vector bundle over $M$ and therefore all components in the expansion of the super-Poisson structure become tensors. The splitting of $T\mathcal{M}$ amounts to choosing a $TM$-on-$E$ connection $\nabla$ with coefficients $\Gamma^b_{\mu a}$, which results in the (noncanonical) isomorphism

$$T\mathcal{M} \overset{\nabla}{\cong} TM \oplus \Pi E\,. \tag{13}$$

We define the tensorial components

$$\mathcal{P}^{\mu a}_{\triangledown} = \mathcal{P}^{\mu a} + \mathcal{P}^{\mu\nu}\,\Gamma^a_{\nu b}\,\theta^b\,, \qquad \mathcal{P}^{ab}_{\triangledown} = \mathcal{P}^{ab} - 2\,\mathcal{P}^{\mu(a}\,\Gamma^{b)}_{\mu c}\,\theta^c + \mathcal{P}^{\mu\nu}\,\Gamma^a_{\mu c}\Gamma^b_{\nu d}\,\theta^c\theta^d\,. \tag{14}$$

The coefficients of these components in powers of $\theta$ are sections of $\wedge E^*$ tensored with $E$ or $TM$. The Jacobi identity takes a form similar to (9), see [10]. The compatibility conditions between $q_S$ and the super-Poisson structure in terms of the covariant components of the latter, take the form

$$0 = \mathrm{d}_{\overline{\triangledown}}\mathcal{P}^{\mu\nu} + 2\,\partial_a\rho^{[\mu}\,\mathcal{P}^{\nu]a}_{\triangledown}\,, \qquad 0 = \mathrm{d}_{\overline{\triangledown}}\mathcal{P}^{\mu a}_{\triangledown} - \partial_b\rho^{\mu}\,\mathcal{P}^{ba}_{\triangledown} - \mathcal{P}^{\mu\nu}S^{\triangledown a}_{\nu}\,, \qquad 0 = \mathrm{d}_{\overline{\triangledown}}\mathcal{P}^{ab}_{\triangledown} - 2\,\mathcal{P}^{\mu(a}_{\triangledown}S^{\triangledown b)}_{\mu}\,, \tag{15}$$

where we introduced the operator

$$\mathrm{d}_{\overline{\triangledown}}T^{\mu\dots a\dots}(x,\theta) := \big(\rho^\nu\partial_\nu + V^b\partial_b\big)T^{\mu\dots a\dots}(x,\theta) + \overline{\Gamma}^{\mu}{}_{\nu}\,T^{\nu\dots a\dots}(x,\theta) + \overline{\Gamma}^{a}{}_{b}\,T^{\mu\dots b\dots}(x,\theta) + \dots\,, \tag{16}$$

where the dots denote similar terms for additional indices left implicit, and we defined

$$\overline{\Gamma}^{\nu}{}_{\mu} := -\partial_\mu\rho^\nu + \partial_a\rho^\nu\Gamma^a_{\mu b}\theta^b\,, \qquad \overline{\Gamma}^{a}{}_{b} := \partial_b V^a + \partial_b\rho^\mu\Gamma^a_{\mu c}\theta^c\,, \qquad S^{\triangledown a}_{\mu} := \nabla_\mu V^a_{\triangledown} + \rho^\nu R_{\nu\mu}{}^a{}_b\theta^b\,, \tag{17}$$

where $V^a_{\triangledown} := V^a + \rho^\mu\Gamma^a_{\mu b}\theta^b$ and $R_{\mu\nu}{}^a{}_b$ are the components of the curvature of $\nabla$. These structures acquire an interesting geometric meaning when $q_S$ is simply the Chevalley–Eilenberg differential associated with a Lie algebroid on $E$,

$$\rho^\mu = \theta^a t_a{}^\mu(x)\,, \qquad V^a = -\tfrac{1}{2}\theta^b\theta^c C_{bc}{}^a(x)\,. \tag{18}$$

This is the case when the $\mathbb{Z}_2$-degree is compatible with the $\mathbb{Z}$-degree, so that the HVF does not involve arbitrary powers of the odd coordinate but only the ones that correspond to the anchor map $t = (t_a{}^\mu)$ and the Lie bracket of a Lie algebroid with structure functions $C_{ab}{}^c(x)$. In that case, $\overline{\Gamma}$ become the coefficients of the basic $E$-connection on the adjoint complex of the Lie algebroid [12] and $S^{\nabla}$ is identified with the basic curvature of the connection $\nabla$, entailing a relationship with the (co)adjoint representation of the Lie algebroid, which is a representation up to homotopy. Consequently, the compatibility conditions of $\mathcal{P}$ with $q_S$ can be written as

$$0 = \mathrm{d}_{\overline{\triangledown}}\mathcal{P}^{\mu\nu} + 2\,t_a{}^{[\mu}\,\mathcal{P}^{\nu]a}_{\triangledown}\,, \quad 0 = \mathrm{d}_{\overline{\triangledown}}\mathcal{P}^{\mu a}_{\triangledown} - \partial_b t^{\mu}\,\mathcal{P}^{ba}_{\triangledown} + \tfrac{1}{2}\mathcal{P}^{\mu\nu}\theta^b\theta^c S_{bc\nu}{}^a\,, \quad 0 = \mathrm{d}_{\overline{\triangledown}}\mathcal{P}^{ab}_{\triangledown} + \mathcal{P}^{\mu(a}_{\triangledown}\theta^c\theta^d S_{cd\mu}{}^{b)}\,, \tag{19}$$

where $\mathrm{d}_{\overline{\triangledown}}$ denotes the Lie algebroid differential defined by the basic connection, acting on the covariant components of $\mathcal{P}$, which are cochains in the complex $\wedge E^* \otimes \wedge^2(TM \oplus E)$. Focusing on such HVFs $q_S$ from now on, the goal is to find solutions of the Jacobi identity together with the compatibility conditions (19) and construct sigma-models with $\mathcal{P}$-induced gauge symmetries and $q_S$-induced global supersymmetries.

## 4 Supersymmetric Poisson sigma-models with global supersymmetry

Given the geometrical data of a Poisson Q-supermanifold, one can associate to them a 2D supersymmetric topological sigma model, in the same fashion as an ordinary Poisson manifold is the target space of the PSM. In 1993, Ikeda proposed a nonlinear super-gauge theory that includes $\mathcal{N} = 1$ dilaton supergravity [1]. In the general approach that we develop here, this corresponds simply to a Poisson supermanifold, namely to the case where $q_S = 0$ (no additional global supersymmetry). In [8], the authors constructed a "differential Poisson sigma model", which in our approach corresponds to the split

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tangent bundle case, namely to $E = TM$ – so that $T\Pi TM$ splits into $TM \oplus \Pi TM$ – together with the compatible tangent Lie algebroid structure, for which $q_S$ is the de Rham differential. Both cases are solutions of the general conditions, and we will see below that there is more in this solution space than just these two examples.

A brief description of how to construct the general supersymmetric PSM follows. We consider the 2D spacetime $(T[1]\Sigma, \mathrm{d})$, $\Sigma$ being a Riemann surface and d the de Rham differential associated to the source data, and target space data $(T^*[1]\Pi E, \mathcal{Q}, \mathcal{Q}_S, \omega)$ corresponding to a Poisson Q-supermanifold. The latter is intrinsically defined through the data $(\mathcal{M} = \Pi E, \mathcal{P}, q_S)$, as described in Section 2. Since $\mathcal{Q}$ is compatible with the symplectic form $\omega$, it admits a Hamiltonian function of $\mathbb{Z}$-degree 2 and even parity, defined via

$$\mathcal{Q} = \{\mathcal{H}, -\}\,, \tag{20}$$

in terms of the odd (degree $-1$) Poisson bracket $\{-,-\}$ associated to the symplectic form $\omega$. This Hamiltonian function takes the form

$$\mathcal{H} = \tfrac{1}{2}\,(-1)^{\alpha(\beta+1)}\,\mathcal{P}^{\alpha\beta}\,p_\alpha\,p_\beta = \tfrac{1}{2}\,\mathcal{P}^{\mu\nu}\,a_\mu a_\nu + \mathcal{P}^{\mu a}\,a_\mu\chi_a + \tfrac{1}{2}\,\mathcal{P}^{ab}\,\chi_a\chi_b\,, \tag{21}$$

where $p_\alpha = (a_\mu, \chi_a)$ are the conjugate momenta to $x^\alpha = (x^\mu, \theta^a)$ with respect to the symplectic structure. The $\mathbb{Z}\times\mathbb{Z}_2$-degree of $a_\mu$ and $\chi_a$ is $(1,0)$ and $(1,1)$, respectively. The symplectic form in Darboux coordinates and the Hamiltonian define a sigma-model with action

$$S[X^\alpha, P_\alpha] = \int \left(P_\alpha \wedge \mathrm{d}X^\alpha + \tfrac{1}{2}(-1)^{\alpha(\beta+1)}\,\mathcal{P}^{\alpha\beta}(X)P_\alpha \wedge P_\beta\right), \tag{22}$$

where $P_\alpha$ and $X^\alpha$ are the fields of the theory obtained by pulling back $p_\alpha$ and $x^\alpha$ through the sigma-model map. By construction, this model has gauge symmetries generated by $\mathcal{Q}$. These include both bosonic and fermionic gauge symmetries (supersymmetries). We refer to [10] for more details on their explicit form. In addition, the model may have global supersymmetry associated to $q_S$, as long as the consistency conditions (19) are satisfied, which guarantees invariance of the action under the transformations generated by $q_S$.

We now proceed to examples. First, it is useful to expand the super-Poisson structure in powers of the odd coordinate on the supermanifold. The expansion is finite for finite-rank vector bundle $E$. Consider

$$\mathcal{P}^{\mu\nu}(x,\theta) = \underbrace{\Pi^{\mu\nu}(x)}_{0} + \underbrace{\tfrac{1}{2}\,\theta^a\theta^b\,\mathcal{P}_{ab}{}^{\mu\nu}(x)}_{2} + \dots\,, \tag{23a}$$

$$\mathcal{P}^{\mu a}(x,\theta) = \underbrace{\theta^b\,\Gamma^{\mu a}_{b}(x)}_{0} + \dots\,, \tag{23b}$$

$$\mathcal{P}^{ab}(x,\theta) = \underbrace{g^{ab}(x)}_{-2} + \underbrace{\tfrac{1}{2}\,\theta^c\theta^d\,\mathcal{P}_{cd}{}^{ab}(x)}_{0} + \dots\,. \tag{23c}$$

We have indicated the $\mathbb{Z}$-degree of each term in underbraces. Observe that with respect to the $\mathbb{Z}$-degree that distinguishes between different even powers of the odd coordinate, the super-Poisson bracket does not have a definite degree. Specifically, there is only one term that corresponds to a degree $-2$ super-Poisson bracket, which is a symmetric tensor $g^{ab}$, not necessarily nondegenerate. There are three terms corresponding to degree 0 and also three terms for any even degree above it, contained in the ellipses. We will highlight below four different cases/classes of models that solve all the conditions studied so far. The first is Jackiw-Teitelboim supergravity and it is obtained from a super-Poisson structure whose nonvanishing components are the leading ones in the above expansion; this corresponds to a mixed degree $-2$ and 0 case, in particular to a Lie superalgebra. The second class of examples comprises differential PSMs, where the super-Poisson structure is strictly of degree 0. The third model corresponds to a strictly degree $-2$ structure. Finally, we will present a new class, not studied in [10], with higher-order bracket in $\theta$.

### *4.1 Jackiw-Teitelboim supergravity*

As a first example, we show how to obtain the well-known $\mathcal{N} = 1$ Jackiw-Teitelboim supergravity with cosmological constant in this general setting. First recall that the dual of a Lie superalgebra is a Poisson supermanifold. Presently we consider the dual of the orthosymplectic algebra $\mathfrak{osp}(1|2,\mathbb{R})$ to serve as the target space of the sigma-model. The super-Poisson structure is one with only leading order terms in the expansion (23), namely the ones containing $\Pi^{\mu\nu}$, $\Gamma^{\mu a}_{b}$ and $g^{ab}$. Clearly $\Pi^{\mu\nu}$ is the linear Poisson structure on the base, given by the structure constants $C^{\mu\nu}{}_{\rho}$ of the bosonic subalgebra of the Lie superalgebra. Collectively, the graded Jacobi identity is solved by the following components of the super-Poisson bracket

$$\mathcal{P}^{\mu\nu} = C^{\mu\nu}{}_{\rho}\, x^{\rho}\,, \qquad \mathcal{P}^{\mu a} = \Gamma^{\mu\, a}{}_{b}\, \theta^{b}\,, \qquad \mathcal{P}^{ab} = -\tfrac{1}{2}\varepsilon^{(a}{}_{c}\Gamma^{|c|b)}_{\mu} x^{\mu}\,, \tag{24}$$

where $\Gamma^{\mu a}_{b} = (\frac{1}{2}\gamma^{\bar{\mu}a}_{b}, \frac{1}{2}\gamma^{5a}_{b})$ with $\gamma^{\bar{\mu}}$ 2D gamma matrices and $\gamma^{5}$ the 2D chirality operator. Observe that the contravariant connection in the present case corresponds to a representation of the bosonic subalgebra on the fermionic one. The action functional reduces to

$$\begin{aligned} S = \int \big( & e_{\bar{\mu}} \wedge \mathrm{d}X^{\bar{\mu}} + \omega \wedge \mathrm{d}X + \tfrac{1}{2} X\, \varepsilon_{\bar{\mu}\bar{\nu}}\, e^{\bar{\mu}} \wedge e^{\bar{\nu}} + X^{\bar{\mu}} \varepsilon_{\bar{\mu}\bar{\nu}}\, \omega \wedge e^{\bar{\nu}} \\ & - \tfrac{1}{4} X^{\bar{\mu}} \chi^{a} \varepsilon_{ab} \gamma_{\bar{\mu}}{}^{bc} \chi_{c} + \tfrac{1}{4} X \chi^{a} \varepsilon_{ab} \gamma_{5}{}^{bc} \chi_{c} + \theta^{a} \wedge \mathrm{d}\chi_{a} + \tfrac{1}{2}\theta^{a} \gamma_{5a}{}^{b} \omega \wedge \chi_{b} + \tfrac{1}{2}\theta^{a} \gamma_{\bar{\mu}a}{}^{b} e^{\bar{\mu}} \wedge \chi_{b}\,, \end{aligned} \tag{25}$$

where $X^{\mu} = (X^{\bar{\mu}}, X)$ and $P_{\mu} = (e_{\bar{\mu}}, \omega)$, hence $X$ is the dilaton field and $\omega$ is the spin connection in 2D, not to be confused with the unrelated graded symplectic form that we denoted with the same letter earlier. In this case, there is no additional compatible Q-structure corresponding to $q_S$, the target is just a Poisson supermanifold, and therefore there is no additional global supersymmetry. More details on the structure of dilaton supergravity may be found, for example, in [3].

### *4.2 Differential Poisson sigma-models with de Rham & Lichnerowicz–Poisson global supersymmetry*

A second class of models is obtained when only degree 0 terms are considered in the super-Poisson structure (meaning that $g^{ab} = 0$, as well as all higher-order terms). The geometrical problem amounts to determining Lie algebroid structures on $E$ compatible with this super-Poisson bracket. This problem was studied in [10], where it was found that there exist two distinguished cases of particular significance. They correspond to the following choices of super-Poisson structure and HVF $q_S$.

*Tangent bundle.* Choosing $\Pi E = \Pi TM$, the parity-shifted tangent bundle, we have coordinates $x^{\mu}$ and $\theta^{\mu}$. To avoid confusion with index types, we rename $\mathcal{P}^{\mu a} := C^{\mu a}$ and $\mathcal{P}^{ab} := R^{ab}$ for this example. Then we set the following super-Poisson structure and HVF:

$$\mathcal{P}^{\mu\nu} = \Pi^{\mu\nu}\,, \qquad C^{\mu\nu} = -\Pi^{\mu\kappa}\Gamma^{\nu}_{\kappa\lambda}\theta^{\lambda}\,, \qquad R^{\mu\nu} = -\tfrac{1}{2}\big(\Pi^{(\mu|\kappa} R^{\overline{\nabla}}_{\rho\sigma}{}^{|\nu)}{}_{\kappa} - 2\Pi^{\kappa\lambda}\Gamma^{(\mu}_{\kappa\rho}\Gamma^{\nu)}_{\lambda\sigma}\big)\theta^{\rho}\theta^{\sigma}\,, \tag{26a}$$

$$q_S = \theta^{\mu}\frac{\partial}{\partial x^{\mu}}\,, \tag{26b}$$

where $\Pi^{\mu\nu}$ is a Poisson structure on the base, which is covariantly constant with respect to the basic connection $\overline{\nabla}$ with curvature $R^{\overline{\nabla}}$, $\Gamma^{\nu}_{\kappa\lambda}$ are the coefficients of an affine connection on $M$, and $q_S$ is the de Rham differential. Note that the basic connection in the present case is just an ordinary one: the opposite of the connection $\nabla$, defined as $\overline{\nabla}_X Y := \nabla_Y X + [X,Y]$ with $\overline{\Gamma}^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$. The Jacobi identity imposes further algebraic and differential conditions on the curvature, which may be found in [8, 10].

*Cotangent bundle (contravariant case).* Choosing instead $\Pi E = \Pi T^{*}M$, the parity-shifted cotangent bundle, we have coordinates $x^{\mu}$ and $\theta_{\mu}$ and we take the super-Poisson structure and HVF $q_S$ to be

$$\mathcal{P}^{\mu\nu} = \Pi^{\mu\nu}\,, \qquad \mathcal{P}^{\mu}{}_{\nu} = \Pi^{\mu\kappa}\Gamma^{\lambda}_{\kappa\nu}\theta_{\lambda}\,, \qquad \mathcal{P}_{\mu\nu} = -\tfrac{1}{2}\big(S^{\kappa\lambda}{}_{(\mu\nu)} + 2\Pi^{\rho\sigma}\Gamma^{\kappa}_{\rho(\mu}\Gamma^{\lambda}_{\sigma\nu)}\big)\theta_{\kappa}\theta_{\lambda}\,, \tag{27a}$$

$$q_S = \Pi^{\mu\nu}\theta_{\mu}\tfrac{\partial}{\partial x^{\nu}} - \tfrac{1}{2}\partial_{\mu}\Pi^{\kappa\lambda}\theta_{\kappa}\theta_{\lambda}\tfrac{\partial}{\partial\theta_{\mu}}\,. \tag{27b}$$

We have once again a Poisson structure on the base, which is covariantly constant with respect to the basic connection. The compatible Lie algebroid differential is now the Lichnerowicz—Poisson differential.

Aside these two cases, there is a host of further examples that solve the compatibility conditions when the anchor of the Lie algebroid associated to $q_S$ is invertible. In this sense, the contravariant case is special, since it is the only one where the anchor is not necessarily an invertible map. In all cases, the target space covariant formulation has $\mathcal{P}^{\mu a}_{\nabla} = 0$ and $\mathcal{P}^{ab}_{\nabla}$ is a tensor that involves the basic curvature.[4] Specifically, the action for the above two models in the target space covariant formulation

[4] We note that the basic curvature appears in the *BV action* of (twisted) PSMs [16], whereas in the supersymmetric case it appears already at the level of the classical action.

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reads, respectively,

$$S_{TM} = \int \left( A^{\nabla}_{\mu} \wedge \mathrm{d}X^{\mu} + \chi_{\mu} \wedge \nabla\theta^{\mu} + \tfrac{1}{2}\Pi^{\mu\nu} A^{\nabla}_{\mu} \wedge A^{\nabla}_{\nu} + \tfrac{1}{4}\Pi^{\mu\rho} R^{\overline{\nabla}}_{\kappa\lambda}{}^{\nu}{}_{\rho} \chi_{\mu} \wedge \chi_{\nu}\, \theta^{\kappa}\theta^{\lambda} \right), \tag{28a}$$

$$S_{T^*M} = \int \left( A^{\nabla}_{\mu} \wedge \mathrm{d}X^{\mu} + \chi^{\mu} \wedge \nabla\theta_{\mu} + \tfrac{1}{2}\Pi^{\mu\nu} A^{\nabla}_{\mu} \wedge A^{\nabla}_{\nu} + \tfrac{1}{4} S^{\kappa\lambda}{}_{\mu\nu} \chi^{\mu} \wedge \chi^{\nu} \theta_{\kappa}\theta_{\lambda} \right), \tag{28b}$$

where $\nabla$ is the covariant exterior derivative on spacetime and $A^{\nabla}_{\mu} = A_{\mu} + \Gamma^{b}_{\mu a}\theta^{a}\chi_{b}$ is the redefined 1-form. The tangent model is precisely the one of [8], whereas the cotangent one was found in [10]. The lift of the HVF $q_S$ to a HVF $\mathcal{Q}_S$ on $T^*[1]\mathcal{M}$ is in both cases the coadjoint representation of the corresponding Lie algebroid. It generates a global supersymmetry, which for the tangent model was called de Rham supersymmetry in [8] and for the cotangent model is a new kind, a Lichnerowicz–Poisson supersymmetry.

### *4.3 Symmetric Poisson sigma model*

Up to this point, we saw how to solve the compatibility conditions for a super-Poisson bracket of mixed $-2$ and 0 degree and for one with all terms at 0 degree. A further option is to ask for a strictly degree $-2$ super-Poisson bracket, which involves only one term, the term with the symmetric bilinear form $g^{ab}$. In that case, one may consider a HVF for a bundle of Lie algebras, for which the anchor map vanishes. If in addition $g$ is nondegenerate, then it is an ad$^*$-invariant fiberwise metric with respect to the pointwise bracket from the fiber Lie algebras. Every fiber then becomes a quadratic Lie algebra. The supersymmetric PSM was constructed in [10] and we refer to the original paper for technical details.

### *4.4 Higher-order super-Poisson structure*

Up to *this* point, all examples were associated to a vanilla Poisson structure on the base manifold $M$, given by the first term in the expansion of $\mathcal{P}^{\mu\nu}$. We now ask whether examples exist that go beyond this structure. We choose $\Pi E = \Pi TM$ as in the differential PSM on the tangent bundle. Assume that

$$\mathcal{P}^{\mu\nu} = \Pi^{\mu\nu}(x) + \tfrac{1}{2}\Pi^{\mu\nu}{}_{\kappa\lambda}(x)\theta^{\kappa}\theta^{\lambda} \qquad \text{and} \qquad q_S = \theta^{\mu}\partial_{\mu}\,, \tag{29}$$

i.e. an NLO expansion for the components of the super-Poisson structure along the body and $q_S$ the de Rham differential. The compatibility conditions (19) completely determine the other two components:

$$C^{\mu\nu}_{\nabla} = \tfrac{1}{2}\mathrm{d}_{\overline{\nabla}}\mathcal{P}^{\mu\nu}\,, \qquad R^{\mu\nu}_{\nabla} = \tfrac{1}{2}\,\theta^{\kappa}\theta^{\lambda}\bar{R}_{\kappa\lambda}{}^{(\mu|}{}_{\rho}\mathcal{P}^{\rho|\nu)}\,. \tag{30}$$

In this case, the component $C_{\nabla}$ has a linear and a cubic term in $\theta$ and the component $R_{\nabla}$ has a quadratic and a quartic term. In this setting, (30) are in fact true for an arbitrary expansion of $\mathcal{P}^{\mu\nu}$ in even powers of $\theta$, not just for NLO.[5] We observe that the super-Poisson structure is of mixed degree 0 and 2. To simplify the analysis, we assume that $C^{\mu\nu}_{\nabla}$ vanishes; this assumption was also made in [8] and in the simpler differential PSMs described in 4.2. Then the Jacobi identities impose differential and algebraic conditions on the components of $\mathcal{P}^{\mu\nu}$:

$$\Pi^{\kappa[\mu}\nabla_{\kappa}\Pi^{\nu\lambda]} = 0\,, \qquad \Pi^{\kappa[\mu}\nabla_{\kappa}\Pi^{\nu\lambda]}{}_{\alpha\beta} + \Pi^{\kappa[\mu}{}_{\alpha\beta}\nabla_{\kappa}\Pi^{\nu\lambda]} = 0\,, \qquad \Pi^{\kappa[\mu}{}_{[\alpha\beta}\nabla_{\kappa}\Pi^{\nu\lambda]}{}_{\gamma\delta]} = 0\,,$$

$$\Pi^{\mu\kappa}\Pi^{\nu\lambda}R_{\kappa\lambda}{}^{\rho}{}_{\sigma} = 0\,, \quad \Pi^{[\mu|\kappa}\Pi^{\nu]\lambda}{}_{[\alpha\beta}R_{|\kappa\lambda|}{}^{\sigma}{}_{\gamma]} - \tfrac{1}{2}\Pi^{\rho(\tau}\Pi^{|\mu\nu|}{}_{\tau[\gamma}\bar{R}_{\alpha\beta]}{}^{\sigma)}{}_{\rho} = 0\,, \quad \Pi^{\mu\kappa}{}_{[\alpha\beta}\Pi^{\nu\rho}{}_{\gamma\delta}R_{|\kappa\rho|}{}^{\lambda}{}_{\epsilon]} = 0\,,$$

$$\Pi^{\mu\nu}\nabla_{\nu}(\bar{R}_{\rho\sigma}{}^{(\alpha}{}_{\tau}\Pi^{\beta)\tau}) = 0\,, \qquad \Pi^{\rho(\alpha}\Pi^{|\sigma|\beta)}\bar{R}_{[\kappa\lambda}{}^{\delta}{}_{|\rho|}\bar{R}_{\kappa']\delta}{}^{\epsilon}{}_{\sigma} = 0\,,$$

plus four more, higher-order ones, that we suppress. The first conditions of the upper two lines and the two conditions in the third line are precisely the ones found for the differential PSM on the tangent bundle. We observe that $\Pi^{\mu\nu}$ is a Poisson bivector, accompanied by a bivector-valued 2-form in involution with both the Poisson bivector and itself with respect to the trivial extension of the Schouten-Nijenhuis bracket to multivector-valued $2p$-forms,[6] i.e.

$$[X \otimes \eta, Y \otimes \lambda] = [X, Y]_{\mathrm{SN}} \otimes \eta \wedge \lambda\,, \qquad \forall\, X, Y \in \mathfrak{X}^{\bullet}(M),\ \eta, \lambda \in \Omega^{\bullet}(M)\,. \tag{31}$$

This pattern may easily be extended to all admissible orders in the expansion of $\mathcal{P}_{\mu\nu}$ in the odd coordinate. This gives rise to a host of supersymmetric PSMs with interaction terms between even number of fermions $\theta^{\mu}$ and two fermionic 1-forms $\chi_{\mu}$, beyond the quartic terms of the differential PSMs of Section 4.2.

[5] This only works for the tangent bundle and the solution that gave rise to the differential PSM of [8] is a special case.

[6] This ceases to be true when the assumption that $\mathcal{P}^{\mu\nu}$ is covariantly constant with respect to the basic connection is relaxed. In that case, a simultaneous extension of the Frölicher-Nijenhuis and the Schouten-Nijenhuis bracket is needed, which is a nontrivial problem [17].

Journal of Physics: Conference Series **3152** (2025) 012033 doi:10.1088/1742-6596/3152/1/012033

## 5 Conclusions and outlook

We revisited the construction of supersymmetric Poisson sigma models within the systematic context of Poisson Q-supermanifolds. Motivated by Ikeda's model that uses a Poisson supermanifold to construct dilaton supergravity and by the model of [8] that uses a differential Poisson algebra to construct a differential PSM with global supersymmetry given by the de Rham differential, we studied the geometrical conditions that place these models under the same roof and reveal the existence of further models in the same family. We focused on Poisson supermanifolds with an additional, compatible Q-structure generated by a homological vector field and constructed a general model which contains a variety of special cases. In particular, ($\alpha$) $\mathcal{N} = 1$ dilaton supergravity, ($\beta$) a class of differential PSMs based on the tangent/cotangent bundle of a Poisson supermanifold, with global supersymmetry generated by the de Rham/Lichnerowicz–Poisson differential. This global supersymmetry corresponds to the coadjoint representation (up to homotopy) of the associated Lie algebroid, as defined in [12], ($\gamma$) a symmetric differential PSM based on a degree $-2$ super-Poisson structure, which contains a fiberwise symmetric bilinear form, and ($\delta$) a class of higher-order differential PSMs with de Rham supersymmetry, where the super-Poisson structure contains even, non-negative powers of the odd coordinate.

There are several further directions to be explored. First, the quantization of the various differential PSMs and its relation to deformation quantization and the relative formality theorem of [18], see also [19]. Second, models with global supersymmetry that is not based on a Lie algebroid, but on more general HVFs on the $\mathbb{Z}_2$-graded manifold. Third, the extension of this approach to higher dimensions, such as to supersymmetric extensions of Courant sigma models in 3D [20, 21] or higher dimensional Hamiltonian mechanics [22].

*Acknowledgments.* This work is funded by the Croatian Science Foundation project "Higher Structures and Symmetries in Gauge and Gravity Theories" (IP-2024-05-7921). The work of T.B. was supported by the European Research Council under the European Union's Horizon 2020 research and innovation programme grant agreement No 101002551.

## References

[1] Ikeda N 1994 *Ann. Phys.* **235** 435–464 (*Preprint* `hep-th/9312059`)
[2] Schaller P and Strobl T 1994 *Mod. Phys. Lett. A* **9** 3129–3136 (*Preprint* `hep-th/9405110`)
[3] Grumiller D, Kummer W and Vassilevich D V 2002 *Phys. Rept.* **369** 327–430 (*Preprint* `hep-th/0204253`)
[4] Cattaneo A S and Felder G 2000 *Commun. Math. Phys.* **212** 591–611 (*Preprint* `math/9902090`)
[5] Alexandrov M, Schwarz A, Zaboronsky O and Kontsevich M 1997 *Int. J. Mod. Phys. A* **12** 1405–1429 (*Preprint* `hep-th/9502010`)
[6] Schwarz A S 1993 *Commun. Math. Phys.* **155** 249–260 (*Preprint* `hep-th/9205088`)
[7] Batalin I A and Vilkovisky G A 1981 *Phys. Lett. B* **102** 27–31
[8] Arias C, Boulanger N, Sundell P and Torres-Gomez A 2015 *JHEP* **08** 095 (*Preprint* `1503.05625`)
[9] Arias C, Sundell P and Torres-Gomez A, (*Preprint* `1607.00727`).
[10] Basile T, Chatzistavrakidis A and Lavau S 2025 (*Preprint* `2504.13114`)
[11] Kotov A and Strobl T 2014 *Int. J. Geom. Meth. Mod. Phys.* **12** 1550006 (*Preprint* `0711.4106`)
[12] Abad C A and Crainic M 2012 *J. Reine Angew. Math.* **2012** 91–126 (*Preprint* `0901.0319`)
[13] Batchelor, M 1979 *Trans. Am. Math. Soc.* **253** 329–338
[14] Vaintrob A Y 1997 *Russ. Math. Surv.* **52** 428
[15] Roytenberg D 2002 *Workshop on Quantization, Deformations, and New Homological and Categorical Methods in Mathematical Physics* (*Preprint* `math/0203110`)
[16] Ikeda N and Strobl T 2021 *Ann. Henri Poincaré* **22** 1267–1316 (*Preprint* `1912.13511`)
[17] Dubois-Violette M and Michor P W 1995 *Indag. Math.* **6** 51–66 (*Preprint* `alg-geom/9401006`)
[18] Cattaneo A S and Felder G 2007 *Adv. Math.* **208** 521–548 (*Preprint* `math/0501540`)
[19] Lyakhovich S L and Sharapov A A 2005 *JHEP* **03** 011 (*Preprint* `hep-th/0411247`)
[20] Ikeda N 2003 *Int. J. Mod. Phys. A* **18** 2689–2702 (*Preprint* `hep-th/0203043`)
[21] Roytenberg D 2007 *Lett. Math. Phys.* **79** 143–159 (*Preprint* `hep-th/0608150`)
[22] Ševera P 2001 (*Preprint* `math/0105080`)