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Realizations and star-product of doubly κ-deformed Yang models

T. Martinić-Bilać[1,a], **S. Meljanac**[2,b], **S. Mignemi**[3,4,c]

[1] Faculty of Science, University of Split, Ruđera Boškovića 33, 21000 Split, Croatia
[2] Division of Theoretical Physics, Ruđer Bošković Institute, Bijenička cesta 54, 10002 Zagreb, Croatia
[3] Dipartimento di Matematica e Informatica, Università di Cagliari, Via Ospedale 72, 09124 Cagliari, Italy
[4] INFN, Sezione di Cagliari, Cittadella Universitaria, 09042 Monserrato, Italy

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Abstract The Yang algebra was proposed a long time ago as a generalization of the Snyder algebra to the case of curved background spacetime. It includes as subalgebras both the Snyder and the de Sitter algebras and can therefore be viewed as a model of noncommutative curved spacetime. A peculiarity with respect to standard models of noncommutative geometry is that it includes translation and Lorentz generators, so that the definition of a Hopf algebra and the physical interpretation of the variables conjugated to the primary ones is not trivial. In this paper we consider the realizations of the Yang algebra and its κ-deformed generalization on an extended phase space and in this way we are able to define a Hopf structure and a twist.

1 Introduction

The Yang model [1] is an extension of the Snyder model [2] to the full phase space, obtained assuming that both position and momentum operators do not commute among themselves. It can therefore be interpreted as a noncommutative geometry defined on a spacetime of constant curvature. Algebraically, it is based on an $so(1, 5)$ algebra which includes the Lorentz generators and the position and momentum operators, together with a further generator, necessary to close the algebra. Contrary to most models of noncommutative geometry, the action of the Lorentz algebra on phase space is not deformed. It also enjoys an invariance under a (generalized) Born duality [3].

The relevance of noncommutative geometries for Planck-scale physics has been originally highlighted in [4] and since then the subject has become fashionable. In particular the formalism of Hopf algebras has shown to be useful for the description of the physical implications of the theory [5–7]. Among these, particular attention have received the deformation of the symmetries of spacetime [8–13] and the applications to phenomenology and to the search for observable effects, especially in the context of Doubly Special Relativity [10,14,15]. In particular, the introduction of a curved background can be useful in cosmological contexts, especially when considering phenomenological effects on the propagation of photons from distant sources [16,17].

Thus, after a long oblivion, the Yang model was resumed in recent years. Some generalizations were presented by Khruschev and Leznov (KL) [18] and in ref. [19], where an extension of the model that includes also the related triply special relativity (TSR) theory [20,21] was introduced. Further investigations concerning in particular its realizations on a canonical phase space have been recently performed in [22–24], using the methods introduced in [25–27] for the study of Snyder space. Other contributions to the study of Yang model are given in [28–30], while different models of noncommutative geometry in curved spaces can be found in [31–34].

Finally, in [35] the Yang model has been further generalized by deforming the $so(1, 5)$ algebra to an $so(1, 5; g)$ algebra, so that it also includes κ-Poincaré deformations of the kind introduced in [8,9] for the standard Poincaré algebra, both in position and momentum spaces. Although from a mathematical point of view the two algebras are isomorphic, their physical interpretation is of course different, since, for example, in $so(1, 5; g)$ the Poincaré algebra is deformed.

In this paper, we study the explicit transformations leading from the generators of $so(1, 5)$ to those of $so(1, 5; g)$. This will allow us to obtain a Hopf algebra structure for the Yang model, by defining a coproduct and a twist and then an associative star product. This will be done by using the meth-

[a] e-mail: teamar@pmfst.hr
[b] e-mail: meljanac@irb.hr
[c] e-mail: smignemi@unica.it (corresponding author)

ods that have been proved successful in the case of Snyder [25,26] and κ-deformed Snyder [36–39] spaces. In particular, exploiting the isomorphism of the Yang model to an orthogonal algebra, one can use the results of [40], where the Hopf structure associated to orthogonal groups has been discussed in detail.

As shown in [24] a realization of the Yang model can be obtained on an extended phase space, that includes momenta canonically conjugated to both the original position and momentum variables. Hence, in contrast with standard models of noncommutative geometry, in our case a coproduct must be introduced also for the momentum variables. The physical interpretation of this fact is presently under investigation.

2 Generalized Yang models

The Yang model [1] is based on the Yang algebra, which is a Lie algebra generated by $\hat{x}_i$, $\hat{p}_i$, $\hat{M}_{ij}$ and $\hat{h}$ and is isomorphic to the orthogonal algebra $so(1,5)$, $so(2,4)$ or $so(3,3)$, depending on the signature chosen for the metric tensor [41]. It is defined by the commutation relations[1]

$$[\hat{x}_i,\hat{x}_j]=i\beta^2\hat{M}_{ij},\quad [\hat{p}_i,\hat{p}_j]=i\alpha^2\hat{M}_{ij},$$
$$[\hat{x}_i,\hat{p}_j]=i\eta_{ij}\hat{h},$$
$$[\hat{h},\hat{x}_i]=i\beta^2\hat{p}_i,\quad [\hat{h},\hat{p}_i]=-i\alpha^2\hat{x}_i, \tag{1}$$

where the $\hat{M}_{ij}$ generate the Lorentz algebra $so(3,1)$ and $\hat{x}_i$, $\hat{p}_i$ transform as vectors under the Lorentz algebra, while $\hat{h}$ is a scalar. The parameters α and β are constant with dimensions $[L]^{-1}$ and $[M]^{-1}$ respectively and are usually identified with the square root of the cosmological constant and with the inverse of the Planck mass. In spite of the notation, α^2 and β^2 can be taken negative leading to the $so(2,4)$ and $so(3,3)$ cases, which we shall not consider in detail here.

The operators $\hat{x}_i$ and $\hat{p}_i$ can be interpreted as the position and the momentum operators in a quantum phase space. The operator $\hat{h}$ is necessary to close the algebra and generates rotations in the x–p hyperplane. The Yang algebra satisfies the Born duality [3], being invariant for $\alpha\leftrightarrow\beta$, $\hat{x}_i\to-\hat{p}_i$, $\hat{p}_i\to\hat{x}_i$, $\hat{M}_{ij}\leftrightarrow\hat{M}_{ij}$, $\hat{h}\leftrightarrow\hat{h}$.

Defining

$$\hat{M}_{i4}=\frac{\hat{x}_i}{\beta},\qquad \hat{M}_{i5}=\frac{\hat{p}_i}{\alpha},\qquad \hat{M}_{45}=\frac{\hat{h}}{\alpha\beta}. \tag{2}$$

then the algebra (1) can be put in the explicit $so(1,5)$ form

$$[\hat{M}_{\mu\nu},\hat{M}_{\rho\sigma}]=i\big(\eta_{\mu\rho}\hat{M}_{\nu\sigma}-\eta_{\mu\sigma}\hat{M}_{\nu\rho}-\eta_{\nu\rho}\hat{M}_{\mu\sigma}+\eta_{\nu\sigma}\hat{M}_{\mu\rho}\big). \tag{3}$$

One can find alternative realizations of this algebra by defining linear combinations of the generators. The most general new generators linear in $\hat{x}_i$, $\hat{p}_i$, $\hat{M}_{ij}$ are

$$\tilde{X}_i=A\left(\cos\varphi\,\hat{x}_i+\frac{\beta}{\alpha}\sin\varphi\,\hat{p}_i\right)+\beta a_k\hat{M}_{ik},$$
$$\tilde{P}_i=B\left(\cos\psi\,\hat{p}_i+\frac{\alpha}{\beta}\sin\psi\,\hat{x}_i\right)+\alpha b_k\hat{M}_{ik}, \tag{4}$$

with $\tilde{M}_{ij}=\hat{M}_{ij}$. The parameters A, B, φ, ψ, a_i, b_i are dimensionless with $AB\neq0$. The transformations inverse to (4) are

$$\hat{x}_i=\frac{A^{-1}\alpha\cos\psi(\tilde{X}_i-\beta a_j\tilde{M}_{ij})-B^{-1}\beta\sin\varphi(\tilde{P}_i-\alpha b_j\tilde{M}_{ij})}{\alpha\cos(\varphi+\psi)}$$
$$\hat{p}_i=\frac{B^{-1}\beta\cos\varphi(\tilde{P}_i-\alpha b_j\tilde{M}_{ij})-A^{-1}\alpha\sin\psi(\tilde{X}_i-\beta a_j\tilde{M}_{ij})}{\beta\cos(\varphi+\psi)} \tag{5}$$

The new generators $\tilde{X}_i$ and $\tilde{P}_i$ generate a new class of Lie algebras isomorphic to the initial Yang algebra. The new algebra generated by $\tilde{X}_i$, $\tilde{P}_i$, $\tilde{M}_{ij}$ and $\tilde{H}$ is given by the following commutation relations

$$[\tilde{X}_i,\tilde{X}_j]=i\left(\beta^2\tilde{A}\tilde{M}_{ij}+\beta(a_i\tilde{X}_j-a_j\tilde{X}_i)\right),$$
$$[\tilde{P}_i,\tilde{P}_j]=i\left(\alpha^2\tilde{B}\tilde{M}_{ij}+\alpha(b_i\tilde{P}_j-b_j\tilde{P}_i)\right),$$
$$[\tilde{X}_i,\tilde{P}_j]=i\left(\eta_{ij}\tilde{H}+\alpha b_i\tilde{X}_j-\beta a_j\tilde{P}_i+\alpha\beta\tilde{\rho}\tilde{M}_{ij}\right),$$
$$[\tilde{M}_{ij},\tilde{X}_k]=i\Big(\eta_{ik}\tilde{X}_j-\eta_{jk}\tilde{X}_i+\beta(a_i\tilde{M}_{kj}-a_j\tilde{M}_{ki})\Big),$$
$$[\tilde{M}_{ij},\tilde{P}_k]=i\Big(\eta_{ik}\tilde{P}_j-\eta_{jk}\tilde{P}_i+\alpha(b_i\tilde{M}_{kj}-b_j\tilde{M}_{ki})\Big),$$
$$[\tilde{M}_{ij},\tilde{H}]=i\Big(\alpha(b_j\tilde{X}_i-b_i\tilde{X}_j)-\beta(a_j\tilde{P}_i-a_i\tilde{P}_j)\Big),$$
$$[\tilde{H},\tilde{X}_i]=i\left(\beta^2\tilde{A}\tilde{P}_i-\alpha\beta\tilde{\rho}\,\tilde{X}_i-\beta a_i\tilde{H}\right),$$
$$[\tilde{H},\tilde{P}_i]=i\left(-\alpha^2\tilde{B}\tilde{X}_i+\alpha\beta\tilde{\rho}\,\tilde{P}_i+\alpha b_i\tilde{H}\right), \tag{6}$$

where we have defined

$$\tilde{H}=AB\cos(\varphi+\psi)\hat{h}+\beta a\cdot\tilde{P}-\alpha b\cdot\tilde{X}-\alpha\beta a_h b_k\hat{M}_{hk}, \tag{7}$$

$$\tilde{\rho}=AB\rho+a\cdot b,\qquad \tilde{A}=A^2+a^2,\qquad \tilde{B}=B^2+b^2, \tag{8}$$

with $\rho=\sin(\varphi+\psi)$. A generalized Born duality still holds for $\alpha\leftrightarrow\beta$, $a_i\to-b_i$, $b_i\to a_i$, $\tilde{A}\leftrightarrow\tilde{B}$, $\tilde{\rho}\leftrightarrow-\tilde{\rho}$, $\tilde{X}_i\to-\tilde{P}_i$, $\tilde{P}_i\to\tilde{X}_i$, $\tilde{M}_{ij}\leftrightarrow\tilde{M}_{ij}$, $\tilde{H}\leftrightarrow\tilde{H}$.

These commutation relations are of the kind introduced in [24]. They can be put in the form of as an $so(1,5;g)$ algebra

[1] Latin indices run from 0 to 3, Greek indices from 0 to 5. For definiteness in the following we consider the $so(1,5)$ case with metric $\eta_{ij}=\mathrm{diag}(-1,1,1,1,1,1)$, but our considerations trivially extend to the other cases. We use natural units, in particular we set $\hbar=1$.

if one defines the generators $\tilde{M}_{\mu\nu}$ as

$$\tilde{M}_{i4} = \frac{\tilde{X}_i}{\beta}, \quad \tilde{M}_{i5} = \frac{\tilde{P}_i}{\alpha}, \quad \tilde{M}_{45} = \frac{\tilde{H}}{\alpha\beta}. \tag{9}$$

They satisfy the algebra

$$[\tilde{M}_{\mu\nu}, \tilde{M}_{\rho\sigma}] = i\big(g_{\mu\rho}\tilde{M}_{\nu\sigma} - g_{\mu\sigma}\tilde{M}_{\nu\rho} - g_{\nu\rho}\tilde{M}_{\mu\sigma} + g_{\nu\sigma}\tilde{M}_{\mu\rho}\big), \tag{10}$$

with $g_{\mu\nu}$ a symmetric matrix of the form

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 & a_0 & b_0 \\ 0 & 1 & 0 & 0 & a_1 & b_1 \\ 0 & 0 & 1 & 0 & a_2 & b_2 \\ 0 & 0 & 0 & 1 & a_3 & b_3 \\ a_0 & a_1 & a_2 & a_3 & \tilde{A} & \tilde{\rho} \\ b_0 & b_1 & b_2 & b_3 & \tilde{\rho} & \tilde{B} \end{pmatrix}. \tag{11}$$

Notice that $\det g = A^2B^2\cos^2(\varphi+\psi)$, hence one must require that $\cos(\varphi+\psi) \neq 0$, otherwise the matrix g becomes singular.

The matrix $g_{\mu\nu}$ can be reduced to a diagonal form by a transformation with a matrix S such that $g = S\,\eta\,S^T$. This matrix is defined up to multiplication by an orthogonal matrix O such that $O\,\eta\,O^T = \eta$. One can choose S in a lower triangular form, as

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -a_0 & a_1 & a_2 & a_3 & \sigma & 0 \\ -b_0 & b_1 & b_2 & b_3 & \upsilon & \tau \end{pmatrix}, \tag{12}$$

with

$$\sigma = A, \quad \upsilon = B\sin(\varphi+\psi), \quad \tau = B\cos(\varphi+\psi), \tag{13}$$

and $\det S = \sigma\tau = AB\cos(\varphi+\psi)$.

Clearly, the generators $\hat{M}_{\mu\nu}$ such that $\tilde{M}_{\mu\nu} = (S\,\hat{M}\,S^T)_{\mu\nu}$, satisfy the commutation relations (3) and can then be identified with those defined in (2). The explicit relation between their components are then $\tilde{M}_{ij} = \hat{M}_{ij}$, and

$$\begin{aligned} \tilde{X}_i &= \frac{1}{\sigma}(\hat{x}_i - a^k\hat{M}_{ik}), \\ \tilde{P}_i &= \frac{1}{\tau}(\hat{p}_i - b^k\hat{M}_{ik}) - \frac{\upsilon}{\sigma\tau}(\hat{x}_i - a^k\hat{M}_{ik}), \\ \tilde{H} &= \frac{1}{\sigma\tau}(\hat{h} + b^k\hat{x}_k - a^k\hat{p}_k + a^hb^k\hat{M}_{hk}), \end{aligned} \tag{14}$$

with inverse transformations

$$\begin{aligned} \hat{x}_i &= \sigma\tilde{X}_i + a^k\tilde{M}_{ik}, \\ \hat{p}_i &= \tau\tilde{P}_i + \upsilon\tilde{X}_i + b^k\tilde{M}_{ik}, \\ \hat{h} &= \sigma\tau\,\tilde{H} + (\sigma b^k + \upsilon a^k)\tilde{X}_k + \tau a^k\tilde{P}_k + a^hb^k\tilde{M}_{hk}. \end{aligned} \tag{15}$$

Note that if $a_k = b_k = 0$, then the KL model with $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\rho}$, $\tilde{H}$ is isomorphic to the original Yang model with A, B, ρ, h if $\alpha^2 = \tilde{\alpha}^2/B^2$, $\beta^2 = \tilde{\beta}^2/A^2$ and $h = \tilde{H}/(AB\cos(\varphi+\psi))$ where A, B are real numbers and $A^2B^2 > \tilde{\rho}^2$.

The same construction as (4)–(6) holds also for Yang models isomorphic to $so(3,3)$. For models isomorphic to $so(2,4)$, in Eq. (4) the trigonometric functions should be replaced by the corresponding hyperbolic functions and the parameter ρ can be arbitrary; in particular, when $\varphi = \psi$ it follows that $\rho = 0$.

3 Weyl representation of $so(1,5;g)$

We shall now obtain representations for the κ-deformed Yang model by using the previous relations between $so(1,5)$ and $so(1,5;g)$. Notice that from a four-dimensional point of view our primary fields are the generators $\tilde{M}_{ij}$, $\tilde{X}_i$, $\tilde{P}_i$ and $\tilde{H}$, to each of which we shall assign a conjugate momentum. The formalism is therefore analog to that introduced for the extended Snyder model [38,39].

Consider then the generalized Heisenberg algebra

$$\begin{aligned} &[x_{\mu\nu}, x_{\rho\sigma}] = [k^{\mu\nu}, k^{\rho\sigma}] = 0, \\ &[x_{\mu\nu}, k^{\rho\sigma}] = i(\delta_\mu^{\ \rho}\delta_\nu^{\ \sigma} - \delta_\mu^{\ \sigma}\delta_\nu^{\ \rho}), \end{aligned} \tag{16}$$

where $k_{\mu\nu}$ are momenta conjugated to the $x_{\mu\nu}$, and define $X_{\mu\nu} = (S\,x\,S^T)_{\mu\nu}$ and $K^{\mu\nu} = (S^\ddagger\,k\,S^{-1})^{\mu\nu}$, where $S^\ddagger = (S^{-1})^T$.

These variables satisfy commutation relations analogous to (16),

$$\begin{aligned} &[X_{\mu\nu}, X_{\rho\sigma}] = [K^{\mu\nu}, K^{\rho\sigma}] = 0, \\ &[X_{\mu\nu}, K^{\rho\sigma}] = i(\delta_\mu^{\ \rho}\delta_\nu^{\ \sigma} - \delta_\mu^{\ \sigma}\delta_\nu^{\ \rho}), \end{aligned} \tag{17}$$

but their indices are raised and lowered by the metric $g_{\mu\nu}$, for example $K_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}K^{\rho\sigma}$.

The variables $X_{\mu\nu}$ can be decomposed as

$$X_{i4} = \frac{X_i}{\beta}, \quad X_{i5} = \frac{P_i}{\alpha}, \quad X_{45} = \frac{H}{\alpha\beta}. \tag{18}$$

Moreover, the explicit relations between the four-dimensional components of $K_{\mu\nu}$ and $k_{\mu\nu}$ can be written as

$$\begin{aligned} K_{ij} &= k_{ij} + \frac{\beta}{\sigma}(a_iq_j - a_jq_i) + \frac{\alpha}{\tau}(b_iy_j - b_jy_i) \\ &\quad - \frac{\alpha\upsilon}{\sigma\tau}(a_iy_j - a_jy_i) + \alpha\beta(a_ib_j - a_jb_i)w, \\ Q_i &= \frac{1}{\sigma\tau}\left(\upsilon q_i - \frac{\alpha\tau}{\beta}y_i + \alpha b_iw\right), \\ Y_i &= \frac{1}{\sigma\tau}(\sigma y_i - \alpha a_iw), \\ W &= \frac{w}{\sigma\tau}, \end{aligned} \tag{19}$$

with inverse

$$k_{ij} = K_{ij} - \beta(a_i Q_j - a_j Q_i) - \alpha(b_i Y_j - b_j Y_i) + \alpha\beta(a_i b_j - a_j b_i)W,$$
$$q_i = \sigma(Q_i - \alpha b_i W) + \upsilon\left(\frac{\alpha}{\beta} Y_i + \alpha a_i W\right),$$
$$y_i = \tau(Y_i + \beta a_i W),$$
$$w = \sigma\tau W, \tag{20}$$

where we have defined $k^{i4} = \beta q^i$, $k^{i5} = \alpha y^i$, $k^{45} = \alpha\beta w$ and

$$K^{i4} = \beta Q^i, \quad K^{i5} = \alpha Y^i, \quad K^{45} = \alpha\beta W, \tag{21}$$

so that

$$[X_i, Q_j] = i\delta_{ij}, \quad [P_i, Y_j] = i\delta_{ij}, \quad [H, W] = i. \tag{22}$$

We now want to find a realisation of the $\tilde{M}_{\mu\nu}$ defined in Sect. 1 in terms of the Heisenberg algebra generated by $X_{\mu\nu}$ and $K^{\mu\nu}$. The $\tilde{M}_{\mu\nu}$ satisfy the $so(1, N; g)$ algebra (10), that we write as

$$[\tilde{M}_{\mu\nu}, \tilde{M}_{\rho\sigma}] = i\, C_{\mu\nu,\,\rho\sigma}{}^{\alpha\beta}\, \tilde{M}_{\alpha\beta}. \tag{23}$$

The structure constants are given by

$$C_{\mu\nu,\,\rho\sigma}{}^{\alpha\beta} = \frac{1}{2}\Big[-g_{\nu\rho}(\delta_\mu^\alpha\delta_\sigma^\beta - \delta_\mu^\beta\delta_\sigma^\alpha) + g_{\mu\sigma}(\delta_\rho^\alpha\delta_\nu^\beta - \delta_\rho^\beta\delta_\nu^\alpha) - (\mu \leftrightarrow \nu)\Big], \tag{24}$$

and obey the symmetry properties $C_{\mu\nu,\rho\sigma}{}^{\alpha\beta} = -C_{\nu\mu,\rho\sigma}{}^{\alpha\beta} = -C_{\mu\nu,\sigma\rho}{}^{\alpha\beta} = -C_{\mu\nu,\rho\sigma}{}^{\beta\alpha} = -C_{\rho\sigma,\mu\nu}{}^{\alpha\beta}$. Note that structure constants in (23) are multiplied by $\hbar$, which is set to 1 in our conventions, and in the classical limit $\hbar = 0$ all generators commute.

In general, if the operators $M_{\mu\nu}$ generate a Lie algebra with structure constants $C_{\mu\nu,\rho\sigma}{}^{\alpha\beta}$, the universal realization of $M_{\mu\nu}$ in terms of the Heisenberg algebra (17), corresponding to Weyl-symmetric ordering, is given by [40,42]

$$M_{\mu\nu} = X_{\alpha\beta}\left[\frac{\mathcal{C}}{1 - e^{-\mathcal{C}}}\right]_{\mu\nu}^{\;\;\alpha\beta}, \tag{25}$$

where $\mathcal{C}_{\mu\nu}{}^{\alpha\beta} = -\frac{1}{2}\, C_{\mu\nu,\,\rho\sigma}{}^{\alpha\beta} K^{\rho\sigma}$.

This realization enjoys the property

$$e^{\frac{i}{2}t^{\mu\nu}\tilde{M}_{\mu\nu}} \triangleright 1 = e^{\frac{i}{2}t^{\mu\nu}X_{\mu\nu}}, \tag{26}$$

where the $t_{\mu\nu}$ are real numbers transforming as tensors under $so(1, N; g)$ and the action $\triangleright$ is defined as

$$X_{\mu\nu} \triangleright f(X_{\alpha\beta}) = X_{\mu\nu} f(X_{\alpha\beta}),$$
$$K^{\mu\nu} \triangleright f(X_{\alpha\beta}) = -i\frac{\partial f(X_{\alpha\beta})}{\partial X_{\mu\nu}} = [K^{\mu\nu}, f(X_{\alpha\beta})]. \tag{27}$$

In particular,

$$X_{\mu\nu} \triangleright 1 = X_{\mu\nu}, \qquad K_{\mu\nu} \triangleright 1 = 0,$$
$$K^{\mu\nu} \triangleright e^{\frac{i}{2}t^{\alpha\beta}X_{\alpha\beta}} = t^{\mu\nu} e^{\frac{i}{2}t^{\alpha\beta}X_{\alpha\beta}} \tag{28}$$

We can now expand (25) in powers of the structure constants $C_{\mu\nu,\rho\sigma}{}^{\alpha\beta}$ (i.e. in terms of $\hbar$), Then the Weyl realization of $\tilde{M}_{\mu\nu}$ in terms of the generalized Heisenberg algebra generated by $X_{\mu\nu}$ and $K^{\mu\nu}$ reads up to second order,

$$\tilde{M}_{\mu\nu} = X_{\mu\nu} + \frac{1}{2} X_{\alpha\beta}\, \mathcal{C}_{\mu\nu}{}^{\alpha\beta} + \frac{1}{12} X_{\alpha\beta}\left(\mathcal{C}^2\right)_{\mu\nu}^{\;\;\alpha\beta}. \tag{29}$$

where

$$\mathcal{C}_{\mu\nu}{}^{\alpha\beta} = \frac{1}{2}\Big(\delta_\mu^{\ \alpha} K_\nu^{\ \beta} + \delta_\nu^{\ \beta} K_\mu^{\ \alpha} - (\alpha \leftrightarrow \beta)\Big),$$
$$\left(\mathcal{C}^2\right)_{\mu\nu}^{\;\;\alpha\beta} = \frac{1}{2}\Big(2K_\mu^{\ \alpha} K_\nu^{\ \beta} + \delta_\nu^{\ \beta} K_{\mu\rho} K^{\rho\alpha} + \delta_\mu^{\ \alpha} K_{\nu\rho} K^{\rho\beta} - (\alpha \leftrightarrow \beta)\Big), \tag{30}$$

and the indices are lowered by means of the metric $g_{\mu\nu}$.

Inserting $\mathcal{C}$ in (29), we find up to first order,

$$\tilde{M}_{\mu\nu} = X_{\mu\nu} + \frac{1}{2}\left(X_{\mu\alpha} K_\nu^{\ \alpha} - X_{\nu\alpha} K_\mu^{\ \alpha}\right), \tag{31}$$

and

$$[\tilde{M}_{\mu\nu}, K^{\rho\sigma}] = i(\delta_\mu^{\ \rho}\delta_\nu^{\ \sigma} - \delta_\mu^{\ \sigma}\delta_\nu^{\ \rho}) + \frac{i}{2}(\delta_\mu^{\ \rho} K_\nu^{\ \sigma} - \delta_\nu^{\ \rho} K_\mu^{\ \sigma} + \delta_\nu^{\ \sigma} K_\mu^{\ \rho} - \delta_\mu^{\ \sigma} K_\nu^{\ \rho}). \tag{32}$$

We can write (31) in terms of four-dimensional variables, defined by (9), (18) and (21), as

$$\tilde{M}_{ij} = X_{ij} + \frac{1}{2}\Big(X_{ik}(K_j^{\ k} - \beta a_j Q^k - \alpha b_j Y^k) + X_i(Q_j - \alpha b_j W) + P_i(Y_j + \beta a_j W) - (i \leftrightarrow j)\Big),$$
$$\tilde{X}_i = X_i + \frac{1}{2}\Big(-\beta X_{ij}(a_k K^{jk} + \beta\tilde{A} Q^j + \alpha\tilde{\rho} Y^j) + \beta X_i(a_j Q^j - \alpha\tilde{\rho} W) + X_j(K_i^{\ j} - \beta a_i Q^j - \alpha b_i Y^j) + \beta P_i(a_j Y^j + \beta\tilde{A}\, W) - H(Y_i + \beta a_i W)\Big),$$
$$\tilde{P}_i = P_i + \frac{1}{2}\Big(-\alpha X_{ij}(b_k K^{jk} + \alpha\tilde{B} Y^j + \beta\tilde{\rho} Q^j) + \alpha P_i(b_j Y^j + \beta\tilde{\rho} W) + P_j(K_i^{\ j} - \alpha b_i Y^j - \beta a_i Q^j) + \alpha X_i(\beta b_j Q^j - \alpha\tilde{B}\, W) + H(Q_i - \alpha b_i W)\Big),$$
$$\tilde{H} = H + \frac{1}{2}\Big(\alpha X_i(\alpha\tilde{B} Y^i - \beta\tilde{\rho} Q^i + b_j K^{ij}) - \beta P_i(\beta\tilde{A} Q^i + \alpha\tilde{\rho} Y^i + a_j K^{ij}) + H(\beta a_i Q^i + \alpha b_i Y^i)\Big), \tag{33}$$

where Latin indices are lowered using the flat metric.

Other realizations can be obtained from the Weyl realization $\tilde{M}^W_{\mu\nu}$ using similarity transformations of the type

$$\tilde{M}_{\mu\nu} = S\, \tilde{M}^W_{\mu\nu}\, S^{-1} \tag{34}$$

where $S = \exp(G)$ with G of the form $G = XF(K)$. Then the corresponding realizations will be linear in X and can be written as series in K. The corresponding coproducts, star products and twists can be obtained using the same similarity transformations [19].

4 Coproduct and star product in Weyl realization

Formulae for coproduct and deformed addition of momenta can be deduced using the results of [38,39,42], with the difference that now the sums are performed with the curved metric $g_{\mu\nu}$ instead of the flat metric. This formalism allows us to construct a coproduct for the (κ-deformed) Yang model, in particular, both the momenta conjugated to $\tilde{X}_i$ and to $\tilde{P}_i$ will admit a coproduct. We remark that the coproduct so defined is coassociative, contrary to some realization of the Snyder model (see [25]).

Defining

$$e^{\frac{i}{2}s^{\mu\nu}\tilde{M}_{\mu\nu}}e^{\frac{i}{2}t^{\rho\sigma}\tilde{M}_{\rho\sigma}} = e^{\frac{i}{2}(s^{\mu\nu}\oplus t^{\mu\nu})\tilde{M}_{\mu\nu}} \equiv e^{\frac{i}{2}\mathcal{D}^{\mu\nu}(s,t)\tilde{M}_{\mu\nu}}, \quad (35)$$

where $s^{\mu\nu}$ and $t^{\mu\nu}$ transform as $so(1,5;g)$ tensors, one has at first order

$$\mathcal{D}^{\mu\nu}(s^{\alpha\beta}, t^{\alpha\beta}) = s^{\mu\nu} + t^{\mu\nu} - \frac{1}{2}\Big(s^{\mu\alpha}t^{\nu}_{\ \alpha} - s^{\nu\alpha}t^{\mu}_{\ \alpha}\Big). \quad (36)$$

In the following, we shall write all the formulas up to first order, without explicitly mentioning it.

The coproduct $\Delta K^{\mu\nu}$ is then

$$\Delta K^{\mu\nu} = \mathcal{D}^{\mu\nu}(K^{\mu\nu}\otimes 1, 1\otimes K^{\mu\nu}) = \Delta_0 K^{\mu\nu} - \frac{1}{2}\Big(K^{\mu\alpha}\otimes K^{\nu}_{\ \alpha} - K^{\nu\alpha}\otimes K^{\mu}_{\ \alpha}\Big), \quad (37)$$

where $\Delta_0 K^{\mu\nu} = K^{\mu\nu}\otimes 1 + 1\otimes K^{\mu\nu}$. The coproduct (37) is coassociative.

In components, it reads

$$\begin{aligned}
\Delta K^{ij} = \Delta_0 K^{ij} - \frac{1}{2}\Big(&K^{ik}\otimes K^{j}_{\ k} + \beta^2\tilde{A}Q^i\otimes Q^j + \alpha^2\tilde{B}Y^i\otimes Y^j\\
&+\alpha\beta\tilde{\rho}(Q^i\otimes Y^j + Y^i\otimes Q^j)\\
&+\beta a_k(K^{ik}\otimes Q^j + Q^i\otimes K^{jk})\\
&+\alpha b_k(K^{ik}\otimes Y^j + Y^i\otimes K^{jk}) - (i\leftrightarrow j)\Big),
\end{aligned}$$

$$\begin{aligned}
\Delta Q^i = \Delta_0 Q^i - \frac{1}{2}\Big(&K^{ik}\otimes Q_k - Q_k\otimes K^{ik}\\
&+\alpha\beta\tilde{\rho}(Q^i\otimes W - W\otimes Q^i)\\
&+\alpha^2\tilde{B}(Y^i\otimes W - W\otimes Y^i)\\
&+\alpha b_k(K^{ik}\otimes W - W\otimes K^{ik})\Big),\\
\Delta Y^i = \Delta_0 Y^i - \frac{1}{2}\Big(&K^{ik}\otimes Y_k - Y_k\otimes K^{ik}\\
&-\alpha\beta\tilde{\rho}(Y^i\otimes W - W\otimes Y^i)\\
&+\beta^2\tilde{A}(Q^i\otimes W - W\otimes Q^i)\\
&-\beta a_k(K^{ik}\otimes W - W\otimes K^{ik})\Big),\\
\Delta W = \Delta_0 W - \frac{1}{2}\Big(&Q^k\otimes Y_k - Y^k\otimes Q_k\Big).
\end{aligned} \quad (38)$$

Using the relations (19), the coproduct can also be written in terms of tensors transforming under $so(1,5)$. It is also easy to see that the antipodes are trivial.

The star product can easily be deduced from the previous relations, since it is defined as

$$e^{\frac{i}{2}s^{\mu\nu}X_{\mu\nu}} \star e^{\frac{i}{2}t^{\rho\sigma}X_{\rho\sigma}} = e^{\frac{i}{2}\mathcal{D}^{\mu\nu}(s,t)X_{\mu\nu}}, \quad (39)$$

with $\mathcal{D}^{\mu\nu}(s,t)$ given in (36). This star product is associative.

It may be useful to explicitly write down the four-dimensional expression of $\mathcal{D}^{\mu\nu}(s,t)$: setting $\mathcal{D}^i = \mathcal{D}^{i4}$, $\bar{\mathcal{D}}^i = \mathcal{D}^{i5}$, $\mathcal{D} = \mathcal{D}^{45}$, one has

$$\begin{aligned}
\mathcal{D}^{ij}(s,t) &= s^{ij} + t^{ij} - \frac{1}{2}\Big(s^{ik}t^{j}_{\ k} + \beta^2\tilde{A}s^i t^j + \alpha^2\tilde{B}\bar{s}^i\bar{t}^j\\
&\quad+\alpha\beta\tilde{\rho}(s^i\bar{t}^j + \bar{s}^i t^j) + \beta a_k(s^{ik}t^j + s^i t^{jk})\\
&\quad+\alpha b_k(s^{ik}\bar{t}^j + \bar{s}^i t^{jk}) - (i\leftrightarrow j)\Big),\\
\mathcal{D}^i(s,t) &= s^i + t^i - \frac{1}{2}\Big(s^{ik}t_k - t^{ik}s_k + \alpha\beta\tilde{\rho}(s^i t - s t^i)\\
&\quad+\alpha^2\tilde{B}(\bar{s}^i t - s\bar{t}^i) + \alpha b_k(s^{ik}t - s t^{ik})\Big)\\
\bar{\mathcal{D}}^i(s,t) &= \bar{s}^i + \bar{t}^i - \frac{1}{2}\Big(s^{ik}\bar{t}_k - \bar{s}_k t^{ik} - \alpha\beta\tilde{\rho}(\bar{s}^i t - s\bar{t}^i)\\
&\quad+\beta^2\tilde{A}(s^i t - s t^i) - \beta a_k(s^{ik}t - s t^{ik})\Big)\\
\mathcal{D}(s,t) &= s + t - \frac{1}{2}\Big(s^k\bar{t}_k - \bar{s}^k t_k\Big),
\end{aligned} \quad (40)$$

where we have defined the components of the $so(1,5;g)$ tensors $t^{\mu\nu}$ as $t^i = t^{i4}$, $\bar{t}^i = t^{i5}$, $t = t^{45}$ and analogously for s^{ij}.

5 The twist for the Weyl realization

In this section, we construct the twist operator at first order, again using the results of [38,42]. The twist is defined as a bilinear operator such that $\Delta m = \mathcal{F}\Delta_0 m\mathcal{F}^{-1}$ for each m

belonging to $so(1,5;g)$. Its use in the context of noncommutative geometries was introduced in [43,44] as a tool useful in the construction of quantum field theories.

The twist in a Hopf algebroid sense can be computed by means of the formula [45]

$$\mathcal{F}^{-1} \equiv e^F = e^{-\frac{i}{2}K^{\mu\nu}\otimes X_{\mu\nu}} e^{\frac{i}{2}K^{\rho\sigma}\otimes\tilde{M}_{\rho\sigma}}. \tag{41}$$

Using the Campbell–Baker–Hausdorff formula one gets

$$F = \frac{i}{2}K^{\mu\nu}\otimes(\tilde{M}_{\mu\nu} - X_{\mu\nu}). \tag{42}$$

and substituting (31) in (42), one obtains

$$F = \frac{i}{2}K^{\alpha\gamma}\otimes X_{\alpha\beta}K_\gamma{}^\beta. \tag{43}$$

It is easy to check that

$$\mathcal{F}\Delta_0 K^{\mu\nu}\mathcal{F}^{-1} = \Delta K^{\mu\nu}, \tag{44}$$

with $\Delta K^{\mu\nu}$ given in (37).

In terms of components, one can write

$$\begin{aligned}
F = K^{ij}\otimes\Big[&X_{ik}(K_j{}^k - \beta a_j Q^k - \alpha b_j Y^k)\\
&+X_i(Q_j - \alpha b_j W) + P_i(Y_j + \beta a_j W)\Big]\\
&+Q^i\otimes\Big[-\beta X_{ij}(\beta\tilde{A}Q^j + \alpha\tilde{\rho}Y^j + a_k K^{jk})\\
&+\beta X_i(a_j Q^j - \alpha\tilde{\rho}W) + X_j(K_i{}^j - \beta a_i Q^j - \alpha b_i Y^j)\\
&+\beta P_i(a_j Y^j + \beta\tilde{A}\,W) - H(Y_i + \beta a_i W)\Big]\\
&+Y^i\otimes\Big[-\alpha X_{ij}(\alpha\tilde{B}Y^j + \beta\tilde{\rho}Q^j + b_k K^{jk})\\
&+\alpha P_i(b_j Y^j + \beta\tilde{\rho}W) + P_j(K_i{}^j - \alpha b_i Y^j - \beta a_i Q^j)\\
&+\alpha X_i(\beta b_j Q^j - \alpha\tilde{B}\,W) + H(Q_i - \alpha b_i W)\Big]\\
&+W\otimes\Big[\alpha X_i(b_j K^{ij} + \alpha\tilde{B}Y^i - \beta\tilde{\rho}Q^i) - \beta P_i(\beta\tilde{A}Q^i\\
&+\alpha\tilde{\rho}Y^i + a_j K^{ij}) + H(\beta a_i Q^i + \alpha b_i Y^i)\Big].
\end{aligned} \tag{45}$$

6 Coproduct and twist for the original Yang model

Of course the Hopf structure for the original Yang model can be derived from the previous results simply setting $A = B = 1$, $\varphi = \psi = 0$ and $a = b = 0$. Since these results are not discussed in the literature we briefly report them here.

The coproduct can be written in terms of the four-dimensional variables defined in section 3 as

$$\begin{aligned}
\Delta K^{ij} &= \Delta_0 K^{ij} - \frac{1}{2}\Big(K^{ik}\otimes K^j{}_k + \beta^2 Q^i\otimes Q^j + \alpha^2 Y^i\otimes Y^j\\
&\quad+\alpha\beta(Q^i\otimes Y^j + Y^i\otimes Q^j) - (i\leftrightarrow j)\Big),\\
\Delta Q^i &= \Delta_0 Q^i - \frac{1}{2}\Big(K^{ik}\otimes Q_k - Q_k\otimes K^{ik}\\
&\quad+\alpha\beta(Q^i\otimes W - W\otimes Q^i)\\
&\quad+\alpha^2(Y^i\otimes W - W\otimes Y^i)\Big),\\
\Delta Y^i &= \Delta_0 Y^i - \frac{1}{2}\Big(K^{ik}\otimes Y_k - Y_k\otimes K^{ik}\\
&\quad-\alpha\beta(Y^i\otimes W - W\otimes Y^i)\\
&\quad+\beta^2(Q^i\otimes W - W\otimes Q^i)\Big),\\
\Delta W &= \Delta_0 W - \frac{1}{2}\Big(Q^k\otimes Y_k - Y^k\otimes Q_k\Big).
\end{aligned} \tag{46}$$

It is evident that one cannot disentangle the various components of the conjugated momenta.

Analogously, the twist takes the form

$$\begin{aligned}
F = K^{ij}\otimes&\Big[X_{ik}K_j{}^k + X_i Q_j + P_i Y_j\Big]\\
&+Q_i\otimes\Big[-\beta X_{ik}(\beta Q^k + \alpha Y^k) + X_k K_i{}^k\\
&+\beta(\beta P_i - \alpha X_i)W - HY^i\Big]\\
&+Y_i\otimes\Big[-\alpha X_{ik}(\alpha Y^k + \beta Q^k) + P_k K_i{}^k\\
&+\alpha(\beta P_i - \alpha X_i)W + HQ^i\Big]\\
&+W\otimes\Big[\alpha X_k(\alpha Y^k - \beta Q^k) - \beta P_k(\beta Q^k + \alpha Y^k)\Big].
\end{aligned} \tag{47}$$

7 Conclusions

The Yang model represents a noncommutative geometry defined on a curved background, which is interesting because of possible applications to quantum cosmology and for its dual nature for the interchange of positions and momenta.

In this paper we have discussed a generalization of that model, called doubly κ-deformed Yang model, originally proposed in [35] performing a deformation of the flat metric appearing in the definition of the Yang algebra, so that both the de Sitter and Snyder subalgebras are deformed.

The formalism introduced in this paper, inspired to the one used in Refs. [25,26] for the Snyder model, permits to define an associative star product and a coassociative coproduct, together with a twist. To our knowledge, a Hopf algebra structure for the Yang model has never been discussed before in the literature. Using this formalism, we have been able to calculate in a straightforward way several properties of the associated Hopf algebra. The achievement of these results necessitates the introduction as primary fields of extended tensorial coordinates and a scalar coordinate, besides position and momentum. In addition also the momenta conjugated to these variables must be considered.

The problem of the interpretation of all these extended coordinates is crucial. One possibility is that they derive from the symmetry breaking of an $so(1, 5, g)$ algebra to an $so(1, 3)$ algebra as proposed in [35]. Also the possibility to relate Yang models to Kaluza–Klein theories, interpreting the extra degrees of freedom as higher dimensions is under investigation.

At this point, the question of what are the physical consequences of such models and the possible physical predictions of new observable effects that could be measured arises. A first step in this direction would be to define a dynamics for the theory, writing down a suitable Hamiltonian for particles leaving in Yang spacetime. Further developments may include the definition of a quantum field theory compatible with this structure.

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